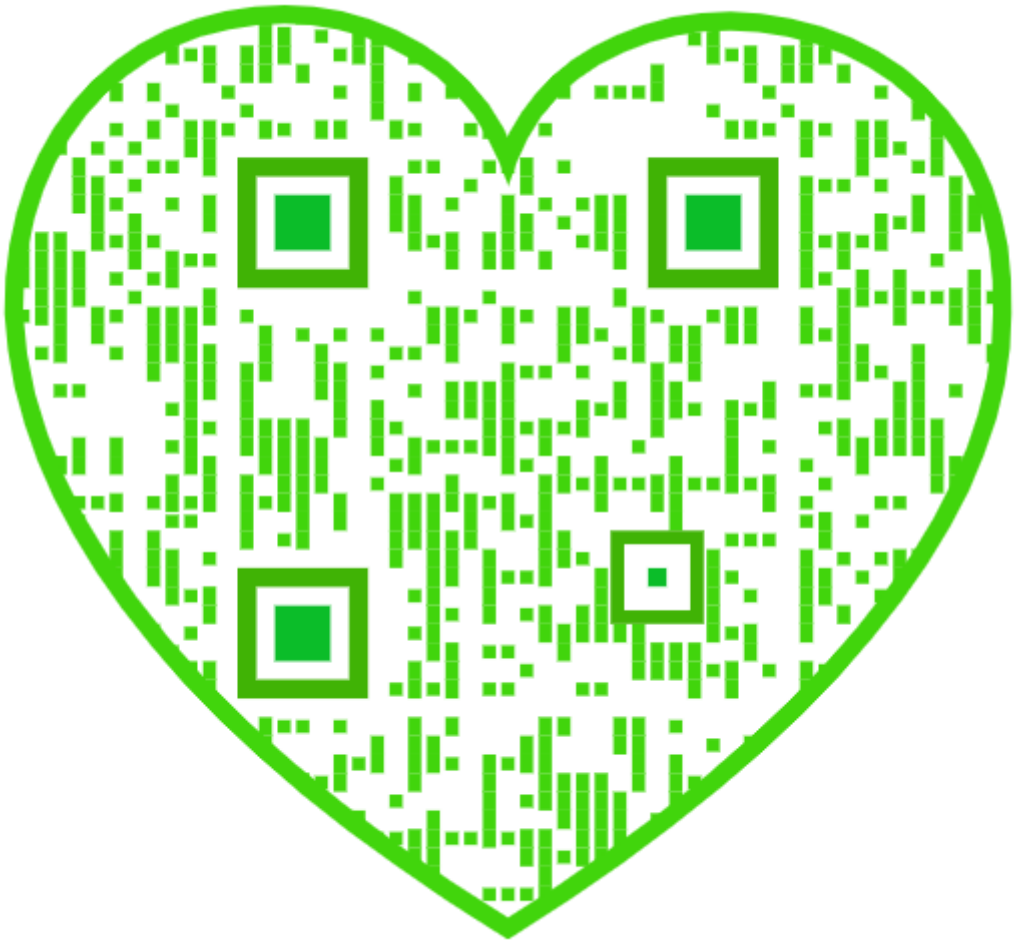

ALGEBRA

AN

ELEMENTARY TEXT-BOOK

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AN ELEMENTARY TEXT-BOOK

FOR THE

HIGHER CLASSES OF SECONDARY SCHOOLS

AND FOR COLLEGES

BY

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PREFACE TO PART II.

THE delay in the appearance of this volume finds an apology partly in circumstances of a private character, partly in public engagements that could not be declined, but most of all in the growth of the work itself as it progressed in my hands. I have not, as some one prophesied, reached ten volumes; but the present concluding volume is somewhat larger and has cost me infinitely more trouble than I expected.

The main object of Part II. is to deal as thoroughly as possible with those parts of Algebra which form, to use Euler's title, an *Introductio in Analysin Infinitorum*. A practice has sprung up of late (encouraged by demands for premature knowledge in certain examinations) of hurrying young students into the manipulation of the machinery of the Differential and Integral Calculus before they have grasped the preliminary notions of a *Limit* and of an *Infinite Series*, on which all the meaning and all the uses of the Infinitesimal Calculus are based. Besides being to a large extent an educational sham, this course is a sin against the spirit of mathematical progress. The methods of the Differential and Integral Calculus which were once

an outwork in the progress of pure mathematics threatened for a time to become its grave. Mathematicians had fallen into a habit of covering their inability to solve many particular problems by a vague wave of the hand towards some generality, like Taylor's Theorem, which was supposed to give "an account of all such things," subject only to the awkwardness of practical inapplicability. Much has happened to remove this danger and to reduce d/dx and $\int dx$ to their proper place as servants of the pure mathematician. In particular, the brilliant progress on the continent of Function-Theory in the hands of Cauchy, Riemann, Weierstrass, and their followers has opened for us a prospect in which the symbolism of the Differential and Integral Calculus is but a minor object. For the proper understanding of this important branch of modern mathematics a firm grasp of the Doctrine of Limits and of the Convergence and Continuity of an Infinite Series is of much greater moment than familiarity with the symbols in which these ideas may be clothed. It is hoped that the chapters on Inequalities, Limits, and Convergence of Series will help to give the student all that is required both for entering on the study of the Theory of Functions and for rapidly acquiring intelligent command of the Infinitesimal Calculus. In the chapters in question, I have avoided trenching on the ground already occupied by standard treatises: the subjects taken up, although they are all important, are either not treated at all or else treated very perfunctorily in other English text-books.

Chapters xxix. and xxx. may be regarded as an

elementary illustration of the application of the modern Theory of Functions. They are intended to pave the way for the study of the recent works of continental mathematicians on the same subject. Incidentally they contain all that is usually given in English works under the title of Analytical Trigonometry. If any one should be scandalised at this traversing of the boundaries of English examination subjects, I must ask him to recollect that the boundaries in question were never traced in accordance with the principles of modern science, and sometimes break the canon of common sense. One of the results of the old arrangement has been that treatises on Trigonometry, which is a geometrical application of Algebra, have been gradually growing into fragments more or less extensive of Algebra itself: so that Algebra has been disorganised to the detriment of Trigonometry; and a consecutive theory of the elementary functions has been impossible. The timid way, oscillating between ill-founded trust and unreasonable fear, in which functions of a complex variable have been treated in some of these manuals is a little discreditable to our intellectual culture. Some expounders of the theory of the exponential function of an imaginary argument seem even to have forgotten the obvious truism that one can prove no property of a function which has not been defined. I have concluded chapter xxx. with a careful discussion of the Reversion of Series and of the Expansion in Power-Series of an Algebraic Function—subjects which have never been fully treated before in an English text-book, although we have in Frost's *Curve Tracing* an admirable collection of examples of their use.

The other innovations call for little explanation, as they aim merely at greater completeness on the old lines. In the chapter on Probability, for instance, I have omitted certain matter of doubtful soundness and of questionable utility; and filled its place by what I hope will prove a useful exposition of the principles of actuarial calculation.

I may here give a word of advice to young students reading my second volume. The matter is arranged to facilitate reference and to secure brevity and logical sequence; but it by no means follows that the volume should be read straight through at a first reading. Such an attempt would probably sicken the reader both of the author and of the subject. Every mathematical book that is worth anything must be read "backwards and forwards," if I may use the expression. I would modify Lagrange's advice a little and say, "Go on, but often return to strengthen your faith." When you come on a hard or dreary passage, pass it over; and come back to it after you have seen its importance or found the need for it further on. To facilitate this skimming process, I have given, after the table of contents, a suggestion for the course of a first reading.

The index of proper names at the end of the work will show at a glance the main sources from which I have drawn my materials for Part II. Wherever I have consciously borrowed the actual words or the ideas of another writer I have given a reference. There are, however, several works to which I am more indebted than appears in the bond. Among these I may mention, besides Cauchy's

Analyse Algébrique, Serret's *Algèbre Supérieure*, and Schlömilch's *Algebraische Analysis*, which have become classical, the more recent work of Stolz, to which I owe many indications of the sources of original information—a kind of help that cannot be acknowledged in footnotes.

I am under personal obligations for useful criticism, for proof-reading, and for help in working exercises, to my assistant, Mr. R. E. ALLARDICE, to Mr. G. A. GIBSON, to Mr. A. Y. FRASER, and to my present or former pupils—Messrs. B. B. P. BRANDFORD, J. W. BUTTERS, J. CROCKETT, J. GOODWILLIE, C. TWEEDIE.

In taking leave of this work, which has occupied most of the spare time of five somewhat busy years, I may be allowed to express the hope that it will do a little in a cause that I have much at heart, namely, the advancement of mathematical learning among English-speaking students of the rising generation. It is for them that I have worked, remembering the scarcity of aids when I was myself a student; and it is in their profit that I shall look for my reward.

G. CHRYSTAL.

EDINBURGH, 1*st November* 1889.

CONTENTS.

The principal technical terms are printed in italics in the following table.

CHAPTER XXIII.

PERMUTATIONS AND COMBINATIONS.

	PAGE
Definition of *r-permutation* and *r-combination*	1
Methods of Demonstration	2
Permutations	2–6
Number of r-permutations of n letters	2
Kramp's Notation for Factorial-n ($n!$)	4
Linear and *Circular Permutations*	4
Number of r-permutations with repetition	4
Permutations of letters having groups alike	5
Examples	6
Combinations	6–12
Combinations from Sets	6
Number of r-combinations of n letters	6
Various properties of ${}_nC_r$—Vandermonde's Theorem	8–9
Combinations when certain letters are alike	10
Combinations with repetition	10
Properties of ${}_nH_r$—Number of r-ary Products	12
Exercises I.	12
Binomial and Multinomial Theorem	14–18
Examples	16
Exercises II.	18
Examples of the application of the Law of Distribution	21–22
Distributions and Derangements	22–25
Distribution Problem	22
Derangement Problem	24
Subfactorial n ($n_!$) defined	25
Theory of Substitutions	25–32
Notation for Substitutions	26

PAGE

Order and *Group* 27
Cyclic Substitutions and *Transpositions* 27
Cycles of a Substitution 27
Decomposition into Transpositions 28
Odd and *even Substitutions* 29
Exercises III. 32
Exercises IV. 33

CHAPTER XXIV.

GENERAL THEORY OF INEQUALITIES.

Definition of Algebraic Inequality 35
Elementary Theorems 36
Examples 38
Derived Theorems 41–50
A Mean-Theorem for Fractions 41
$(x^p-1)/p > < (x^q-1)/q$ 42
$mx^{m-1}(x-1) \gtrless (x^m-1) \gtrless m(x-1)$ 43
$ma^{m-1}(a-b) \gtrless a^m-b^m \gtrless mb^{m-1}(a-b)$ 45
Inequality of Arithmetic and Geometric Means . . . 46
$\Sigma pa^m/\Sigma p > < (\Sigma pa/\Sigma p)^m$ 48
Exercises V. 50
Applications to Maxima- and Minima-Theorems . . . 52–64
Fundamental Theorem 52
Reciprocity Theorem 53
Ten Theorems deduced 53–59
Grillet's Method 59
Method of Increments 61
Purkiss's Theorem 61
Exercises VI. 63

CHAPTER XXV.

LIMITS.

Definition of a Limiting Value and Corollaries . . . 66
Enumeration of Elementary Indeterminate Forms . . . 69
Extension of Fundamental Operations to Limiting Values . . 69
Limit of a Sum 70
Limit of a Product 70
Limit of a Quotient 71
Limit of a Function of Limits 71
Limiting forms for Rational Functions 73
Forms $0/0$ and ∞/∞ 73

PAGE

Fundamental Algebraic Limit $\mathrm{L}(x^m-1)/(x-1)$ when $x=1$ . . 74
Examples—$\mathrm{L}l^r(x+1)/l^rx$, $\mathrm{L}\lambda^rx/l^rx$, when $x=\infty$, &c. . . 76
Exponential Limits 77–81
$\mathrm{L}(1+1/x)^x$ when $x=\infty$, Napierian Base 77
$\mathrm{L}_{x=0}(1+x)^{1/x}$, $\mathrm{L}_{x=\infty}(1+y/x)^x$, $\mathrm{L}_{x=0}(1+xy)^{1/x}$, $\mathrm{L}_{x=0}(a^x-1)/x$. 79
Exponential and Logarithmic Inequalities 80–81
Euler's Constant 81
General Limit Theorems 82
$\mathrm{L}\{f(x)\}^{\phi(x)}=\{\mathrm{L}f(x)\}^{\mathrm{L}\phi(x)}$ 82
$\mathrm{L}\{f(x+1)-f(x)\}=\mathrm{L}f(x)/x$ when $x=\infty$ 83
$\mathrm{L}f(x+1)/f(x)=\mathrm{L}\{f(x)\}^{1/x}$ when $x=\infty$ 84
Exponential Limits Resumed 85
$\mathrm{L}_{x=\infty}a^x/x$, $\mathrm{L}_{x=\infty}\log_a x/x$, $\mathrm{L}_{x=0}x\log_a x$ . . . 85
Examples—$\mathrm{L}_{n=\infty}x^n/n!$, $\mathrm{L}_{n=\infty}m(m-1)\ldots(m-n+1)/n!$ . . 86
$\mathrm{L}_{x=+0}x^x=1$ 87
General Theorem regarding the form 0° 88
Cases where $0^\circ\neq 1$ 88
Forms ∞° and 1^∞ 89
Trigonometrical Limits 89
Fundamental Inequalities 90
$\mathrm{L}\sin x/x$, $\mathrm{L}\tan x/x$, when $x=0$ 91
$\mathrm{L}\left(\sin\frac{a}{x}\Big/\frac{a}{x}\right)^x$, $\mathrm{L}\left(\cos\frac{a}{x}\right)^x$, $\mathrm{L}\left(\tan\frac{a}{x}\Big/\frac{a}{x}\right)^x$, when $x=\infty$ . . 91
Limit of the Sum of an Infinite Number of Infinitely Small Terms . 92
$\mathrm{L}(1^r+2^r+\ldots+n^r)/n^{r+1}$ 92
Dirichlet's Theorem 94
Geometrical Applications 95
Exercises VII. 97

CHAPTER XXVI.

CONVERGENCE OF INFINITE SERIES AND OF INFINITE PRODUCTS.

Definition of the terms *Convergent*, *Divergent*, *Oscillating*, *Non-Convergent* 101
Necessary and Sufficient Conditions for Convergency . . . 102
Residue and *Partial Residue* 103
Four Elementary Comparison Theorems 105
Ratio of Convergence 106
Absolutely Convergent and *Semi-Convergent* Series . . . 107
Special Tests of Convergency for Series of Positive Terms . 107–118
$\mathrm{L}u_n^{1/n}<>1$ 107
$\mathrm{L}u_{n+1}/u_n<>1$ 107

PAGE

Examples—Integro-Geometric, Logarithmic, Exponential, Binomial Series 108–109
Cauchy's *Condensation Test* 109
Logarithmic Criteria, first form 112
Logarithmic Scale of Convergency 113
Logarithmic Criteria, second form 115
Examples—Hypergeometric and Binomial Series . . 116–118
Historical Note 118
Semi-Convergent Series 119–123
Example of Direct Discussion 120
$u_1 - u_2 + u_3 - \ldots$ 121
Trigonometrical Series 122
Abel's Inequality 122
Convergence of a Series of Complex Terms 123
Necessary and Sufficient Condition for Convergency . . 124
Convergence of the Series of Moduli sufficient . . . 124
Examples—Exponential and Logarithmic Series, &c. . . 125
Application of the Fundamental Laws of Algebra to Infinite Series 125–128
Law of Association 125
Law of Commutation 125
Addition of Infinite Series 127
Law of Distribution 127
Theorem of Cauchy and Mertens 127
Special Discussion of the *Power-Series* 129
Condition for Convergency of Power-Series 129
Circle and Radius of Convergence 129
Discontinuity and *Infinitely Slow Convergency* . . . 129
Uniform and *Non-Uniform Convergency* 130
Du Bois-Reymond's Theorem 131
Abel's Theorem 133
Principle of Indeterminate Coefficients 135
Infinite Products 135–144
Convergent, Divergent, and Oscillating Products . . . 136
Discussion by means of $\Sigma \log(1+u_n)$ 136
Criteria from Σu_n 137
Independent Criteria 139
Convergence of Complex Products 139
General Properties of Infinite Products 140
Estimation of the Residue of a Series or Product . . . 145
Convergence of Double Series 147–157
Four ways of Summation 148
Double Series of Positive Terms 150
Cauchy's Test for Absolute Convergency 153
Examples of Exceptional Cases 155
Imaginary Double Series 157
General Theorem regarding Double Power-Series . . . 158
Exercises VIII. 158

CHAPTER XXVII.

BINOMIAL AND MULTINOMIAL SERIES FOR ANY INDEX.

	PAGE
Binomial Series	162–175
Determination of Coefficients, validity being assumed	162
Euler's Proof	164
Addition Theorem for the Binomial Series	165
Examples	168
Ultimate Sign of the Terms	169
Integro-Binomial Series	170
Examples	172
Exercises IX.	175
Series deduced by Expansion of Rational Functions	176–186
Expression of $\alpha^n+\beta^n$ and $(\alpha^{n+1}-\beta^{n+1})/(\alpha-\beta)$ in terms of $\alpha\beta$ and $\alpha+\beta$, and connected series	177
Sum and Number of r-ary Products	181
Examples	184
Exercises X.	186
Multinomial Series	189–191
Numerical Approximation by Binomial Series	191–194
Numerically Greatest Term	191
Limits for the Residue	192
Exercises XI.	195

CHAPTER XXVIII.

EXPONENTIAL AND LOGARITHMIC SERIES.

Exponential Series	197–204
Determination of Coefficients, validity being assumed	197
Deduction from Binomial Theorem	198
Calculation of e	200
Cauchy's Proof	202
Addition Theorem for the Exponential Series	203
Bernoulli's Numbers	204–209
Expansions of $x/(1-e^{-x})$, $x(e^x+e^{-x})/(e^x-e^{-x})$, &c.	205, 208
Bernoulli's Expression for $1^r+2^r+\ldots+n^r$	208
Summations by means of Exponential Theorem	209–211
Integro-Exponential Series	209
Examples	210
Exercises XII.	212
Logarithmic Series	213–226
Expansion of $\log(1+x)$	214
Derived Expansions	215
Calculation of $\log 2$, $\log 3$, &c.	216

PAGE
Factor Method for calculating Logarithms 219
First Difference of $\log x$ 221
Summations by Logarithmic Series 221–226
$\Sigma\phi(n)x^n/(n+a)(n+b)\ldots$ 222
Examples—Certain Semi-Convergent Series, &c. . . . 224
Inequality and Limit Theorems 226
Exercises XIII. 227

CHAPTER XXIX.

SUMMATION OF THE FUNDAMENTAL POWER-SERIES FOR COMPLEX VALUES OF THE VARIABLE.

Preliminary Matter 230–248
Definition and Properties of the Circular Functions . . 230–238
Evenness, Oddness, Periodicity 231
Graphs of the Circular Functions 232
Addition Formulæ for the Circular Functions . . . 234
Inverse Circular Functions 234
Multiple-valuedness 236
Principal and other *Branches* 236
Inversion of $w=z^n$ and $w^p=z^q$ 238–247
Circulo-Spiral Graphs for $w=z^3$ 240
Multiplicity and Continuity of $\sqrt[n]{w}$ 241
Riemann's Surface 241
Principal and other Branches of $\sqrt[n]{w}$ 243
Circulo-Spiral Graphs for $w^3=z^4$ 245
Principal and other Values of $\sqrt[q]{w^p}$ 246
Exercises XIV. 247
Geometric and Integro-Geometric Series 248
$\Sigma r^n\cos(\alpha+n\theta)$, &c. 249
Formulæ connected with Demoivre's Theorem and the Binomial Theorem for an Integral Index 250–255
Generalisation of the Addition Theorems for the Circular Functions 251
Expansions of $\cos n\theta$, $\sin n\theta/\sin\theta$, &c., in powers of $\sin\theta$ or $\cos\theta$. 252
Expression of $\cos^m\theta\sin^n\theta$ in the form $\Sigma a_p\cos p\theta$ or $\Sigma a_p\sin p\theta$. 253
Exercises XV. 255
Expansion of $\cos\theta$ and $\sin\theta$ in powers of θ . . . 256–259
Exercises XVI. 260
Binomial Theorem for a Complex Variable . . . 261–264
Most general case of all (Abel) 263
Exponential and Logarithmic Series for a Complex Variable . 264–273
Definition of $\mathrm{Exp}\,z$ 264
$\mathrm{Exp}\,(x+yi)=e^x(\cos y+i\sin y)$ 266
Graphic Discussion of $w=\mathrm{Exp}\,z$ 266
Imaginary Period of $\mathrm{Exp}\,z$ 268
$\mathrm{Log}\,w=\log(\mathrm{mod}\,w)+i\,\mathrm{amp}\,w$ 269

PAGE
Principal and other Branches of Log w 269
Definition of $\mathrm{Exp}_a z$ 270
Addition Theorem for Log w 271
Expansion of ${}_\epsilon\mathrm{Log}(1+z)$ 272
Generalisation of the Circular Functions . . . 273–289
General Definitions of Cos z, Sin z, &c. 273
Euler's Exponential Formulæ for Cos z and Sin z . . . 274
Properties of the Generalised Circular Functions . . . 275
Introduction of the *Hyperbolic Functions* . . . 276–289
Expressions for the Hyperbolic Functions 276
Graphs of the Hyperbolic Functions 277
Inverse Hyperbolic Functions 279
Properties of the General Hyperbolic Functions . . 279–283
Inequality and Limit Theorems 283
Geometrical Analogies between the Circular and the Hyperbolic Functions 284
Gudermannian Function 287
Historical Note 288
Exercises XVII. 289
Graphical Discussion of the Generalised Circular Functions . 292–301
Cos $(x+yi)$ 292
Sin $(x+yi)$ 295
Tan $(x+yi)$ 297
Graphs of $f(x+yi)$ and $1/f(x+yi)$ 298
General Theorem regarding *Orthomorphosis* . . . 299
Exercises XVIII. 301
Special Applications to the Circular Functions . . 302–310
Series derived from the Binomial Theorem 303
Series for $\cos m\phi$ and $\sin m\phi$ (m not integral) . . . 303
Expansion of $\sin^{-1}x$, Quadrature of the Circle . . . 305
Examples—Series from Abel, &c. 306
Series derived from the Exponential Series 307
Series derived from the Logarithmic Series 307
$\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \ldots = \frac{1}{2}\theta$ 308
Remarkable Discontinuity of this last Series . . . 308
Series for $\tan^{-1}x$, Gregory's Quadrature of the Circle . . 309
Note on the Arithmetical Quadrature of the Circle . . 309
Exercises XIX., XX. 310, 311

CHAPTER XXX.

GENERAL THEOREMS REGARDING THE EXPANSION OF FUNCTIONS IN INFINITE FORMS.

Expansion in Infinite Series 313–320
Expansion of a Function of a Function 313

PAGE

Expansion of an Infinite Product in the form of an Infinite Series 313
Examples—Theorems of Euler and Cauchy 315
Expansion of Sech x and Sec x 317
Euler's Numbers 318
Expansion of Tanh x, x Coth x, Cosech x; Tan x, x Cot x, Cosec x . 319
Exercises XXI. 320
Expression of Certain Functions as Infinite Products . . 322–333
General Theorem regarding the Limit of an Infinite Product . 322
Products for $\sinh pu$, $\sinh u$; $\sin p\theta$, $\sin\theta$ 324
Wallis's Theorem 327
Products for $\cosh pu$, $\cosh u$; $\cos p\theta$, $\cos\theta$ 327
Products for $\cos\phi+\sin\phi\cot\theta$, $\cos\phi-\sin\phi\tan\theta$, $1+\operatorname{cosec}\theta\sin\phi$. 330
Product for $\cos\phi-\cos\theta$ 331
Remark regarding a Certain Fallacy 332
Exercises XXII. 333
Expansion of Circular and Hyperbolic Functions in an *Infinite Series of Partial Fractions* 335–338
Expressions for $\tan\theta$, $\theta\cot\theta$, $\theta\operatorname{cosec}\theta$, $\sec\theta$ . . . 336
Expressions for $\tanh u$, $u\coth u$, $u\operatorname{cosech} u$, $\operatorname{sech} u$ . . 338
Expressions for the Numbers of Bernoulli and Euler . . 338–343
Series for B_m 339
Product for B_m 340
Certain Properties of B_m 340
Radii of Convergence of the Power-Series for $\tan\theta$, $\theta\cot\theta$, $\theta\operatorname{cosec}\theta$ 340
Series and Product for E_m 341
Certain Properties of E_m 342
Radius of Convergence of the Power-Series for $\sec\theta$ . . 342
Sums of Certain Series involving Powers of Integers . . 343
Power-Series for $\log\sin\theta$, &c. 343
Stirling's Theorem 344
Exercises XXIII. 348
Reversion of Series—Expansion of an *Algebraic Function* . 349–370
General Expansion-Theorem regarding $\Sigma(m,n)x^m y^n=0$ . . 350
Reversion of Series 354
Expansion of the various Branches of an Algebraic Function . 355
Irreducibility, *Ordinary* and *Singular Points*, *Multiple Points*, *Zero Points*, *Poles*, *Zeros*, and *Infinities* of an Algebraic Function 356
Expansion at an Ordinary Point 358
Expansion at a Multiple Point 359
Newton's Parallelogram, *Degree-Points*, *Effective Group of Degree-Points* 362
All the Branches of an Algebraic Function expansible . . 365
Algebraic Zeros and *Infinities* and their *Order* . . . 367
Method of *Successive Approximation* 367
Historical Note 370
Exercises XXIV. 370

CHAPTER XXXI.

SUMMATION AND TRANSFORMATION OF SERIES IN GENERAL.

PAGE
The Method of *Finite Differences* 372–383
Difference Notation 372
Two Fundamental Difference Theorems 375
Summation by Differences 376
Examples—Factorial Series, $\Sigma \sin(\alpha + n\beta)$ 377
Expression of $\sum_1^n u_n$ in terms of the Differences of u_1 . . 379
Montmort's Theorem 381
Euler's Theorem 382
Exercises XXV. 383
Recurring Series 385–389
Scale of Relation 385
Generating Function 386
To find the General Term 387
Solution of *Linear Difference Equation* with Constant Coefficients 388
Summation of Recurring Series 388
Exercises XXVI. 389
Waring's Method for Summation by taking every kth term of a Series whose sum is known 390–392
Miscellaneous Methods 392–394
Use of Partial Fractions 392
Euler's Identity 392
Exercises XXVII. 394

CHAPTER XXXII.

SIMPLE CONTINUED FRACTIONS.

Nature and Origin of Continued Fractions . . . 396–402
Terminating, Non-Terminating, Recurring or *Periodic, Simple* Continued Fractions 396
Component Fractions and *Partial Quotients* . . . 396, 397
Every Number convertible into a S.C.F. 397
Every Commensurable Number convertible into a Terminating S.C.F. 399
Conversion of a Surd into a S.C.F. 401
Exercises XXVIII. 403
Properties of the Convergents to a S.C.F. . . . 404–414
Complete Quotients and *Convergents* 404
Recurrence-Formulæ for Convergents 405
Properties of p_n and q_n 406

PAGE
Fundamental Properties of the Convergents . . . 408
Approximation to S.C.F. 410
Condition that p_n/q_n be a Convergent to x_1 . . . 412
Arithmetical Utility of S.C.FF. 413
Convergence of S.C.F. 414
Exercises XXIX. 415
Closest Rational Approximation of Given Complexity . 417–424
Closeness of Approximation of p_n/q_n 418
Principal and *Intermediate Convergents* 419
Historical Note 421
Examples—Calendar, Eclipses, &c. 423
Exercises XXX. 424

CHAPTER XXXIII.

ON RECURRING CONTINUED FRACTIONS.

Every Simple Quadratic Surd Number convertible into a Recurring S.C.F. 425–430
Recurrence-Formulæ for P_n and Q_n 426
Expressions for P_n and Q_n 428
Cycles of P_n, Q_n, a_n 429
Every Recurring S.C.F. equal to a Simple Quadratic Surd Number 430–432
On the S.C.F. which represents $\sqrt{(C/D)}$ . . . 432–441
Acyclic Quotient of $\sqrt{N/M}$ 434
Cycle of the Partial Quotients of $\sqrt{N/M}$ 435
Cycles of the *Rational Dividends* and *Divisors* of $\sqrt{N/M}$ . . 436
Tests for the Middle of the Cycles 439
Examples—Rapid Calculation of High Convergents, &c. . . 440
Exercises XXXI. 441
Applications to the Solution of *Diophantine Problems* . 445–460
$ax - by = c$ 446
$ax + by = c$ 447
$ax + by + cz = d$, $a'x + b'y + c'z = d'$ 449
Solutions of $x^2 - Cy^2 = \pm H$ and $x^2 - Cy^2 = \pm 1$ . . . 450
General Solution of $x^2 - Cy^2 = \pm H$ when $H < \sqrt{C}$ . . . 451
General Formulæ for the Groups of Solutions of $x^2 - Cy^2 = \pm 1$ and $x^2 - Cy^2 = \pm H$ 452
Lagrange's Reduction of $x^2 - Cy^2 = \pm H$ when $H > \sqrt{C}$ . . 454
Remaining Cases of the Binomial Equation . . . 458
General Equation of the Second Degree 458
Exercises XXXII. 461

CHAPTER XXXIV.

GENERAL CONTINUED FRACTIONS.

PAGE
Fundamental Formulæ 463–466
Meaning of G.C.F. 464
G.C.FF. of *First* and *Second Class* 464
Properties of the Convergents 464
Continuants 466–474
Continuant Notation—*Simple Continuant* . . . 466, 467
Functional Nature of a Continuant 467
Euler's Construction 468
Euler's Continuant-Theorem 470
Henry Smith's Proof of Fermat's Theorem that a Prime of the form $4\lambda+1$ is the Sum of Two Integral Squares . . . 471
Every Continuant reducible to a Simple Continuant . . 472
C.F. in terms of Continuants 473
Equivalent Continued Fractions 473
Reduction of G.C.F. to a form having Unit Numerators . . 474
Exercises XXXIII. 474
Convergence of Infinite C.FF. 477
Convergence, *Divergence*, *Oscillation* of C.F. . . . 477
Partial Criterion for C.F. of First Class 478
Complete Criterion for C.F. of First Class 479
Partial Criterion for C.F. of Second Class 482
Incommensurability of certain C.FF. 484–486
Legendre's Propositions 484
Conversion of Series and Continued Products into C.FF. . 486–496
Euler's Transformation of a Series into an equivalent C.F. . 486
Examples—Brouncker's Quadrature of the Circle, &c. . . 488
C.F. equivalent to a given Continued Product . . . 489
Lambert's Transformation of an Infinite Series into an Infinite C.F. 489
Example—after Legendre 492
C.FF. for $\tan x$ and $\tanh x$ 494
Incommensurability of π and e 495
Gauss's Conversion of the Hypergeometric Series into a C.F. . 495
Exercises XXXIV. 497

CHAPTER XXXV.

GENERAL PROPERTIES OF INTEGRAL NUMBERS.

Numbers which are congruent with respect to a given Modulus 500–506
Modulus and *Congruence* 500
Periodicity of Integers 501
Examples of Properties deduced from Periodicity—Integrality of $x(x+1)\ldots(x+p-1)/p!$, Pythagorean Problem, &c. . . 501

PAGE

Property of an Integral Function 504
Test for Divisibility of $f(x)$ 504
$f(x)$ represents an Infinity of Composites 504
Difference Test of Divisibility 505
Exercises XXXV. 506
On the Divisors of a given Integer 508–518
Limit for the Least Factor of N 508
Sum and Number of the Divisors of a Composite . . . 509
Examples—Perfect Number, &c. 510
Number of Integers $<$N and prime to N, ϕ(N) . . . 511
Euler's Theorem regarding ϕ(N) 513
Gauss's Theorem $\Sigma\phi(d_n)=N$ 514
Properties of $m!$ 515–518
Exercises XXXVI. 518
On the Residues of a Series of Integers in Arithmetical Progression 519–526
Periodicity of the Residues of an A.P. . . . 520, 521
Fermat's Theorem 522
Historical Note 522
Euler's Generalisation of Fermat's Theorem . . . 523
Wilson's Theorem 524
Historical Note 525
Theorem of Lagrange including the Theorems of Fermat and Wilson 525
Exercises XXXVII. 526
Partition of Numbers 527–536
Notation for the Number of Partitions 528
Expansions and Partitions 528
Euler's Table for $P(n \mid * \mid \gtrless q)$ 530
Partition Problems solvable by means of Euler's Table . 531–533
Constructive Theory of Partitions 533–536
Graph of a Partition, Regular Graphs, Conjugate Partitions . 534
Franklin's Proof that $(1-x)(1-x^2)(1-x^3)\ldots = \sum_{p=0}^{\infty}(-)^p x^{\frac{1}{2}(3p^2\pm p)}$ 535
Exercises XXXVIII. 536

CHAPTER XXXVI.

PROBABILITY, OR THE THEORY OF AVERAGES.

Fundamental Notions, *Event, Universe, Series*, &c. . . . 538
Definition of *Probability* or *Chance*, and Remarks thereon . . 539
Corollaries on the Definition 541
Odds on or against an Event 542
Direct Calculation of Probabilities 543–547
Elementary Examples 543
Use of the Law of Distribution 545

PAGE

Examples—Demoivre's Problem, &c. 546
Addition and Multiplication of Probabilities . . . 547–553
Addition Rule for *Mutually Exclusive Events* . . . 547
Multiplication Rule for *Mutually Independent Events* . . 548
Examples 549–553
General Theorems regarding the Probability of Compound Events 553–561
Probability that an Event happen on exactly r out of n occasions . 553
More General Theorem of a Similar Kind 554
Probability that an Event happen on at least r out of n occasions . 555
Pascal's Problem 556
Some Generalisations of the Foregoing Problems . . . 557
The *Recurrence Method* for calculating Probabilities . . . 558
"Duration of Play" 559
Evaluation of Probabilities involving Factorials of Large Numbers . 561
Exercises XXXIX. 562
Mathematical Measure of an Expectation . . . 565–567
Value of an Expectation 566
Addition of Expectations 566
Life Contingencies 567–576
Mortality Table 568
Examples of the Use of a Mortality Table 569
Annuity Problems, *Notation*, and *Terminology*, *Average Accounting* 572–573
Calculation of *Life Insurance Premium* 574
Recurrence-Method for calculating Annuities . . . 575
Columnar or Commutation Method 575
Remarks, General and Bibliographical 577
Exercises XL. 577

RESULTS OF EXERCISES 581

SUGGESTION FOR THE COURSE OF A FIRST READING OF PART II.

Chap. xxiii., §§ 1–15. Chap. xxxvi., §§ 1–4. Chap. xxiv., §§ 1–9. Chap. xxv. Chap. xxvi., §§ 1–5, 12–19, 32–35. Chap. xxvii. Chap. xxviii., §§ 1–5, 8–15. Chap. xxix., §§ 1–19, 23–31. Chap. xxxi. Chap. xxxii. Chap. xxxiii., §§ 10–14. Chap. xxxv. Chap. xxxvi., §§ 5–22.

CORRIGENDA.

P. 122, l. 20, *for* "as n increases" *read* "as n increases and has the same sign."
P. 133, l. 16, *for* "mod $x=\mathrm{R}$" *read* "mod $x=\mathrm{R}$ $(\mathrm{R}<1)$."
P. 243, ll. 25 and 27, *for* "$w^{1/n}$" *read* "$\sqrt[n]{w}$."

CHAPTER XXIII.

Permutations and Combinations.

§ 1.] We have already seen the importance of the statistic of combinations in the elementary theory of integral functions. It was found, for example, that the problem of finding the coefficients in the expansion of a binomial is identical with the problem of finding the numbers of combinations of a certain number of things taken 1, 2, 3, &c., at a time. Besides its theoretical use, the theory of permutations and combinations has important practical applications; for example, to economic statistics, to the calculus of probabilities, to fire and life assurance, and to the theory of voting.

Beginners usually find the subject somewhat difficult. This arises in part from the fineness of the distinctions between the different problems, distinctions which are not always easy to express clearly in ordinary language. Close attention should therefore be paid to the terminology we are now to introduce.

§ 2.] For our present purpose we may represent individual things by letters.

By an *r-permutation* of n letters we mean r of those letters arranged in a certain order, in a straight line. An n-permutation, which means all the letters in a certain order, is sometimes called a *permutation* simply.

Example. The 2-permutations of the three letters a, b, c are $bc, cb; ac, ca; ab, ba$. The permutations of the three letters are $abc, acb; bac, bca; cab, cba$.

By an *r-combination* of n letters we mean r of those letters considered without reference to order.

Example. The 2-combinations of a, b, c are bc, ac, ab.

Unless the contrary is stated, the same letter is not supposed to occur more than once in each combination or permutation. In other words, if the n letters were printed on n separate counters each permutation or combination could be actually selected and set down before our eyes.

Another point to be attended to is that in some problems certain sets of the given letters may be all alike or indifferent; that is to say, it may be supposed that no alteration in any permutation or combination is produced by interchanging them.

§ 3.] The fundamental part of every demonstration of a theorem in the theory of permutations and combinations is an enumeration. It is necessary that this enumeration be systematic and exhaustive. If possible it should also be simplex, that is, each permutation or combination should occur only once; but it may be multiplex, provided the degree of multiplicity be ascertained (see § 8, below).

Along with the enumeration there often occurs the process of reasoning step by step, called *mathematical induction*.

The results of the law of distribution, as applied both to closed functions and to infinite series, are often used (after the manner of chap. iv., §§ 5, 11, and exercise vi. 30) to lighten the labour of enumeration.

All these methods of proof will be found illustrated below. We have called attention to them here in order that the student may know what tools are at his disposal.

PERMUTATIONS.

§ 4.] *The number of r-permutations of n letters* $({}_nP_r)$ *is*

$$n(n-1)(n-2)\ \ldots\ (n-r+1).$$

1*st Proof*.—Suppose that we have r blank spaces, the problem is to find in how many different ways we can fill these with n letters all different.

We can fill the first blank in n different ways, namely, by putting into it any one of the n letters. Having put any one letter into the first blank, we have $n-1$ to choose from in filling the

second blank. Hence we can fill the second blank in $n-1$ different ways for each way we can fill the first. Hence we can fill the two first in $n(n-1)$ ways.

When any two particular letters have been put into the first two blanks, there are $n-2$ left to choose from in filling the third. Hence we can fill the first three blanks in $n(n-1)$ times $(n-2)$ ways.

Reasoning in this way, we see that we can fill the r blanks in $n(n-1)(n-2) \ldots (n-r+1)$ ways.

Hence $${}_nP_r = n(n-1) \ldots (n-r+1).$$

2nd Proof.—We may enumerate, exhaustively and without repetition, the ${}_nP_r$ r-permutations as follows:—

1st. All those in which the first letter a_1 stands first;

2nd. All those in which a_2 stands first: and so on.

There are as many permutations in which a_1 stands first as there are $(r-1)$-permutations of the remaining $n-1$ letters, that is, there are ${}_{n-1}P_{r-1}$ permutations in the first class. The same is true of each of the other n classes.

Hence $${}_nP_r = n_{n-1}P_{r-1}.$$

Now this relation is true for any positive integral values of n and r, so long, of course, as $r \not> n$. Hence we may write successively

$$\begin{aligned} {}_nP_r &= n_{n-1}P_{r-1}, \\ {}_{n-1}P_{r-1} &= (n-1)_{n-2}P_{r-2}, \\ &\ldots\ldots\ldots\ldots \\ {}_{n-r+2}P_2 &= (n-r+2)_{n-r+1}P_1. \end{aligned}$$

If now we multiply all these equations together, and observe that all the P's cancel each other except ${}_nP_r$ and ${}_{n-r+1}P_1$, and observe further that the value of ${}_{n-r+1}P_1$ is obviously $n-r+1$, we see that

$${}_nP_r = n(n-1) \ldots (n-r+2)(n-r+1) \qquad (1).$$

The second proof is not so simple as the first, but it illustrates a kind of reasoning which is very useful in questions regarding permutations and combinations.

Cor. 1. *The number of different ways in which a set of n letters can be arranged in linear order is*

$$n(n-1) \ldots 3.2.1,$$

that is, the product of the first n integral numbers.

This follows at once from (1), for the number required is the number of n-permutations of the n letters. Putting $r = n$ in (1), we have

$$_nP_n = n(n-1) \ldots 2.1 \qquad (2).$$

The product of the first n consecutive integers may be regarded as a function of the integral variable n. It is called *factorial-n*, and is denoted by $n!$.*

Cor. 2. $_nP_r = n!/(n-r)!$.

For
$$_nP_r = n(n-1) \ldots (n-r+1),$$
$$= \frac{n(n-1) \ldots (n-r+1)(n-r) \ldots 2.1}{(n-r) \ldots 2.1},$$
$$= \frac{n!}{(n-r)!}.$$

Cor. 3. *The number of ways of arranging n letters in circular order is* $(n-1)!$, *or* $(n-1)!/2$, *according as clock-order and counter-clock-order are or are not distinguished.*

Since the circular order merely, and not actual position, is in question, we may select any one letter and keep it fixed. We have thus as many different arrangements as there are $(n-1)$-permutations of the remaining $n-1$ letters, that is $(n-1)!$.

If, however, the letters written in any circular order clockwise be not distinguished from the letters written in the same order counter-clock-wise, it is clear that each arrangement will be counted twice over. Hence the number in this case is $(n-1)!/2$.

§ 5.] *When each of the n letters may be repeated, the number of r-permutations is* n^r.

* This is Kramp's notation. Formerly $\underline{|n}$ was used in English works, but this is now being abandoned on account of the difficulty in printing the $\underline{|}$. The value of 1! is of course 1. Strictly speaking, 0! has no meaning. It is convenient, however, to use it, with the understanding that its value is 1; by so doing we avoid the exceptional treatment of initial terms in many series.

Suppose that we have r blanks before us. We may fill the first in n ways; the second also in n ways, since there is now no restriction on the choice of the letter. Hence the first two may be filled in $n \times n$, that is, n^2 ways. With each of these n^2 ways of filling the first two blanks we may combine any one of the n ways of filling the third; hence we may fill the first three blanks in $n^2 \times n$, that is, n^3 ways, and so on. Hence we can fill the r blanks in n^r ways.

§ 6.] *The number of permutations of n letters of which a group of α are all alike, a group of β all alike, a group of γ all alike, &c., is*

$$n!/\alpha!\beta!\gamma! \ldots,$$ where $\alpha+\beta+\gamma+\ldots = n$

Let us suppose that x denotes the number in question. If we take any one of the x permutations and keep all the rest of the letters fixed in their places, but make the α letters unlike and permutate them in every possible way among themselves, we shall derive $\alpha!$ permutations in which the α letters are all unlike. Hence the effect of making the α letters unlike is to derive $x\alpha!$ permutations from the x permutations.

If we now make all the β letters unlike, we derive $x\alpha!\beta!$ permutations from the $x\alpha!$.

Hence, if we make all the letters unlike, we derive $x\alpha!\beta!\gamma!\ldots$ permutations. But these must be exactly all possible permutations of n letters all unlike, that is, we must have

$$x\alpha!\beta!\gamma! \ldots = n!.$$

Hence
$$x = n!/\alpha!\beta!\gamma! \ldots$$

Cor. *The number of ways in which n things can be put into r pigeon-holes, so that α shall go into the first, β into the second, γ into the third, and so on, is*

$$n!/\alpha!\beta!\gamma! \ldots$$

N.B.—*The order of the pigeon-holes is fixed, and must be attended to, but the order of the things inside the holes is indifferent.*

Putting the things into the holes is evidently the same as allowing them to stand in a line and affixing to them labels marked with the names of the holes. There will thus be α

labels each marked 1, β each marked 2, γ each marked 3, and so on.

The problem is now to find in how many ways n labels, α of which are alike, β alike, γ alike, &c., can be distributed among n things standing in a given order. The number in question is $n!/\alpha!\beta!\gamma!\ \ldots$, by the above proposition.

Example 1. In arranging the crew of an eight-oared boat the captain has four men that can row only on the stroke-side and four that can row only on the bow-side. In how many different ways can he arrange his boat—1st, when the stroke is not fixed; 2nd, when the stroke is fixed?

In the first case, the captain may arrange his stroke-side in as many ways as there are 4-permutations of 4 things, that is, in 4! ways, and he may arrange the bow-side in just as many ways. Since the arrangements of the two sides are independent, he has, therefore, $4!\times 4!(=576)$ different ways of arranging the whole crew.

In the second case, since stroke is fixed, there are only 3! ways of arranging the stroke-side. Hence, in this case, there are $3!\times 4!(=144)$ different ways of arranging the crew.

Example 2. Find the number of permutations that can be made with the letters of the word *transalpine*.

The letters are *traannslpie*, there being two sets, each containing two like letters. The number required is therefore (by § 6) $11!/2!2!=11.10.9.8.7.6.5.3.2=9979200$.

Example 3. In how many different ways can n different beads be formed into a bracelet?

Since merely turning the bracelet over turns a clock-arrangement of the stones into the corresponding counter-clock-arrangement, it follows, by § 4, that the number required is $(n-1)!/2$.

COMBINATIONS.

§ 7.] *The number of ways in which s things can be selected by taking one out of a set of n_1, one out of a set of n_2, &c., is $n_1 n_2 \ldots n_s$.*

The first thing can be selected in n_1 ways; the second in n_2 ways; and so on. Hence, since the selection of each of the things does not depend in any way on the selection of the others, the number of ways in which the s things can be selected is $n_1 \times n_2 \times \ldots \times n_s$.

§ 8.] *The number of r-combinations of n letters ($_nC_r$) is*

$$n(n-1)\ \ldots\ (n-r+1)/1.2\ \ldots\ r.$$

1st Proof.—We may enumerate the combinations as follows:—

1st. All those that contain the letter a_1;
2nd. „ „ „ a_2;
. . . . . .
nth. „ „ „ a_n.

In each of these classes there is the same number of combinations; namely, as many combinations as there are $(r-1)$-combinations of $n-1$ letters; for we obviously form all the r-combinations in which a_1 occurs by forming all possible $(r-1)$-combinations of $a_2, a_3, \ldots, a_n$ and adding a_1 to each of them.

This enumeration, though exhaustive, is not simplex; for each r-combination will be counted once for every letter it contains, that is, r times. Hence

$$r\,{}_nC_r = n\,{}_{n-1}C_{r-1} \qquad (1).$$

This relation holds for all values of n and r, so long as $r \not> n$. Hence we have successively—

$${}_nC_r = \frac{n}{r}\,{}_{n-1}C_{r-1},$$

$${}_{n-1}C_{r-1} = \frac{n-1}{r-1}\,{}_{n-2}C_{r-2},$$

$${}_{n-2}C_{r-2} = \frac{n-2}{r-2}\,{}_{n-3}C_{r-3},$$

. . . . . .

$${}_{n-r+2}C_2 = \frac{n-r+2}{2}\,{}_{n-r+1}C_1.$$

If we multiply these $r-1$ equations together, and observe that the C's cancel, except ${}_nC_r$ and ${}_{n-r+1}C_1$, and that the value of ${}_{n-r+1}C_1$ is obviously $n-r+1$, we have

$${}_nC_r = \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} \qquad (2).$$

2nd Proof.—Since every r-combination of n letters, if permutated in every possible way, would give $r!$ r-permutations, and all the r-permutations of the n letters can be got once and only once by dealing in this way with all the r-combinations, it follows

that ${}_nC_r r! = {}_nP_r$. Hence ${}_nC_r = {}_nP_r/r! = n(n-1)\ldots(n-r+1)/1.2\ldots r$.

Cor. 1. If we multiply both numerator and denominator of the expression for ${}_nC_r$ by $(n-r)(n-r-1)\ldots 2.1$, we deduce

$$ {}_nC_r = n!/r!(n-r)! \qquad (3). $$

Cor. 2. $$ {}_nC_r = {}_nC_{n-r} $$

This follows at once from (3). It may also be proved by enumeration; for it is obvious that for every r-combination of the n things we select we leave behind an $(n-r)$-combination; there are, therefore, just as many of the latter as of the former.

Cor. 3. $$ {}_nC_r = {}_{n-1}C_r + {}_{n-1}C_{r-1} \qquad (4). $$

This can be proved by using the expressions for ${}_nC_r$, ${}_{n-1}C_r$, ${}_{n-1}C_{r-1}$, and this is important, because it shows that the property holds for functions of n having the form (2) irrespective of any restriction on the value of n.

The theorem (when n is a positive integer) also follows at once by classifying the r-combinations of n letters $a_1, a_2, \ldots, a_n$ into, 1st, those that contain a_1, ${}_{n-1}C_{r-1}$ in number, and, 2nd, those that do not contain a_1, ${}_{n-1}C_r$ in number.

Cor. 4. $$ {}_{n-1}C_s + {}_{n-2}C_s + {}_{n-3}C_s + \ldots + {}_sC_s = {}_nC_{s+1} \qquad (5). $$

Since the order of letters in any combination is indifferent, we may arrange them in alphabetical order, and enumerate the $(s+1)$-combinations of n letters by counting, 1st, those in which a_1 stands first; 2nd, those in which a_2 stands first, &c. This enumeration is clearly both exhaustive and simplex; and we observe that a_1 cannot occur in any of the combinations of the 2nd class, neither a_1 nor a_2 in any of the 3rd class, and so on. Hence the number of combinations in the 1st class is ${}_{n-1}C_s$; in the 2nd, ${}_{n-2}C_s$; in the 3rd, ${}_{n-3}C_s$; and so on. Thus the theorem follows.

Cor. 5. $$ {}_pC_s + {}_pC_{s-1}\,{}_qC_1 + {}_pC_{s-2}\,{}_qC_2 + \ldots + {}_pC_1\,{}_qC_{s-1} + {}_qC_s = {}_{p+q}C_s \qquad (6). $$

If we divide $p+q$ letters into two groups of p and q respectively, the ${}_{p+q}C_s$ s-combinations of the $p+q$ letters may be classified exhaustively and simplexly as follows:—

1st. All the s-combinations of the p letters. The number of these is $_pC_s$.

2nd. All the combinations found by taking every one of the $(s-1)$-combinations of the p things with every one of the 1-combinations of the q things. The number of these is

$$_pC_{s-1} \times {}_qC_1.$$

3rd. All the combinations found by taking every one of the $(s-2)$-combinations of the p things with every one of the 2-combinations of the q things. The number of these is

$$_pC_{s-2} \times {}_qC_2.$$

And so on. Thus the theorem follows.

It should be noticed that Cor. 4 and Cor. 5 furnish propositions in the summation of series. For example, we may write Cor. 5 thus—

$$\begin{aligned}\frac{p(p-1)\ldots(p-s+1)}{1.2\ldots s} &+ \frac{p(p-1)\ldots(p-s+2)}{1.2\ldots(s-1)} \cdot \frac{q}{1} \\ &+ \frac{p(p-1)\ldots(p-s+3)}{1.2\ldots(s-2)} \cdot \frac{q(q-1)}{1.2} \\ &\ldots\ldots\ldots\ldots \\ &+ \frac{p}{1} \cdot \frac{q(q-1)\ldots(q-s+2)}{1.2\ldots(s-1)} \\ &+ \frac{q(q-1)\ldots(q-s+1)}{1.2\ldots s} \\ = \frac{(p+q)(p+q-1)\ldots(p+q-s+1)}{1.2\ldots s} &\qquad (7).\end{aligned}$$

It is obvious that (7) is an algebraical identity which could be proved by actually transforming the left-hand side into the right (see chap. v., § 16). If we take this view, it is clear that the only restriction upon p, q, s is that s shall be a positive integer. Thus generalised, (7) becomes of importance in the establishment of the Binomial Theorem for fractional and negative indices.

Cor. 6. If we multiply both sides of (7) by $1.2\ldots s$, and denote $p(p-1)\ldots(p-s+1)$ by p_s, we deduce

$$(p+q)_s = p_s + {}_sC_1\, p_{s-1}q_1 + {}_sC_2\, p_{s-2}q_2 + \ldots + q_s \qquad (8),$$

which is often called *Vandermonde's theorem*, although the result was known before Vandermonde's day.

§ 9.] *To find the number of r-combinations of $p+q$ letters p of which are alike.*

1st. With the q unlike letters we can form ${}_qC_r$ r-combinations.

2nd. Taking one of the p letters, and $r-1$ of the q, we can form ${}_qC_{r-1}$ r-combinations.

3rd. Taking two of the p, and $r-2$ of the q, we can form ${}_qC_{r-2}$ r-combinations; and so on, till at last we take r of the p (supposing $p>r$), and form one r-combination.

We thus find for the number required

$$ {}_qC_r + {}_qC_{r-1} + {}_qC_{r-2} + \ldots + {}_qC_1 + 1 $$
$$ = q!\left\{\frac{1}{r!(q-r)!} + \frac{1}{(r-1)!(q-r+1)!} + \ldots + \frac{1}{1!(q-1)!} + \frac{1}{q!}\right\}. $$

Cor. *The number of r-permutations of $p+q$ things p of which are alike is*

$$ q!r!\left\{\frac{1}{r!(q-r)!} + \frac{1}{1!(r-1)!(q-r+1)!} + \frac{1}{2!(r-2)!(q-r+2)!} + \right. $$
$$ \left. \ldots + \frac{1}{(r-1)!1!(q-1)!} + \frac{1}{r!q!}\right\}. $$

For, with the ${}_qC_r$ combinations of the 1st class above we can form ${}_qC_r\, r!$ permutations;

With the ${}_qC_{r-1}$ combinations of the 2nd class, ${}_qC_{r-1}\, r!$ permutations;

With the ${}_qC_{r-2}$ combinations of the 3rd class (in each of which two letters are alike), ${}_qC_{r-2}\, r!/2!$ permutations: and so on.

Hence the whole number of permutations is

$$ {}_qC_r\, r! + {}_qC_{r-1}\, r!/1! + {}_qC_{r-2}\, r!/2! + \ldots + {}_qC_1\, r!/(r-1)! + 1, $$

whence the result follows.

A similar process will give the number of r-combinations, or of r-permutations, when we have more than one group of like letters; but the general formula is very complicated.

§ 10.] *The number of r-combinations of n letters (${}_nH_r$), when each letter may be repeated any number of times up to r, is*

$$ n(n+1)(n+2)\ldots(n+r-1)/1.2.3\ldots r \quad (1). $$

In the first place, we remark that the number of $(r+1)$-combinations, in each of which the letter a_1 occurs at least once, is the same as the number of r-combinations not subject to this restriction. This is obvious if we reflect that every $(r+1)$-combination of the kind described leaves an r-combination when a_1 is removed, and, conversely, every r-combination of the n letters gives, when a_1 is added to it, an $(r+1)$-combination of the kind described.

It follows, then, that if we add to each of the r-combinations of the theorem all the n letters, we get all the $(n+r)$-combinations of the n letters, in each of which each letter appears at least once, and not more than $r+1$ times. We may therefore enumerate the latter instead of the former.

This new problem may be reduced to a question of permutations as follows. Instead of writing down all the repeated letters, we may write down each letter once, and write after it the letter s (initial of same) as often as the letter is repeated. Thus, we write *asssbsscs* . . . instead of *aaaabbbcc* . . . With this notation there will occur in each of the $(n+r)$-combinations the n letters $a_1, a_2, \ldots, a_n$ along with r s's. The problem now is to find in how many ways we can arrange these $n+r$ letters. It must be remembered that there is no meaning in the occurrence of s at the beginning of the series; hence, since the order of the letters $a_1, a_2, \ldots, a_n$ is indifferent, we may fix a_1 in the first place. We have now to consider the different arrangements of the $n-1$ letters $a_2, a_3, \ldots, a_n$ along with r s's. In so doing we must observe that nothing depends on the order of $a_2, a_3, \ldots, a_n$ *inter se*; so that in counting the permutations they must be regarded as all alike. We have, therefore, to find the number of permutations of $n-1+r$ things, $n-1$ of which are alike, and r of which are alike. Hence we have

$$ {}_nH_r = \frac{(n+r-1)!}{(n-1)!\,r!} \qquad (2), $$

$$ = \frac{n(n+1)\ldots(n+r-1)}{1.2\ldots r}. $$

Cor. 1. ${}_nH_r = {}_{n+r-1}C_r$.

This follows at once from (2).

Cor. 2. ${}_nH_r = {}_{n-1}H_r + {}_nH_{r-1}$.

For the r-combinations consist, 1st, of those in which a_1 occurs at least once, the number of which we have seen to be ${}_nH_{r-1}$; 2nd, of those in which a_1 does not occur at all, the number of which is ${}_{n-1}H_r$.

Cor. 3. ${}_nH_r = {}_{n-1}H_r + {}_{n-1}H_{r-1} + {}_{n-1}H_{r-2} + \ldots + {}_{n-1}H_1 + 1$.

This follows from the consideration that we may classify the r-combinations into

1st. Those in which a_1 does not occur at all, ${}_{n-1}H_r$ in number;

2nd. Those in which a_1 occurs once, ${}_{n-1}H_{r-1}$ in number;

3rd. Those in which a_1 occurs twice, ${}_{n-1}H_{r-2}$ in number: and so on.

Cor. 4. *The number of different r-ary products that can be made with n different letters is* $n(n+1) \ldots (n+r-1)/1 \cdot 2 \ldots r$; *and the number of terms in a complete integral function of the rth degree in n variables is* $(n+1)(n+2) \ldots (n+r)/1 \cdot 2 \ldots r$.

The first part of the corollary is of course obvious. The second follows from the consideration that the complete integral function is the sum of all possible terms of the degrees 0, 1, 2, . . ., r respectively. Hence the number of its terms is

$$1 + {}_nH_1 + {}_nH_2 + \ldots + {}_nH_r.$$

But, by Cor. 3, this sum is ${}_{n+1}H_r$.

We have thus obtained a general solution of the problems suggested in chap. iv., §§ 17, 19. As a verification, if we put $n=2$, we have for the number of terms in the general integral function of the rth degree in two variables $3 \cdot 4 \ldots (r+2)/1 \cdot 2 \ldots r$, which reduces to $(r+1)(r+2)/2$, in agreement with our former result.

Exercises I.

Combinations and Permutations.

(1.) How many different numbers can be made with the digits 11122333450?

(2.) How many different permutations can be made of the letters of the sentence *Ut tensio sic vis*?

(3.) How many different numbers of 4 digits can be formed with 0123456?

(4.) How many odd numbers can be formed with the digits 3694?

(5.) If ${}_{2n}C_{n-1}/{}_{2n-2}C_n = 132/35$, find n.

(6.) If $m = {}_nC_2$, show that ${}_mC_2 = 3_{n+1}C_4$.

(7.) In any set of n letters, if the number of r-permutations which contain a be equal to the number of those that do not contain a, prove that the same holds of r-combinations.

(8.) In how many ways can the major pieces of a set of chess-men be arranged in a line on the board?

If the pawns be included, in how many ways can the pieces be arranged in two lines?

(9.) Out of 13 men, in how many ways may a guard of 6 be formed in line, the order of the men to be attended to?

(10.) In how many ways can 12 men be selected out of 17—1st, if there be no restriction on the choice; 2nd, if 2 particular men be always included; 3rd, if 2 particular men never be chosen together?

(11.) In how many ways can a bracelet be made by stringing together 5 like pearls, 6 like rubies, and 7 like diamonds?

How many different settings of 3 stones for a ring could be selected from the above?

What modification of the solution of the first part of the above problem is necessary when two, or all three, of the given numbers are even?

(12.) In how many ways can an eight-oared boat be manned out of 31 men, 10 of whom can row on the stroke side only, 12 on the bow side only, and the rest on either side?

(13.) In a regiment there are 10 captains, 20 lieutenants, 30 sergeants, and 60 corporals. In how many ways can a party be selected, consisting of 2 captains, 5 lieutenants, 10 sergeants, and 20 corporals?

(14.) Three persons have 4 coats, 5 vests, and 6 hats between them; in how many different ways can they dress?

(15.) A man has 12 relations, 7 ladies and 5 gentlemen; his wife has 12 relations, 5 ladies and 7 gentlemen. In how many ways can they invite a dinner party of 6 ladies and 6 gentlemen so that there may be 6 of the man's relations and 6 of the wife's?

(16.) In how many ways can 7 ladies and 7 gentlemen be seated at a round table so that no 2 ladies sit together?

(17.) At a dinner-table the host and hostess sit opposite each other. In how many ways can $2n$ guests be arranged so that 2 particular guests do not sit together?

(18.) In how many ways can a team of 6 horses be selected out of a stud of 16, so that there shall always be 3 out of the 6 ABCA'B'C', but never AA', BB', or CC' together?

(19.) With 9 consonants and 7 vowels, how many words can be made, each containing 4 consonants and 3 vowels—1st, when there is no restriction on the arrangement of the letters; 2nd, when two consonants are never allowed to come together?

(20.) In how many ways can 52 cards, all different, be dealt into 4 equal

hands, the order of the hands, but not of the cards in the hands, to be attended to?

In how many cases will 13 particular cards fall in one hand?

(21.) In how many ways can a set of 12 black and 12 white draught-men be placed on the black squares of a draught-board?

(22.) In how many ways can a set of chess-men be placed on a chess-board?

(23.) How many 3-combinations and how many 3-permutations can be made with the letters of *parabola*?

(24.) With an unlimited number of red, white, blue, and black balls at disposal, in how many ways can a bagful of 10 be selected?

In how many of these selections will all the colours be represented?

(25.) In an election under the cumulative system there were p candidates for q seats; (1) in how many ways can an elector give his votes; (2) if there be r voters, how many different states of the poll are there?

If there be 15 candidates and 10 seats, and a voter give one minute to the consideration of each way of giving his vote, how long would it take him to make up his mind how to vote?

BINOMIAL AND MULTINOMIAL THEOREMS.

§ 11.] It has already been shown, in chap. iv., § 11, that

$$(a+b)^n = a^n + {}_nC_1\, a^{n-1}b + \ldots + {}_nC_r\, a^{n-r}b^r + \ldots + b^n,$$

where ${}_nC_1$, ${}_nC_2$, . . ., ${}_nC_r$ . . . denote the numbers of 1-, 2-, . . ., r-combinations of n things. Using the expressions just found for ${}_nC_1$, ${}_nC_2$, &c., we now have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}a^{n-r}b^r + \ldots + b^n \qquad (1).$$

This is the Binomial Theorem as Newton discovered it, proved, of course, as yet for positive integral indices only.

§ 12.] We may establish the Binomial Theorem by a somewhat different process of reasoning, which has the advantage of being applicable to the expansion of an integral power of any multinomial.

Consider

$$(a_1 + a_2 + \ldots + a_m)^n \qquad (2).$$

We have to distribute the product of n factors, namely,

$$(a_1 + a_2 + \ldots + a_m)(a_1 + a_2 + \ldots + a_m)\ldots(a_1 + a_2 + \ldots + a_m) \quad (3);$$

and the problem is to find the coefficient of any given term, say

$$a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_m^{\alpha_m} \qquad (4),$$

where of course $\alpha_1 + \alpha_2 \ldots + \alpha_m = n$. In other words, we have to find how often the partial product (4) occurs in the distribution of (3).

We may write out (4) in a variety of ways, such as

$$a_1 a_1 a_2 a_2 a_2 a_3 a_4 a_4 \ldots \qquad (5),$$

there being always α_1 a_1's, α_2 a_2's, &c.

Written as in (5) we may regard the partial product as formed by taking a_1 from the 1st and 2nd brackets in (3); a_2 from the 3rd, 4th, and 5th; a_3 from the 6th; and so on. It appears, therefore, that the partial product (4) will occur just as often as we can make different permutations of the n letters, such as (5). Now, since α_1 of the letters are all alike, α_2 all alike, &c., the number of different permutations is, by § 6, $n!/\alpha_1!\alpha_2! \ldots \alpha_m!$. Hence we have

$$(a_1 + a_2 + \ldots + a_m)^n = \Sigma \frac{n!}{\alpha_1!\alpha_2! \ldots \alpha_m!} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_m^{\alpha_m} \qquad (6):$$

wherein $\alpha_1, \alpha_2, \ldots \alpha_m$ assume all positive integral values consistent with the relation

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = n \qquad (7).$$

This is the *Multinomial Theorem* for a positive integral index.

The Binomial Theorem is merely the particular case where $m = 2$. We then have, since $\alpha_1 + \alpha_2 = n$, and therefore $\alpha_2 = n - \alpha_1$,

$$(a_1 + a_2)^n = \Sigma \frac{n!}{\alpha_1!(n-\alpha_1)!} a_1^{\alpha_1} a_2^{n-\alpha_1},$$

$$= \Sigma \frac{n(n-1) \ldots (n-\alpha_1+1)}{\alpha_1!} a_1^{\alpha_1} a_2^{n-\alpha_1},$$

which agrees with (1).

Cor. *To find the coefficient of x^r in the expansion of*

$$(b_1 + b_2 x + \ldots + b_m x^{m-1})^n \qquad (8)$$

we have simply to pick out all the terms which contain x^r. The general term is

$$\frac{n!}{\alpha_1!\alpha_2! \ldots \alpha_m!} b_1^{\alpha_1} b_2^{\alpha_2} \ldots b_m^{\alpha_m} x^{\alpha_2 + 2\alpha_3 + \ldots + (m-1)\alpha_m}.$$

Hence we have to take all the terms which are such that

$$a_2 + 2a_3 + \ldots + (m-1)a_m = r \qquad (9).$$

The coefficient of x^r in the expansion of (8) is therefore

$$\Sigma \frac{n!}{a_1!a_2!\ldots a_m!} b_1^{a_1} b_2^{a_2} \ldots b_m^{a_m} \qquad (10),$$

where $a_1, a_2, \ldots, a_m$ have all positive integral values subject to the restrictions (7) and (9).

Example 1. The coefficient of a^3b^2 in the expansion of $(a+b+c+d)^5$ is

$$\frac{5!}{3!2!0!0!} = 10.$$

Example 2. To find the coefficient of x^5 in $(1+2x+x^2)^4$.

Here we must have $a_1 + a_2 + a_3 = 4,$

$$a_2 + 2a_3 = 5.$$

Hence $a_1 = a_3 - 1, \quad a_2 = 5 - 2a_3.$

Since a_1 and a_2 must both be positive, the only two admissible values of a_3 are 1 and 2. We have therefore the following table of values :—

a_1	a_2	a_3
0	3	1
1	1	2

The required coefficient is therefore

$$\frac{4!}{0!3!1!} 1^0 2^3 1^1 + \frac{4!}{1!1!2!} 1^1 2^1 1^2 = 56.$$

The correctness of the result may be easily verified in the present case for $(1+2x+x^2)^4 = (1+x)^8$, the coefficient of x^5 in which is ${}_8C_5 = 56$.

Example 3. To find the greatest coefficient, or coefficients, in the expansion of $(a_1 + a_2 + \ldots + a_m)^n$.

This amounts to determining $x, y, z, \ldots$ so that $n!/x!y!z!\ldots$ shall be a maximum, where $x+y+z+\ldots = n$. This, again, amounts to determining $x, y, z, \ldots$ so that

$$u = x!y!z!\ldots \qquad (1)$$

shall be a minimum, subject to the condition

$$x+y+z+\ldots = n \qquad (2).$$

Let us first consider the case where there are only two variables, x and y. We obtain all possible values of $x!y!$ by giving y successively the values $0, 1, 2, \ldots, n$, x taking in consequence the values $n, n-1, n-2, \ldots, 0$. The consecutive value to $x!y!$ is $(x-1)!(y+1)!$, and the ratio of the latter to the former is $(y+1)/x$; that is (since $x+y=n$), $(n+1-x)/x$, that is,

$(n+1)/x-1$. This ratio is less than unity so long as $(n+1)/x<2$, that is, so long as $x>(n+1)/2$. Until x falls below this value the terms in the series above mentioned will decrease; and after x falls below this limit they will begin to increase.

If n be odd, $=2k+1$ say, then $(n+1)/2=k+1$. Hence, if we make $x=k+1$, the ratio $(n+1)/x-1=1$, and two consecutive values of $x!y!$, viz. $(k+1)!k!$ and $k!(k+1)!$, are equal and less than any of the others.

If n be even, $=2k$ say, then $(n+1)/2=k+\frac{1}{2}$. Hence, if we make $x=k$, we obtain a single term of the series, viz. $k!k!$, which is less than any of the others.

Returning now to the general case, we see that, if u be a minimum for all values of $x, y, z, \ldots$ subject to the restriction (2), it will also be a minimum for values such that x and y alone are variable, $z, \ldots$ being all constant. In other words, the values of x and y for which $x!y!z!\ldots$ is a minimum must be such as render $x!y!$ a minimum. Hence, by what has just been proved, x and y must either be equal or differ only by unity. The like follows for every pair of the variables $x, y, z, \ldots$ Let us therefore suppose that p of these are each equal to ξ; then the remaining $m-p$ must each be equal to $\xi+1$. Further, let q be the quotient and r the remainder when n is divided by m; so that $n=mq+r$. We thus have

$$p\xi+(m-p)(\xi+1)=mq+r.$$

Hence
$$m\xi+(m-p)=mq+r;$$

so that
$$\xi+(m-p)/m=q+r/m.$$

Now $(m-p)/m$ and r/m are proper fractions; hence we must have

$$\xi=q, \quad m-p=r.$$

It follows, therefore, that r of the variables are each equal to $q+1$, and the rest are each equal to q. The maximum coefficient is therefore

$$n!/(q!)^{m-r}\{(q+1)!\}^r;$$

that is,
$$n!/(q!)^m(q+1)^r \qquad (3).$$

This coefficient is, of course, common to all terms of the type $a_1^q a_2^q \ldots a_{m-r}^q a_{m-r+1}^{q+1} \ldots a_m^{q+1}$.

As a special case, consider $(a_1+a_2+a_3)^4$. Here $4=3\times1+1$; $q=1$, $r=1$. Hence the terms that have the greatest coefficient are those of the type $a_1a_2a_3^2$, and the coefficient in question is $4!/(1!)^3 2^1=12$. This is right; for we find by distributing that

$$(a_1+a_2+a_3)^4=\Sigma a_1^4+4\Sigma a_1^3a_2+6\Sigma a_1^2a_2^2+12\Sigma a_1^2a_2a_3.$$

Example 4. Show that

$$1-\frac{n}{1}\frac{1+x}{1+nx}+\frac{n(n-1)}{1.2}\frac{1+2x}{(1+nx)^2}-\frac{n(n-1)(n-2)}{1.2.3}\frac{1+3x}{(1+nx)^3}+\ldots=0.$$

(*Wolstenholme.*)

The left-hand side may be written

$$1-\frac{n}{1}\frac{1}{1+nx}+\frac{n(n-1)}{1.2}\frac{1}{(1+nx)^2}-\frac{n(n-1)(n-2)}{1.2.3}\frac{1}{(1+nx)^3}+\ldots$$
$$-\frac{n}{1}\frac{x}{1+nx}+\frac{n(n-1)}{1.2}\frac{2x}{(1+nx)^2}-\frac{n(n-1)(n-2)}{1.2.3}\frac{3x}{(1+nx)^3}+\ldots$$

$$=1-\frac{n}{1}\frac{1}{1+nx}+\frac{n(n-1)}{1.2}\frac{1}{(1+nx)^2}-\frac{n(n-1)(n-2)}{1.2.3}\frac{1}{(1+nx)^3}+\ldots$$
$$-\frac{nx}{1+nx}\left\{1-\frac{(n-1)}{1}\frac{1}{(1+nx)}+\frac{(n-1)(n-2)}{1.2}\frac{1}{(1+nx)^2}-\ldots\right\},$$
$$=\left\{1-\frac{1}{1+nx}\right\}^n-\frac{nx}{1+nx}\left\{1-\frac{1}{1+nx}\right\}^{n-1},$$
$$=\left\{\frac{nx}{1+nx}\right\}^n-\frac{nx}{1+nx}\left\{\frac{nx}{1+nx}\right\}^{n-1},$$
$$=\left\{\frac{nx}{1+nx}\right\}^n-\left\{\frac{nx}{1+nx}\right\}^n,$$
$$=0.$$

§ 13.] The Binomial Theorem can be used in its turn to establish identities in the theory of combinations; as the two following examples will show:—

Example 1. We have

$$1=(\overline{1+x}-x)^r$$
$$=(1+x)^r-{}_rC_1\,x(1+x)^{r-1}+{}_rC_2\,x^2(1+x)^{r-2}-\ldots(-)^r{}_rC_r x^r.$$

On the right-hand side of this identity the coefficient of every power of x must vanish. Hence, s being any positive integer less than r, we have

$${}_rC_s\times 1-{}_{r-1}C_{s-1}\times{}_rC_1+{}_{r-2}C_{s-2}\times{}_rC_2-\ldots+(-)^{s-1}{}_{r-s+1}C_1\times{}_rC_{s-1}+(-)^s{}_rC_s=0.$$

Example 2. To find the sum of the squares of the binomial coefficients.

We have
$$(1+x)^{2n}=(1+x)^n\times(x+1)^n$$
$$=(1+{}_nC_1x+{}_nC_2x^2+\ldots+{}_nC_nx^n)$$
$$\times(x^n+{}_nC_1x^{n-1}+{}_nC_2x^{n-2}+\ldots+{}_nC_n).$$

If we imagine the product on the right to be distributed, we see that the coefficient of x^n is $1^2+{}_nC_1{}^2+{}_nC_2{}^2+\ldots+{}_nC_n{}^2$; the coefficient of x^n on the left is ${}_{2n}C_n$. Hence

$$1^2+{}_nC_1{}^2+{}_nC_2{}^2+\ldots+{}_nC_n{}^2={}_{2n}C_n=2n!/n!n!.$$

Since

$$2n!=2n(2n-1)(2n-2)\ldots4.3.2.1=2^n.1.2\ldots n\times1.3\ldots(2n-1),$$

we have $1^2+{}_nC_1{}^2+{}_nC_2{}^2+\ldots+{}_nC_n{}^2=2^n.1.3\ldots(2n-1)/n!$.

A great variety of results can be obtained by the above process of equating coefficients in identities derived from the binomial theorem; some specimens are given among the exercises below.

Exercises II.

(1.) Find the third term in the expansion of $(2+3x)^{20}$.

(2.) Find the coefficient of x^5 in the expansion of $(1+x+x^2)(1-x)^{15}$.

(3.) Find the term which is independent of x in the expansion of $(x+1/x)^{2n}$.

(4.) Find the coefficient of x^{2r} in the expansion of $(x-1/x)^{2n}$.

(5.) Find the ratio of the coefficients of x^{2n} in $(1+x)^{4n}$ and $(1+x)^{2n}$.

(6.) Find the middle term in the expansion of $(2+\frac{1}{2}x)^n$.

(7.) The product of the coefficients in $(1+x)^{n+1}$: the product of the coefficients in $(1+x)^n=(n+1)^n : n!$.

(8.) The coefficient of x^r in $\{(r-2)x^2+nx-r\}(x+1)^n$ is $n\,{}_nC_{r-2}$.

(9.) If I denote the integral part and F the proper fractional part of $(3+\sqrt{5})^n$, and if ρ denote the rational part and σ the irrational part of the same, show that

$$I=2\{3^n+{}_nC_2\,3^{n-2}.5+{}_nC_4\,3^{n-4}.5^2+\ldots\}-1,$$
$$F=1-(3-\sqrt{5})^n,$$
$$\rho=\tfrac{1}{2}(I+1),$$
$$\sigma=\tfrac{1}{2}(I+2F-1).$$

(10.) If $(\sqrt{2}+1)^{2m+1}=I+F$, where F is a positive proper fraction and I is integral, show that $F(I+F)=1$.

(11.) Find the integral parts of $(2\sqrt{3}+3)^{2m}$, and of $(2\sqrt{3}+3)^{2m+1}$.

(12.) Show that the greatest term in the expansion of $(a+x)^n$ is the $(r+1)$th, where r is the integral part of $(n+1)/(a/x+1)$.

Exemplify with $(2+3)^{10}$ and with $(2+\frac{1}{2})^9$.

(13.) Find the condition that the greatest term in $(a+x)^n$ shall have the greatest coefficient. Find the limits for x in order that this may be so in $(1+x)^{100}$.

(14.) If the pth term be the greatest in $(a+x)^m$, and the qth the greatest in $(a+x)^n$, then either the $(p+q)$th or the $(p+q-1)$th is the greatest in $(a+x)^{m+n}$.

(15.) Sum the series

$$\frac{{}_nC_1}{1}+2\frac{{}_nC_2}{{}_nC_1}+3\frac{{}_nC_3}{{}_nC_2}+\ldots+n\frac{{}_nC_n}{{}_nC_{n-1}}.$$

(16.) Sum the series

$$1+2\,{}_nC_1+3_n\,C_2+4_n\,C_3+\ldots.$$

(17.) If p_r denote the coefficient of x^r in $(1+x)^n$, prove the following relations:—

$$1^\circ.\quad p_1-2p_2+3p_3-\ldots+n(-1)^{n-1}p_n=0.$$
$$2^\circ.\quad \tfrac{1}{2}p_1-\tfrac{1}{3}p_2\quad+\ldots+\frac{(-1)^{n-1}}{n+1}\quad p_n=\frac{n}{n+1}.$$
$$3^\circ.\quad 1+\frac{p_1}{2}+\frac{p_2}{3}\quad+\ldots+\frac{p_n}{n+1}\qquad=\frac{2^{n+1}-1}{n+1}.$$

(18.) If p_r have the same meaning as in last question, show that

$$p_1-\tfrac{1}{2}p_2+\tfrac{1}{3}p_3-\ldots+\frac{(-1)^{n-1}}{n}p_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

(19.) Show that

$${}_rC_s\times 1+{}_{r-1}C_{s-1}\times{}_rC_1+{}_{r-2}C_{s-2}\times{}_rC_2+\ldots+{}_{r-s+1}C_1\times{}_rC_{s-1}+1\times{}_rC_s={}_rC_s2^s.$$

(20.) Show that

$$(1-{}_nC_2+{}_nC_4-\ldots)^2+({}_nC_1-{}_nC_3+\ldots)^2=1+{}_nC_1+{}_nC_2+\ldots.$$

(21.) Show that

$$1\times {}_nC_2+{}_nC_1\times {}_nC_3+\ldots+{}_nC_{n-2}\times {}_nC_n=(2n)!/(n+2)!\,(n-2)!.$$

(22.) Show that $1-n^2+\left(\frac{n(n-1)}{2!}\right)^2-\left(\frac{n(n-1)(n-2)}{3!}\right)^2+\ldots=0$ if n be odd, and $=(-1)^{n/2}(n+2)(n+4)\ldots 2n/2.4\ldots n$ if n be even.

(23.) Show that

$$1.n(n+1)+\frac{n}{1!}(n-1)n+\frac{n(n+1)}{2!}(n-2)(n-1)+\frac{n(n+1)(n+2)}{3!}(n-3)(n-2)$$
$$+\ldots=2(2n+1)!/(n+2)!\,(n-1)!.$$

(24.) If u_r stand for x^r+1/x^r, show that

$$u_{r+1}+{}_{r+1}C_1\,u_{r-1}+{}_{r+1}C_2\,u_{r-3}+\ldots=u_1(u_r+{}_rC_1\,u_{r-2}+{}_rC_2\,u_{r-4}+\ldots).$$

(25.) If a_r denote the coefficient of x^p in $(1+x)^{2(n-r)}(1-x)^{2r}$, show that $a_0-{}_nC_1\,a_1+{}_nC_2\,a_2-\ldots=0$ for all values of p except $p=n$, in which case the right-hand side of the equation is 4^n.

(26.) Show that

$$\frac{1}{x}-\frac{{}_nC_1}{x+1}+\frac{{}_nC_2}{x+2}-\ldots+\frac{(-1)^n{}_nC_n}{x+n}=\frac{n!}{x(x+1)\ldots(x+n)}.$$

(27.) Find the coefficient of x^r in $(1+x+x^2+\ldots)^3$.

(28.) Find the coefficient of x^{18} in $(1+x^5+x^6+x^9)^4$.

(29.) Find the coefficient of x^n in $(1+x+2x^2+3x^3+\ldots)^2$.

(30.) If $a_0,\ a_1,\ \ldots,\ a_{2n}$ be the coefficients of the powers of x in $(1+2x+2x^2)^n$, show that $a_0a_{2n}-a_1a_{2n-1}+\ldots+a_{2n}a_0=0$ if n be odd, $=2^n n!/\{(\frac{1}{2}n)!\}^2$ if n be even.

(31.) If a_r be the coefficient of x^r in $(1+x+x^2+\ldots+x^p)^n$, show that $a_r-{}_nC_1\,a_{r-1}+{}_nC_2\,a_{r-2}-\ldots=0$, unless n be a multiple of $p+1$. What does the equation become in the latter case?

(32.) Find the coefficient of x^{11} in $(1+2x+3x^2+4x^3)^{12}$.

(33.) Write out the expansion of $(a+b+c+d)^7$.

(34.) Show that

$$\Sigma\frac{1^r2^s\ldots n^k}{r!\,s!\ldots k!}=\frac{1}{p!}\left\{\frac{n(n+1)}{2}\right\}^p,$$

where $r,\ s,\ \ldots,\ k$ have all values between 0 and p, both inclusive, subject to the restriction $r+s+\ldots+k=p$.

(35.) If ${}_nH_r$ have the meaning of § 10 above, prove that

1°. ${}_{m+n}H_r={}_mH_r+{}_mH_{r-1}\times {}_nH_1+{}_mH_{r-2}\times {}_nH_2+\ldots+{}_mH_1\times {}_nH_{r-1}$.

2°. $1-{}_nC_1\times {}_nH_1+{}_nC_2\times {}_nH_2-{}_nC_3\times {}_nH_3+\ldots+(-1)^n{}_nC_n\,{}_nH_n=0$.

(36.) If $x_r=x(x+1)\ldots(x+r-1)$, show that

$$(x+y)_r=x_r+{}_rC_1\,x_{r-1}y_1+{}_rC_2\,x_{r-2}y_2+\ldots+y_r.$$

(37.) Find the largest coefficient in the expansion of $(a+b+c+d+e)^{25}$.

EXAMPLES OF THE APPLICATION OF THE LAW OF DISTRIBUTION.

§ 14.] *If we have r sets, consisting of* $n_1, n_2, \ldots, n_r$ *different letters respectively, the whole number of different ways of making combinations by taking* 1, 2, 3, . . . *up to r of the letters at a time, but never more than one from each set, is*

$$(n_1 + 1)(n_2 + 1) \ldots (n_r + 1) - 1.$$

Consider the product

$$\begin{aligned} &(1 + a_1 + b_1 + \ldots n_1 \text{ letters}) \\ &\times (1 + a_2 + b_2 + \ldots n_2 \text{ letters}) \\ &\ldots\ldots\ldots\ldots \\ &\times (1 + a_r + b_r + \ldots n_r \text{ letters}). \end{aligned}$$

In the distributed product there will occur every possible combination of the letters taken 1, 2, 3, . . ., r at a time, with the term 1 in addition. If we replace each letter by unity, each term in the distributed product will become unity, and the sum of these terms will exceed the whole number of combinations by unity. Hence the number required is

$$\begin{aligned} &(1 + n_1)(1 + n_2) \ldots (1 + n_r) - 1 \\ &\quad = \Sigma n_1 + \Sigma n_1 n_2 + \ldots + n_1 n_2 \ldots n_r. \end{aligned}$$

This result might have been obtained by repeated use of § 7.

§ 15.] *If we have r sets of counters, marked with the following numbers—*

$$\begin{aligned} &\alpha_1, \beta_1, \ldots, \kappa_1, \\ &\alpha_2, \beta_2, \ldots, \kappa_2, \\ &\ldots\ldots\ldots \\ &\alpha_r, \beta_r, \ldots, \kappa_r \end{aligned}$$

the number of counters not being necessarily the same for each set, and the inscribed numbers not necessarily all different, then the number of different ways in which r counters can be drawn, one from each set, so that the sum of the inscribed numbers shall be n, is the coefficient of x^n *in the distribution of the product*

$$(x^{\alpha_1} + x^{\beta_1} + \ldots + x^{\kappa_1})$$
$$\times (x^{\alpha_2} + x^{\beta_2} + \ldots + x^{\kappa_2})$$
$$\ldots\ldots\ldots\ldots$$
$$\times (x^{\alpha_r} + x^{\beta_r} + \ldots + x^{\kappa_r}).$$

This theorem is an obvious result of the principles laid down in chap. iv.

Cor. 1. *If in the first set there be a_1 counters marked with the number α_1, b_1 marked with β_1, &c., in the second a_2 marked with α_2, b_2 marked with β_2, &c., the number of ways in which r counters can be drawn so that the sum of the numbers on them is n, is the coefficient of x^n in the distribution of*

$$(a_1 x^{\alpha_1} + b_1 x^{\beta_1} + \ldots + k_1 x^{\kappa_1})$$
$$\times (a_2 x^{\alpha_2} + b_2 x^{\beta_2} + \ldots + k_2 x^{\kappa_2})$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\times (a_r x^{\alpha_r} + b_r x^{\beta_r} + \ldots + k_r x^{\kappa_r}).$$

Cor. 2. *In a box there are a counters marked α, b marked β, &c. A counter is drawn r times, and each time replaced. The number of ways in which the sum of the drawings can amount to n is the coefficient of x^n in the distribution of*

$$(ax^{\alpha} + bx^{\beta} + \ldots)^r.$$

DISTRIBUTIONS AND DERANGEMENTS.

§ 16.] The variety of problems that arise in connection with the subject of the present chapter is endless, and it would be difficult within the limits of a text-book to indicate all the methods that have been used in solving such of these problems as mathematicians have already discussed. The following have been selected as types of problems which are not, very readily at least, reducible to the elementary cases above discussed.*

§ 17.] *To find the number of ways in which n different letters can be distributed among r pigeon-holes, attention being paid to the order of the pigeon-holes, but not to the order of the letters in any one pigeon-hole, and no hole to contain less than one letter.*

Let D_r denote the number in question.

* For further information see Whitworth's *Choice and Chance*.

If we leave s specified holes vacant and distribute the letters among the remaining $r-s$ holes under the conditions of the question, we should thus get D_{r-s} distributions. Hence, if ${}_rC_s$ have its usual meaning, the number of distributions when s of the holes are blank is ${}_rC_s\, D_{r-s}$.

Again, the whole number of distributions when none, one, two, &c., of the holes may be blank is evidently r^n, for we can distribute the n letters separately among the r holes in r^n ways.

Hence

$$D_r + {}_rC_1\, D_{r-1} + {}_rC_2\, D_{r-2} + \ldots + {}_rC_{r-1}\, D_1 = r^n \quad (A).$$

The equation (A) contains the solution of our problem, for, by putting $r=2$, $r=3$, &c., successively, we could calculate D_2, D_3, &c., and D_1 is known, being simply 1.

We can, however, deduce an expression for D_r in terms of n and r, as follows. Writing $r-1$ in place of r we have

$$D_{r-1} + {}_{r-1}C_1\, D_{r-2} + \ldots + {}_{r-1}C_{r-2}\, D_1 = (r-1)^n \quad (B).$$

From (A) and (B), by subtraction, remembering (§ 8, Cor. 3) that

$${}_rC_s - {}_{r-1}C_{s-1} = {}_{r-1}C_s,$$

we derive

$$D_r + {}_{r-1}C_1\, D_{r-1} + {}_{r-1}C_2\, D_{r-2} + \ldots + {}_{r-1}C_{r-1}\, D_1 = r^n - (r-1)^n \quad (1).$$

From (1), putting $r-1$ in place of r, we derive

$$D_{r-1} + {}_{r-2}C_1\, D_{r-2} + \ldots + {}_{r-2}C_{r-2}\, D_1 = (r-1)^n - (r-2)^n \quad (1').$$

From (1) and (1'), by subtraction, we derive

$$D_r + {}_{r-2}C_1\, D_{r-1} + {}_{r-2}C_2\, D_{r-2} + \ldots + {}_{r-2}C_{r-2}\, D_2 = r^n - 2(r-1)^n + (r-2)^n \quad (2).$$

Treating now (2) exactly as we treated (1) we derive

$$D_r + {}_{r-3}C_1\, D_{r-1} + {}_{r-3}C_2\, D_{r-2} + \ldots + {}_{r-3}C_{r-3}\, D_3 = r^n - 3(r-1)^n + 3(r-2)^n - (r-3)^n \quad (3).$$

The law of formation of the right-hand side is obvious, the coefficients being formed by the addition rule peculiar to the binomial coefficients (see chap. iv., § 14). We shall therefore finally obtain

$$D_r = r^n - {}_rC_1(r-1)^n + {}_rC_2(r-2)^n - \ldots\ (-)^{r-1}{}_rC_{r-1}1^n,$$
$$= r^n - \frac{r}{1}(r-1)^n + \frac{r(r-1)}{1.2}(r-2)^n - \ldots\ (-)^{r-1}\frac{r}{1}1^n \quad (4).$$

Cor. *If the order of the pigeon-holes be indifferent, the number of distributions is* $D_r/r!$. *In other words, the number of partitions of* n *different letters into* r *lots, no vacant lots being allowed, is* $D_r/r!$.

We shall discuss the closely-allied problem to find the number of r-partitions of n—that is, to find the number of ways in which n letters, all alike, may be distributed among r pigeon-holes, the order of the holes being indifferent, and no hole to be empty—when we take up the Theory of the Partition of Numbers.

§ 18.] *Given a series of* n *letters, to find in how many ways the order may be deranged so that no one out of* r *assigned letters shall occupy its original position.*

Let ${}_n\Delta_r$ denote the number in question.

The number of different derangements in which the r assigned letters *do all* occupy their original places is $(n-r)!$. Hence the number of derangements in which the r assigned letters *do not all* occupy their original places is $n! - (n-r)!$. Now, this last number is made up of—

1st. The number of derangements in which no one of the r letters occupies its original place; that is, ${}_n\Delta_r$.

2nd. The number of derangements in which any one of the r letters occupies its original place, and no one of the remaining $r-1$ does so; that is, ${}_rC_1\ {}_{n-1}\Delta_{r-1}$.

3rd. The number of derangements in which any two of the r letters occupy their original places, and no one of the remaining $r-2$ does so; that is, ${}_rC_2\ {}_{n-2}\Delta_{r-2}$. And so on.

Hence

$$n! - (n-r)! = {}_n\Delta_r + {}_rC_1\ {}_{n-1}\Delta_{r-1} + {}_rC_2\ {}_{n-2}\Delta_{r-2} + \ldots + {}_rC_{r-1}\ {}_{n-r+1}\Delta_1 \quad (A).$$

If we write in this equation $n-1$ for n, and $r-1$ for r, and subtract the new equation thus derived from (A), we deduce

$$n! - (n-1)! = {}_n\Delta_r + {}_{r-1}C_1\ {}_{n-1}\Delta_{r-1} + {}_{r-1}C_2\ {}_{n-2}\Delta_{r-2} + \ldots + {}_{r-1}C_{r-1}\ {}_{n-r+1}\Delta_1 \quad (1).$$

We can now treat this equation exactly as we treated equation (1) of § 16. We thus deduce

$$ {}_n\Delta_r = n! - \frac{r}{1}(n-1)! + \frac{r(r-1)}{1.2}(n-2)! - \ldots (-)^r(n-r)! \qquad (2). $$

If we remember that $(n-r)!$, above, stands for the number of derangements in which the r letters all occupy their original positions, we see that, when $r = n$, $(n-r)!$ must be replaced by 1. Hence

Cor. *The number of derangements of a series of n letters in which no one of the original n occupies its original position is*

$$ n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^n}{n!}\right\}. \qquad (3). $$

The expression (3) may be written

$$ n(\ldots(4(3(2(1-1)+1)-1)+1)\ldots-(-1)^n)+(-1)^n. $$

Hence it may be formed as follows:—Set down 1, subtract 1; multiply by 2 and add 1; multiply by 3 and subtract 1; and so on. The function thus formed is of considerable importance in the present branch of mathematics, and has been called by Whitworth *subfactorial n*. He denotes it by $\underline{|n}$. A more convenient notation would be $n|$.

SUBSTITUTIONS.

§ 19.] Hitherto we have merely counted the permutations of a group of letters. If we direct our attention to the actual permutations, and in particular to the process by which these permutations are derived from each other, we are led to an order of ideas which forms the foundation of that important branch of modern algebra which is called the *Theory of Substitutions*.

Consider any two permutations, *becda*, *bcade*, of the five letters *a*, *b*, *c*, *d*, *e*. The latter is derived from the former by replacing *a* by *e*, *b* by *b*, *c* by *a*, *d* by *d*, *e* by *c*. This process may be represented by the operator $\begin{pmatrix} ebadc \\ abcde \end{pmatrix}$; and we may write

$$ \begin{pmatrix} ebadc \\ abcde \end{pmatrix} becda = bcade: $$

or, omitting the letters that are unaltered, and thus reducing the operator to its *simplest form*,

$$\begin{pmatrix} eac \\ ace \end{pmatrix} becda = bcade.$$

The operator $\begin{pmatrix} eac \\ ace \end{pmatrix}$, and the operation which it effects, are called a *Substitution*; and the operator is often denoted by a single capital letter, S, T, &c.

Since the number of different permutations of a group of n letters is $n!$, it is obvious that the number of different substitutions is also $n!$, if we include among them the *identical substitution* $\begin{pmatrix} abcde\ldots \\ abcde\ldots \end{pmatrix}$, (denoted by S^0 or by 1), in which no letter is altered.

We may effect two substitutions in succession upon the same permutation, and represent the result by writing the two symbols representing the substitutions before the permutation in order from right to left. Thus, if $S \equiv \begin{pmatrix} cab \\ abc \end{pmatrix}$, $T \equiv \begin{pmatrix} ea \\ ae \end{pmatrix}$,

$$ST\,aebcd = ecabd.$$

We may also effect the same substitution twice or three times over, and denote SS by S^2, SSS by S^3, &c. Thus, S being as before,

$$S^2 aebcd = Sceabd = becad.$$

It should be observed that the multiplication of substitution symbols is not in general commutative. For example, S and T being as above, $ST\,aebcd = ecabd$, but $TS\,aebcd = caebd$. If, when reduced to their simplest form, the symbols S and T have no letter in common, they are obviously commutative. This condition, although sufficient, is not necessary; for we have

$$\begin{pmatrix} dcab \\ abcd \end{pmatrix} \begin{pmatrix} badc \\ abcd \end{pmatrix} abcde = cdbae = \begin{pmatrix} badc \\ abcd \end{pmatrix} \begin{pmatrix} dcab \\ abcd \end{pmatrix} abcde.$$

§ 20.] Since the number of permutations of n letters is limited, it is obvious that if we repeat the same substitution, S, sufficiently often we shall ultimately reproduce the permutation that we started with. The smallest number, μ, of repetitions for which this happens is called the *order of the substitution* S.

Hence we have $S^{\mu} = 1$, and $S^{p\mu} = 1$, where p is any positive integer.

We may define a negative index in the theory of substitutions by means of the equation $S^{-q} = S^{p\mu-q}$, μ being the order of S, and p such that $p\mu > q$. From this definition we see that $S^{q}S^{-q} = S^{q}S^{p\mu-q} = S^{p\mu} = 1$. In other words, S^{q} and S^{-q} are inverse to each other; in particular, if

$$S = \begin{pmatrix} dabc \\ abcd \end{pmatrix}, \text{ then } S^{-1} = \begin{pmatrix} abcd \\ dabc \end{pmatrix} = \begin{pmatrix} bcda \\ abcd \end{pmatrix}.$$

A set of substitutions which are such that the product of any number of them is always one of the set is called a *group*; and the number of distinct substitutions in the group is called *the order of the group.* The number of letters operated on is called the *degree of the group.*

It is obvious from what has been shown that all the powers of a single substitution, S, form a group whose order is the order of S.

§ 21.] A substitution such as $\begin{pmatrix} bcdefa \\ abcdef \end{pmatrix}$, where each letter is replaced by the one that follows it, and the last by the first, is called a *Cyclic Substitution*, and is usually denoted by the symbol $(abcdef)$.*

The cyclic substitution (a), consisting of one letter, is an identical substitution; it may be held to mean that a passes into itself.

The cyclic substitution of two letters (ab), or what is the same thing (ba), is spoken of as a *Transposition.*

The effect of a cyclic substitution may be represented by writing the n letters at equal intervals round the circumference of a circle, and shifting each through $1/n$th of the circumference. Thus, or otherwise, it is obvious that the order of a cyclic substitution is equal to the number of the letters which it involves.

§ 22.] *Every substitution either is cyclic or is the product of a number of independent cyclic substitutions (cycles).*

Consider, for example, the substitution

* Or, of course, by $(bcdefa)$, $(cdefab)$, &c.

$$S \equiv \begin{pmatrix} bfdcgaeh \\ abcdefgh \end{pmatrix}.$$

This replaces a by b, b by f, f by a; these together constitute the cyclic substitution (abf). Next, c is replaced by d, and d by c; this is equivalent to the cycle (cd). Again, e is replaced by g, and g by e; this gives the cycle (eg). Finally, h is unaltered. Hence we have the following decomposition of the substitution S into cycles—

$$S \equiv (abf)(cd)(eg)(h).$$

The decomposition is obviously unique; and the reasoning by which we have arrived at it is perfectly general. It should be noticed that, since the cycles are independent, that is, have no letters in common, they are commutative, and it is indifferent in what order we write them.

§ 23.] *Every cyclic substitution of n letters can be decomposed into the product of n − 1 transpositions.*

For example, we have $(abcd) = (ab)(bc)(cd)$; and the process is general.

Cor. *Every substitution can be decomposed into n − r transpositions, where n is the number of letters which it displaces, and r the number of its proper cycles.*

Thus,
$$\begin{pmatrix} bfdcgaeh \\ abcdefgh \end{pmatrix} = (abf)(cd)(eg)(h),$$
$$= (ab)(bf)(cd)(eg).$$

This decomposition into transpositions is not unique, as will be seen presently, but the above gives the minimum number.

§ 24.] The following properties of a product of two transpositions are of fundamental importance.

I. *The product of two transpositions which have two letters in common is an identical substitution.*

This is obvious from the meaning of (ab).

II. *In the product of two transpositions,* TT′, *which have a letter in common,* T′ *may be placed first, provided we replace the common letter in* T *by the other letter in* T′.

For we have $(ab)(bc) \equiv \begin{pmatrix} bca \\ abc \end{pmatrix}$, $(bc)(ac) \equiv \begin{pmatrix} bca \\ abc \end{pmatrix}$,

therefore $(ab)(bc) = (bc)(ac)$.

Cor. 1. $(ef)(af) = (ae)(ef)$.

Cor. 2. $(ae)(af) = (af)(ef)$.

III. *If two transpositions,* T *and* T′, *have no letter in common, they are commutative.*

This is a mere particular case of a remark already made regarding two independent substitutions.

§ 25.] *The decomposition of a given substitution into transpositions is not unique.*

For we can always introduce a pair of factors $(ab)(ab)$, and then commutate one or both of them with the others, in accordance with the rules of § 24.

In this way we always increase the number of transpositions by an even number. In fact, we can prove the following important theorem—

The number of the transpositions which represent a given substitution is always odd or always even.

We may prove this by reducing the product of transpositions to a standard form as follows—

Select any one of the letters involved, say a; take the last transposition, T, on the right that involves a, and proceed to commutate this transposition successively with those to the left of it. So long as we come across transpositions that have no letter in common with T, neither T nor the others are affected. If we come to one that has a letter in common with T which is not a, we see (§ 24, II., Cor. 1) that the a in T remains, the other letter being altered, and the transposition passed over remains unaltered. If we come to a transposition that has a, and a only, in common with T, by § 24, II., Cor. 2, T passes to the left unaltered, and the transposition passed over loses its a. Lastly, if we come to a transposition that has both a and its other letter in common with T, then both it and T may be removed. If this last happen, we must now take that remaining transposition containing a which is farthest to the right, and proceed as before.

The result of this process, so far as a is concerned, will be, either that all the transpositions containing a will have disappeared, or that some even number (including 0) will have done so, and one only, say (ab), will remain on the extreme left.

Consider now b. If among the remaining factors b does not occur, then we have obtained a cycle (ab) of the substitution; and we now proceed to consider some other letter.

If, however, b does occur again, we take the factor farthest to the right in which it occurs, and commutate as before; the result being, either that all the transpositions (even in number) containing b disappear, or that an even number of them do, and we are left with, say (bc), in the second place. We now deal with c in like manner; and obtain in the third place, say (cd). This goes on until all the letters are exhausted, or until we come to a letter, say f, that disappears from the factors not yet finally arranged. We thus arrive at a product $(ab)(bc)(cd)(de)(ef)$ on the left.

Now
$$(ab)(bc)(cd)(de)(ef) \equiv \begin{pmatrix} bcdefa \\ abcdef \end{pmatrix} \equiv (abcdef).$$

We have, in fact, arrived at one of the independent cycles of the substitution. If we now take any other letter that occurs in one of the remaining substitutions on the right, we shall in like manner arrive at the cycle to which it belongs, after losing an even number, if any, of the transpositions; and so on, until all the letters are exhausted, and all the cycles arrived at. Since the whole number of transpositions lost is even, the truth of the theorem is now obvious; and our proof furnishes a method for reducing to the minimum number of transpositions.

It appears, therefore, that we may divide all the substitutions of a set of n letters into two classes—namely, *even substitutions*, which are equivalent to an even number of transpositions, and *odd substitutions*, which are equivalent to an odd number of transpositions.

Cor. 1. *If n be the number of letters altered by a substitution, r the number of its cycles, and $2s$ an arbitrary even integer, the number of factors in an equivalent product of transpositions is $n - r + 2s$.*

Cor. 2. *The number of the even is equal to the number of the odd substitutions of a set of n letters.*

For any one transposition, applied in succession to all the different odd substitutions, will give as many even substitutions, all different. Hence there are at least as many even as there are odd substitutions. In like manner we see that there are at least as many odd as there are even. Hence the number of the even is equal to the number of the odd substitutions.

Cor. 3. *A cyclic substitution is even or odd according as the number of the letters which it involves is odd or even.*

For example, $(abc) \equiv (ab)(bc)$ is even.

Cor. 4. *The product of any number of substitutions is even or odd according as the number of odd factors is even or odd. In particular, any power whatever of an even substitution, and any even power of any substitution whatever, form even substitutions.*

Cor. 5. *All the even substitutions of a set of n letters form a group whose order is* $n!/2$.

§ 26. If we select arbitrarily any one, say P, of the $n!$ permutations of a set of n letters, and call it an even permutation, then we can divide all the $n!$ permutations into two classes—1st, $n!/2$ *even permutations*, derived by applying to P the $n!/2$ even substitutions; 2nd, $n!/2$ *odd permutations*, derived by applying to P all the $n!/2$ odd substitutions.

The student who is familiar with the theory of determinants will observe that the above is precisely the classification of the permutations of the indices (or umbræ) which is adopted in defining the signs of the terms in a determinant.

It is farther obvious, from the definitions given in chap. iv., § 20, that *symmetric functions of a set of n variables are unaltered in value by any substitution whatever of the variables*; or, as the phrase is, they are said to "*admit any substitution whatever.*" *Alternating functions, on the other hand, admit only even substitutions of their variables*, the result of any odd substitution being to alter their sign without otherwise affecting their value.

§ 27.] The limits of the present work will not permit us to enter farther into the Theory of Substitutions, or to discuss its

applications to the Theory of Equations. The reader who desires to pursue this subject farther will find information in the following works: Serret, *Cours d'Algèbre Supérieure* (Paris, 1879); Jordan, *Traité des Substitutions* (Paris, 1870); Netto, *Substitutionen-theorie* (Leipzig, 1882).

EXERCISES III.

(1.) There are 10 counters in a box marked 1, 2, . . ., 10 respectively. Three drawings are made, the counter drawn being replaced each time. In how many ways can the sum of the numbers drawn amount—1st, to 9 exactly; 2nd, to 9 at least?

(2.) Out of the integers 1, 2, 3, . . ., 10 how many pairs can be selected so that their sum shall be even?

(3.) How many different throws can be made with n dice?

(4.) In how many ways can 5 black, 5 white, 5 blue balls be equally distributed among three bags, the order of the bags to be attended to?

(5.) A selection of c things is to be made partly from a group of a, the rest from a group of b. Prove that the number of ways in which such a set can be made will never be greater than when the number of things taken from the group of a is next less than $(a+1)(c+1)/(a+b+2)$.

(6.) In how many ways can p +'s and n −'s be placed in a row so that no two −'s come together?

(7.) In the Morse signalling system how many signals can be made without exceeding 5 movements?

(8.) In how many ways can 3 pairs of subscribers be set to talk in a telephone exchange having n subscribers?

(9.) There are 3 colours, and m balls of each. In how many ways can they be arranged in 3 bags each containing m, the order of the bags to be attended to?

(10.) If of $p+q+r$ things p be alike, q alike, and r different, the total number of combinations will be $(p+1)(q+1)2^r-1$.

(11.) In how many ways can $2n$ things be divided into n pairs?

(12.) The number of combinations of $3n$ things (n of which are alike), taken n at a time, is the coefficient of x^n in $(1+x)^{2n}/(1-x)$.

(13.) N boat clubs have $a, b, c, 1, 1, \ldots, 1$ boats each. In how many ways can the boats be arranged subject to the restriction that the 1st boat of any club is to be always above its 2nd, its 2nd always above its 3rd, &c.?

(14.) If there be p things of one sort, q of another, r of another, &c., the number of combinations of the $p+q+r+\ldots$ things, taken k at a time, is the coefficient of x^k in $(1-x^{p+1})(1-x^{q+1})\ldots/(1-x)(1-x)\ldots$

(15.) In how many ways can an arrangement of n things in a row be deranged so that—1st, each thing is moved one place; 2nd, no thing more than one place?

(16.) Given n things arranged in succession, the number of sets of 3

which can be formed under the condition that no set shall contain two things which were formerly contiguous is $(n-2)(n-3)(n-4)$, the order inside the sets to be attended to.

(17.) In how many ways can m white and n black balls be arranged in a row so that there shall be $2r-1$ contacts between white and black balls?

(18.) In how many ways can an examiner give 30 marks to 8 questions without giving less than 2 to any one question?

*(19.) The number of ways in which n letters can be arranged in r pigeon-holes, the order of the holes and of the letters in each hole to be attended to and empty holes admitted, is $r(r+1)(r+2)\ldots(r+n-1)$.

(20.) The same as last, no empty holes being admitted, $n!(n-1)!/(n-r)!(r-1)!$.

(21.) The same as last, the order of the holes not being attended to, $n!(n-1)!/(n-r)!r!(r-1)!$.

(22.) The number of ways in which n letters, all alike, can be distributed into r pigeon-holes, the order of the holes to be attended to, empty holes to be excluded, is ${}_{n-1}C_{r-1}$.

(23.) Same as last, empty holes being admitted, ${}_{n+r-1}C_{r-1}$.

(24.) Same as last, no hole to contain less than q letters, ${}_{n-1-r(q-1)}C_{r-1}$.

(25.) The number of ways of deranging a row of n letters so that no letter may be followed by the letter which originally followed it is $n\lfloor+(n-1)\rfloor$.

(26.) The number of ways of deranging $m+n$ terms so that m are displaced and n not displaced is $(m+n)!\lfloor m\rfloor/m!n!$.

(27.) The number of ways in which r different things can be distributed among $n+p$ persons so that certain n of those persons may each have one at least is

$$S_r=(n+p)^r-n(n+p-1)^r+\frac{n(n-1)}{2!}(n+p-2)^r-\ldots$$

Hence prove that

$$S_1=S_2=\ldots=S_{n-1}=0,\quad S_n=n!,\quad S_{n+1}=\left(\frac{n}{2}+p\right)(n+1)!.$$

(*Wolstenholme.*)

(28.) Fifteen school-girls walk out arranged in threes. How many times can they go out so that no two are twice together? (See Cayley's *Works*, vol. I., p. 481.)

EXERCISES IV.

Topological.

(1.) The number of sides of a complete n-point is $\frac{1}{2}n(n-1)$, and the number of vertices of a complete n-side is the same.

(2.) The number of triangles that can be formed with $2n$ lines of lengths $1, 2, \ldots, 2n$ is $n(n-1)(4n-5)/6$.

(3.) There are n points in a plane, no three of which are collinear, How

* Exercises 19-25 are solved in Whitworth's *Choice and Chance*; q.v.

many closed r-sided figures can be formed by joining the points by straight lines?

(4.) If m points in one straight line be joined to n points in another in every possible way, show that, exclusive of the $m+n$ given points, there are $mn(m-1)(n-1)/2$ points of intersection.

(5.) On three straight lines, A, B, C, are taken l, m, n points respectively, no one of which is a point of intersection. Show that the number of triangles which can be formed by taking three of the $l+m+n$ points is $\frac{1}{2}(m+n)(n+l)(l+m)-mn-nl-lm$.

(6.) There are n points in a plane, no three of which are collinear and no four concyclic. Through every two of the points is drawn a straight line and through every three a circle. Assuming each straight line to cut each circle in two distinct points, find the number of the intersections of straight lines with circles.

(7.) In a convex polygon of n sides the number of exterior intersections of diagonals is $\frac{1}{12}n(n-3)(n-4)(n-5)$, and the number of interior intersections is $\frac{1}{24}n(n-1)(n-2)(n-3)$.

(8.) There are n points in space, no three of which are collinear, and no four coplanar. A plane is drawn through every three. Find, 1st, the number of distinct lines of intersections of these planes; 2nd, the number of these lines of intersection which pass through one of the given n points; 3rd, the number of distinct points of intersection exclusive of the original n points.

(9.) Out of n straight lines $1, 2, \ldots, n$ inches long respectively, four can be chosen to form a pericyclic quadrilateral in $\{2n(n-2)(2n-5)-3+3(-1)^n\}/48$ ways.

(10.) Show that n straight lines, no two of which are parallel and no three concurrent, divide a plane into $\frac{1}{2}(n^2+n+2)$ regions. Hence, or otherwise, show that n planes through the centre of a sphere, no three of which are coaxial, divide its surface into n^2-n+2 regions.

(11.) Show that two pencils of straight lines lying in the same plane, one containing m the other n, divide the plane into $mn+2m+2n-1$ regions, it being supposed that no two of the lines are parallel or coincident.

(12.) If any number of closed curves be drawn in a plane each cutting all the others, and if n_r be the number of points through which r curves pass, the number of distinct closed areas formed by the plexus is

$$1+n_2+2n_3+\ldots+rn_{r+1}+\ldots$$

CHAPTER XXIV.

General Theory of Inequalities. Maxima and Minima.

§ 1.] The subject of the present chapter is of importance in many branches of algebra. We have already met with special cases of inequalities in the theory of Ratio and in the discussion of the Variation of Quadratic Functions of a single variable; and much of what follows is essential as a foundation for the theory of Limits, and for the closely allied theory of Infinite Series. In fact, the theory of inequalities forms the best introduction to the theory of infinite series, and, for that reason, ought to be set as much as possible on an independent basis.

§ 2.] We are here concerned with real algebraical quantity merely. As we have already explained, no comparison of complex numbers as to relative magnitude in the ordinary sense can be made, because any such number is expressed in terms of two absolutely heterogeneous units. Strictly speaking, there is a similar difficulty in comparing real algebraical quantities which have not the same sign; but this difficulty is met (see chap. xiii., § 1) by an extension of the notion of inequality. It will be remembered that a is defined to be algebraically greater or less than b according as the reduced value of $a - b$ is positive or negative. An immediate consequence of this definition is that a positive quantity increases algebraically as it increases numerically, but a negative quantity decreases algebraically as it increases numerically. The neglect of this consideration is a fruitful source of mistakes in the theory of inequalities.

§ 3.] From one point of view the theory of inequalities runs

parallel to the theory of conditional equations. In fact, the approximate numerical solution of equations depends, as we have seen, on the establishment of a series of inequalities.*

The following theorems will bring out the analogies between the two theories, and at the same time indicate the nature of the restrictions that arise owing to the fact that the two sides of an inequality cannot, like the two sides of an equation, be interchanged without altering its nature. For the sake of brevity, we shall, for the most part, write the inequalities so that the greater quantity is on the left, and the sign $>$ alone appears. The modifications necessary when the other sign appears are in all cases obvious.

I. *If* $P > Q$, $Q > R$, $R > S$, *then* $P > S$.

Proof.—$(P - Q) + (Q - R) + (R - S) \equiv P - S$, hence, since $P - Q$, $Q - R$, $R - S$ are all positive, $P - S$ is positive, that is, $P > S$.

II. *If* $P > Q$, *then* $P \pm R > Q \pm R$.

For $(P \pm R) - (Q \pm R) \equiv P - Q$; hence the sign of the former quantity is the same as the sign of the latter.

Cor. 1. *If* $P + Q > R + S$, *then*

$$P + Q - R > S, \quad -R - S > -P - Q, \quad -P - Q < -R - S.$$

It thus appears that *we may transfer a term from one side of an inequality to another, provided we change its sign; and we may change the signs of all the terms on both sides of an inequality, provided we reverse the symbol of inequality.*

Cor. 2. *Every inequality may be reduced to one or other of the forms* $P > 0$ *or* $P < 0$.

In other words, every problem of inequality may be reduced to the determination of the sign of a certain quantity.

III. *If* $P_1 > Q_1$, $P_2 > Q_2$, . . ., $P_n > Q_n$,

then $$P_1 + P_2 + \ldots + P_n > Q_1 + Q_2 + \ldots + Q_n;$$

for $$(P_1 + P_2 + \ldots + P_n) - (Q_1 + Q_2 + \ldots + Q_n) \equiv (P_1 - Q_1) + (P_2 - Q_2) + \ldots + (P_n - Q_n),$$

whence the theorem follows.

It should be noticed that it does not follow that, if $P_1 > Q_1$, $P_2 > Q_2$, then $P_1 - P_2 > Q_1 - Q_2$.

* See, for example, the proof that every equation has a root.

IV. *If* $P > Q$, *then* $PR > QR$, *and* $P/R > Q/R$, *provided* R *be positive; but* $PR < QR$, $P/R < Q/R$, *if* R *be negative.*

For $(P - Q)R$ and $(P - Q)/R$ have both the same sign as $P - Q$ if R be positive, and both the opposite sign if R be negative.

Cor. 1. *If* $P > QR$, *and* $R > S$, *then* $P > QS$, *provided* Q *be positive.*

Cor 2. *Every fractional inequality can be integralised.*

For example, if $P/Q > R/S$, then, provided QS be positive, we have, after multiplying by QS, $PS > QR$; but, if QS be negative, $PS < QR$.

If there be any doubt about the sign of QS, then we may multiply by Q^2S^2, which is certainly positive, and we have $QPS^2 > Q^2RS$.

V. *If* $P_1 > Q_1$, $P_2 > Q_2$, . . ., $P_n > Q_n$, *and all the quantities be positive, then*

$$P_1P_2 \ldots P_n > Q_1Q_2 \ldots Q_n.$$

For $$P_1P_2P_3 \ldots P_n > Q_1P_2P_3 \ldots P_n,$$

since $P_1 > Q_1$ and $P_2P_3 \ldots P_n$ is positive;

$$> Q_1Q_2P_3 \ldots P_n,$$

since $P_2 > Q_2$ and $Q_1P_3 \ldots P_n$ is positive; and so on. Hence, finally, we have

$$P_1P_2 \ldots P_n > Q_1Q_2 \ldots Q_n.$$

Cor. 1. *If* $P > Q$, *and both be positive, then* $P^n > Q^n$, *n being any positive integer.*

Cor. 2. *If* $P > Q$, *and both be positive, then* $P^{1/n} > Q^{1/n}$, *n being any positive integer, and the real positive value of the nth root being taken on both sides.*

For, if $P^{1/n} \lesseqgtr Q^{1/n}$, then, since both are real and positive, $(P^{1/n})^n \lesseqgtr (Q^{1/n})^n$, by Cor. 1; that is, $P \lesseqgtr Q$, which contradicts our hypothesis.

Cor. 3. *If* $P > Q$, *both being positive, and n be any positive quantity, then* $P^{-n} < Q^{-n}$, *where, if the indices are fractional, there is the usual understanding as to the root to be taken.*

Remark.—The necessity for the restrictions regarding the

sign of the members of the inequalities in the present theorem will appear if we consider that, although $-2 > -3$, and $-3 > -4$, yet it is not true that $(-2)(-3) > (-3)(-4)$.

These restrictions might be removed in certain cases; for example, it follows from $-3 > -4$ that $(-3)^3 > (-4)^3$, in other words, that $-27 > -64$: but the importance of such particular cases does not justify their statement at length.

Cor. 4. *An inequality may be rationalised if due attention be paid to the above-mentioned restrictions regarding sign.*

§ 4.] By means of the theorems just stated and the help of the fundamental principle that the product of two real quantities is positive or negative according as these quantities have the same or opposite sign, and, in particular, that the square of any real quantity is positive, we can solve a great many questions regarding inequalities.

The following are some examples of the direct investigation of inequalities; the first four are chosen to illustrate the parallelism and mutual connection between inequalities and equations:—

Example 1. Under what circumstances is

$$(3x-1)/(x-2)+(2x-3)/(x-5) > \text{ or } < 5\ ?$$

1st. Let us suppose that x does not lie between 2 and 5, and is not equal to either of these values. Then $(x-2)(x-5)$ is positive, and we may multiply by this factor without reversing the signs of inequality.

Hence $\quad F \equiv (3x-1)/(x-2)+(2x-3)/(x-5) > < 5,$

according as

$$(3x-1)(x-5)+(2x-3)(x-2) > < 5(x-2)(x-5),$$

according as $\quad 5x^2-23x+11 > < 5x^2-35x+50,$

according as $\quad 12x > < 39,$

according as $\quad x > < 3\tfrac{1}{4}.$

Under our present supposition, x cannot have the value $3\frac{1}{4}$; but we conclude from the above that if $x > 5$, $F > 5$, and if $x < 2$, $F < 5$.

2nd. Suppose $2 < x < 5$. In this case $(x-2)(x-5)$ is negative, and we must reverse all the signs of inequality after multiplying by it.

We therefore infer that if $2 < x < 3\frac{1}{4}$, $F > 5$, and if $3\frac{1}{4} < x < 5$, then $F < 5$.

The student should observe that, as x varies from $-\infty$ to $+\infty$, the sign of the inequality is thrice reversed, namely, when $x=2$, when $x=3\frac{1}{4}$, and when $x=5$; the first and last reversals occur because F changes sign by passage through an infinite value; the second reversal occurs because F passes

through the value 5. The student should draw the graph of the function F.*

Example 2. Under what circumstances is

$$F \equiv (3x-4)/(x-2) \gtrless 1\ ?$$

Multiplying by the positive quantity $(x-2)^2$, we have

$$(3x-4)/(x-2) \gtrless 1,$$

according as $(3x-4)(x-2) \gtrless (x-2)^2$,

according as $\{(3x-4)-(x-2)\}(x-2) \gtrless 0$,

according as $2(x-1)(x-2) \gtrless 0$.

Hence $F > 1$, if $x < 1$ or > 2;

$F < 1$, if $1 < x < 2$.

Example 3. Under what circumstances is $x^3+25x \gtrless 8x^2+26$?

$$x^3+25x \gtrless 8x^2+26,$$

according as $x^3-8x^2+25x-26 \gtrless 0$,

according as $(x-2)(x^2-6x+13) \gtrless 0$,

according as $(x-2)\{(x-3)^2+4\} \gtrless 0$.

Now $(x-3)^2+4$ is positive for all real values of x; hence

$$x^3+25x \gtrless 8x^2+26,$$

according as $x \gtrless 2$.

Example 4. If the positive values of the square roots be taken in all cases, is

$$\sqrt{(2x+1)}+\sqrt{(x-1)} \gtrless \sqrt{(3x)}\ ?$$

Owing to the restriction as to sign, we may square without danger of reversing the inequality. Hence

$$\sqrt{(2x+1)}+\sqrt{(x-1)} \gtrless \sqrt{(3x)},$$

according as $2x+1+x-1+2\sqrt{\{(2x+1)(x-1)\}} \gtrless 3x$,

according as $2\sqrt{\{(2x+1)(x-1)\}} \gtrless 0$.

Now, provided x is such that the value of $\sqrt{\{(2x+1)(x-1)\}}$ is real, that is, provided $x > 1$,

$$2\sqrt{\{(2x+1)(x-1)\}} > 0,$$

therefore $\sqrt{(2x+1)}+\sqrt{(x-1)} > \sqrt{(3x)}$, if $x > 1$.

Negative values of x less than $-\frac{1}{2}$ would also make $\sqrt{\{(2x+1)(x-1)\}}$ real; but such values would make $\sqrt{(2x+1)}$, $\sqrt{(x-1)}$, and $\sqrt{(3x)}$ imaginary, and, in that case, the original inequality would be meaningless.

Example 5. If $x, y, z \ldots$ be n real quantities $(n-1)\Sigma x^2 \prec 2\Sigma xy$.

Since all the quantities are real, $\Sigma(x-y)^2 \prec 0$.

Hence, since x will appear once along with each of the remaining $n-1$ letters, and the same is true of $y, z, \ldots$, we have

$$(n-1)\Sigma x^2 - 2\Sigma xy \prec 0,$$

that is, $$(n-1)\Sigma x^2 \prec 2\Sigma xy.$$

* The graphical study of inequalities involving only one variable will be found to be a good exercise.

In the case where $x=y=z=\ldots$ we have $\Sigma x^2=nx^2$, $2\Sigma xy=2_nC_2x^2=n(n-1)x^2$, so that the inequality just becomes an equality.

When $n=2$, we have the theorem

$$x^2+y^2\leqslant 2xy;$$

or, if we put $x=\sqrt{a}$, $y=\sqrt{b}$, a and b being real and positive,

$$a+b\leqslant 2\sqrt{(ab)},$$

a theorem already established, of which the preceding may be regarded as a generalisation. A more important generalisation of another kind will be given presently.

Example 6. If $x, y, z, \ldots$ be n real positive quantities, and p and q any two real quantities having the same sign, then

$$x^{p+q}+y^{p+q}\leqslant x^py^q+x^qy^p,$$
$$n\Sigma x^{p+q}\leqslant \Sigma x^p\Sigma x^q.$$

We have seen that x^p-y^p and x^q-y^q will both have the same sign as $x-y$, or both opposite signs, according as p and q are both positive or both negative. Hence, in either case, $(x^p-y^p)(x^q-y^q)$ has the positive sign. Therefore

$$(x^p-y^p)(x^q-y^q)\leqslant 0,$$

whence

$$x^{p+q}+y^{p+q}\leqslant x^py^q+x^qy^p.$$

If we write down the ${}_nC_2$ inequalities like the last, obtained by taking every possible pair of the n quantities $x, y, z, \ldots$, and add, we obtain the following result—

$$(n-1)\Sigma x^{p+q}\leqslant \Sigma x^py^q.$$

If we now add Σx^{p+q} to both sides, we deduce

$$n\Sigma x^{p+q}\leqslant \Sigma x^p\Sigma x^q.$$

N.B.—If p and q have opposite signs, then

$$n\Sigma x^{p+q}\geqslant \Sigma x^p\Sigma x^q.$$

These theorems contain a good many others as particular cases. For example, if we put $q=-p$, we deduce

$$\Sigma x^p\Sigma x^{-p}\leqslant n^2,$$

which, when $n=3$, $p=1$, gives

$$(x+y+z)(1/x+1/y+1/z)\leqslant 9;$$

whence

$$(x+y+z)(yz+zx+xy)\leqslant 9xyz;$$

and so on.

Example 7. If x, y, z be real and not all equal, then $\Sigma x^3 \gtrless 3xyz$, according as $\Sigma x \gtrless 0$.

For

$$\Sigma x^3-3xyz\equiv \Sigma x(\Sigma x^2-\Sigma xy),$$
$$\equiv \tfrac{1}{2}\Sigma x\Sigma(x-y)^2.$$

Hence the theorem, since $\Sigma(x-y)^2$ is essentially positive.

Example 8. To show that

$$\frac{1}{\sqrt{(2n+1)}}>\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}>\frac{\sqrt{(n+1)}}{2n+1},$$

where n is any positive integer.

From the inequality $a+b>2\sqrt{(ab)}$ we deduce

$$(2n-1)+(2n+1)>2\sqrt{\{(2n-1)(2n+1)\}};$$

whence $$(2n-1)/2n<\sqrt{\{(2n-1)/(2n+1)\}} \quad (1);$$

similarly $$(2n-3)/2(n-1)<\sqrt{\{(2n-3)/(2n-1)\}} \quad (2);$$

. . . . . .

$$5/2.3<\sqrt{\{5/7\}} \quad (n-2);$$
$$3/2.2<\sqrt{\{3/5\}} \quad (n-1);$$
$$1/2.1<\sqrt{\{1/3\}} \quad (n).$$

Multiplying these inequalities together, we get

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}<\frac{1}{\sqrt{(2n+1)}} \quad (A).$$

Again, $$n+(n+1)>2\sqrt{\{n(n+1)\}},$$

that is, $$2n+1>2\sqrt{\{n(n+1)\}}.$$

Hence we have the following inequalities—

$$(2n+1)/2n>\sqrt{\{(n+1)/n\}} \quad (1)',$$
$$(2n-1)/2(n-1)>\sqrt{\{n/(n-1)\}} \quad (2)',$$

. . . . . .

$$7/2.3>\sqrt{\{4/3\}} \quad (n-2)',$$
$$5/2.2>\sqrt{\{3/2\}} \quad (n-1)',$$
$$3/2.1>\sqrt{\{2/1\}} \quad (n)'.$$

Multiplying these n inequalities together, we get

$$\frac{1.3.5\ldots(2n+1)}{2.4\ldots2n}>\sqrt{(n+1)}.$$

Hence $$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}>\frac{\sqrt{(n+1)}}{2n+1} \quad (B).$$

(A) and (B) together establish the theorem in question.

Since $\sqrt{(n+1)}/(2n+1)>\sqrt{(n+1)}/(2n+2)>1/2\sqrt{(n+1)}$, we may state the above theorem more succinctly thus,

$$\frac{1}{\sqrt{(2n+1)}}>\frac{1.3\ldots(2n-1)}{2.4\ldots2n}>\frac{1}{2\sqrt{(n+1)}}.$$

DERIVED THEOREMS.

§ 5.] We now proceed to prove several theorems regarding inequality which are important for their own sake, and will be of use to us in following chapters.

If $b_1, b_2 \ldots, b_n$ be all positive, the fraction $(a_1+a_2+\ldots+a_n)/(b_1+b_2+\ldots+b_n)$ is not less than the least, and not greater than the greatest, of the n fractions $a_1/b_1, a_2/b_2 \ldots, a_n/b_n$.

Let f be the least, and f' the greatest of the n fractions, then

$$a_1/b_1 \not< f, \quad a_2/b_2 \not< f, \quad \ldots, \quad a_n/b_n \not< f.$$

Hence, since $b_1, b_2, \ldots, b_n$ are all positive,

$$a_1 \lessgtr f b_1, \quad a_2 \lessgtr f b_2, \quad \ldots, \quad a_n \lessgtr f b_n.$$

Adding, we have

$$(a_1 + a_2 + \ldots + a_n) \lessgtr f(b_1 + b_2 + \ldots + b_n);$$

whence

$$(a_1 + a_2 + \ldots + a_n)/(b_1 + b_2 + \ldots + b_n) \lessgtr f.$$

In like manner, it may be shown that

$$(a_1 + a_2 + \ldots + a_n)/(b_1 + b_2 + \ldots + b_n) \gtrless f'.$$

Remark.—This theorem is only one among many of the same kind.* The reader will find no difficulty in demonstrating the following:—

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ *be as before, and* $l_1, l_2, \ldots, l_n$ *be n positive quantities, then* $\Sigma l_1 a_1 / \Sigma l_1 b_1$ *is not less than the least, and not greater than the greatest, among the n fractions* $a_1/b_1, a_2/b_2, \ldots, a_n/b_n$.

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, l_1, l_2 \ldots, l_n$ *be all positive, then* $\{\Sigma l_1 a_1^m / \Sigma l_1 b_1^m\}^{1/m}$ *and* $\{a_1 a_2 \ldots a_n / b_1 b_2 \ldots b_n\}^{1/n}$ *are, each of them, not less than the least, and not greater than the greatest, among the n fractions* $a_1/b_1, a_2/b_2, \ldots, a_n/b_n$.

Example, to prove that

$$\frac{1}{2} < \sqrt[n]{\left\{\frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n}\right\}} < 1.$$

Since the fractions 1/2, 3/4, ... $(2n-1)/2n$ are obviously in ascending order of magnitude, we have, in the second part of the last of the theorems just stated,

$$\frac{1}{2} < \sqrt[n]{\left\{\frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n}\right\}} < \frac{2n-1}{2n}.$$

Now, $(2n-1)/2n = 1 - 1/2n < 1$, hence the theorem follows; and it holds, be it observed, however great n may be.

§ 6.] *If* x, p, q *be all positive, and* p *and* q *be integers, then* $(x^p - 1)/p \gtrless (x^q - 1)/q$ *according as* $p \gtrless q$.

Since p and q are positive,

$$(x^p - 1)/p \gtrless (x^q - 1)/q,$$

according as $\quad q(x^p - 1) \gtrless p(x^q - 1),$

* See the interesting remarks on Mean Values in Cauchy's *Analyse Algébrique.*

according as

$$(x-1)\{q(x^{p-1}+x^{p-2}+\ldots+1)-p(x^{q-1}+x^{q-2}+\ldots+1)\}\gtrless 0.$$

If $p>q$, we have

$$X\equiv(x-1)\{q(x^{p-1}+x^{p-2}+\ldots+1)-p(x^{q-1}+x^{q-2}+\ldots+1)\},$$
$$\equiv(x-1)\{q(x^{p-1}+x^{p-2}+\ldots+x^q)-(p-q)(x^{q-1}+x^{q-2}+\ldots+1)\}.$$

Now, if $x>1$,

$$x^{p-1}+x^{p-2}+\ldots+x^q > (p-q)x^q;$$
$$x^{q-1}+x^{q-2}+\ldots+1 < qx^{q-1};$$

therefore,

$$X > (x-1)\{q(p-q)x^q-(p-q)qx^{q-1}\},$$
$$> q(p-q)x^{q-1}(x-1)^2,$$
$$> 0.$$

Again, if $x<1$,

$$X > (x-1)\{q(p-q)x^{p-1}-(p-q)q\},$$
$$> q(p-q)(x-1)(x^{p-1}-1),$$
$$> 0. \qquad (\S\ 3,\ \text{Th. V., Cor. 1.})$$

Hence, in both cases,

$$(x^p-1)/p > (x^q-1)/q.$$

By the same reasoning, if $q>p$,

$$(x^q-1)/q > (x^p-1)/p,$$

that is, if $p<q$,

$$(x^p-1)/p < (x^q-1)/q.$$

§ 7.] *If x be positive, and $\neq 1$, then*

$$mx^{m-1}(x-1) > x^m-1 > m(x-1),$$

unless m lie between 0 and +1, in which case

$$mx^{m-1}(x-1) < x^m-1 < m(x-1).$$

From § 6, we have

$$(\xi^p-1) \gtrless (p/q)(\xi^q-1) \qquad (1),$$

according as $p \gtrless q$, where ξ is any positive quantity $\neq 1$, and p and q positive integers. In (1) we may put $x^{1/q}$ for ξ, where x is any positive quantity $\neq 1$ (the real positive value of the qth root to be taken), and we may put m for p/q, where m is any positive commensurable quantity. (1) then becomes

$$x^m-1 \gtrless m(x-1) \qquad (2),$$

according as $m > < 1$, which is part of the theorem to be established.

In (2) we may replace x by $1/x$, where x is any positive quantity $\neq 1$, and the inequality will still hold.

Hence $$(1/x)^m - 1 > < m(1/x - 1) \qquad (3),$$
according as $$m > < 1.$$

If we multiply (3) by $-x^m$, we deduce
$$x^m - 1 < > mx^{m-1}(x-1),$$
that is, $$mx^{m-1}(x-1) > < x^m - 1,$$
according as $$m > < 1.$$

We have thus established the theorem for positive values of m.

Next, let $m = -n$ where n is any positive commensurable quantity. Then
$$x^{-n} - 1 > < (-n)(x-1),$$
according as $$1 - x^n > < -nx^n(x-1),$$
according as $$x^n - 1 < > nx^n(x-1),$$
$$nx^{n+1} - nx^n > < x^n - 1.$$
Add $x^{n+1} - x^n$ to both sides, and we see that
$$x^{-n} - 1 > < (-n)(x-1),$$
according as
$$(n+1)x^n(x-1) > < x^{n+1} - 1.$$

Now, since n is positive, $n + 1 > 1$, therefore, by what we have already proved,
$$(n+1)x^n(x-1) > x^{n+1} - 1.$$
Hence $$x^{-n} - 1 > (-n)(x-1) \qquad (4).$$

In (4) we may write $1/x$ for x; and then we have
$$(1/x)^{-n} - 1 > (-n)(1/x - 1).$$

If we multiply by $-x^{-n}$, this last inequality becomes
$$x^{-n} - 1 < (-n)x^{-n-1}(x-1),$$
that is, $$(-n)x^{-n-1}(x-1) > x^{-n} - 1.$$
Hence, if m be negative,
$$mx^{m-1}(x-1) > x^m - 1 > m(x-1);$$
which completes the demonstration.

Cor. If x and y be any two unequal positive quantities, we may replace x in the above theorem by x/y. On multiplying throughout by y^m, we thus deduce the following—

If x and y be positive and unequal, then

$$mx^{m-1}(x-y) > x^m - y^m > my^{m-1}(x-y),$$

unless m lie between 0 and +1, in which case

$$mx^{m-1}(x-y) < x^m - y^m < my^{m-1}(x-y).$$

We have been careful to state and prove the inequality of the present section in its most general form because of its great importance: much of what follows, and many theorems in the following chapter, are in fact consequences of it.*

Example 1. Show that, if x be positive, $(1+x)^m$ always lies between $1+mx$ and $(1+x)/\{1+(1-m)x\}$, provided $mx<1+x$.

Suppose, for example, that m is positive and <1. Then, by the theorem of the present section,

$$m(1+x)^{m-1}x < (1+x)^m - 1 < mx.$$

Hence $$(1+x)^m < 1+mx.$$

Also, $$(1+x)^m - 1 > mx(1+x)^m/(1+x),$$

$$\{1-mx/(1+x)\}(1+x)^m > 1.$$

If $mx<1+x$, $1-mx/(1+x)$ is positive, and we deduce

$$(1+x)^m > 1/\{1-mx/(1+x)\},$$

$$> (1+x)/\{1+(1-m)x\}.$$

The other cases may be established in like manner.

Remark.—It should be observed that

$$(1\pm x)^m \gtrless 1 \pm mx,$$

according as m does not or does lie between 0 and +1.

Example 2. Show that, if $u_1, u_2 \ldots, u_n$ be all positive, then

$$(1+u_1)(1+u_2)\ldots(1+u_n) > 1+u_1+u_2+\ldots+u_n;$$

also that, if $u_1, u_2 \ldots, u_n$ be all positive and each less than 1, then

$$(1-u_1)(1-u_2)\ldots(1-u_n) > 1-u_1-u_2-\ldots-u_n.$$

The first part of the theorem is obvious from the identity

$$(1+u_1)(1+u_2)\ldots(1+u_n) \equiv 1+\Sigma u_1+\Sigma u_1u_2+\Sigma u_1u_2u_3+\ldots+u_1u_2\ldots u_n.$$

The latter part may be proved, step by step, thus—

$$1-u_1 = 1-u_1.$$

$$(1-u_1)(1-u_2) = 1-u_1-u_2+u_1u_2,$$

$$> 1-u_1-u_2.$$

---

* Several mathematical writers have noticed the unity introduced into the elements of algebraical analysis by the use of this inequality. See especially Schlömilch's *Handbuch der Algebraischen Analysis*. The secret of its power lies in the fact that it contains as a particular case the fundamental limit theorem upon which depends the differentiation of an algebraic function. The use of the theorem has been considerably extended in the present volume.

Hence, since $1-u_3$ is positive,

$$(1-u_1)(1-u_2)(1-u_3) > (1-u_3)(1-u_1-u_2),$$
$$> 1-u_1-u_3-u_3+u_3(u_1+u_2),$$
$$> 1-u_1-u_2-u_3.$$

And so on.

These inequalities are a generalisation of $(1\pm x)^n > 1\pm nx$ ($x<1$ and n a positive integer). They are useful in the theory of infinite products.

§ 8.] *The arithmetic mean of n positive quantities is not less than their geometric mean.*

Let us suppose this theorem to hold for n quantities $a, b, c, \ldots, k$, and let l be one more positive quantity. By hypothesis,

$$(a+b+c+\ldots+k)/n \nless (abc\ldots k)^{1/n},$$

that is,

$$a+b+c+\ldots+k \nless n(abc\ldots k)^{1/n}.$$

Therefore

$$a+b+c+\ldots+k+l \nless n(abc\ldots k)^{1/n}+l.$$

Now,

$$n(abc\ldots k)^{1/n}+l \nless (n+1)(abc\ldots kl)^{1/(n+1)},$$

provided

$$n\{abc\ldots k/l^n\}^{1/n}+1 \nless (n+1)\{abc\ldots kl/l^{n+1}\}^{1/(n+1)},$$
$$\nless (n+1)\{abc\ldots k/l^n\}^{1/(n+1)},$$

that is, provided

$$n\xi^{n+1}+1 \nless (n+1)\xi^n,$$

where

$$\xi^{n(n+1)} = abc\ldots k/l^n,$$

that is, provided

$$(n+1)\xi^n(\xi-1) \nless \xi^{n+1}-1,$$

which is true by § 7.

Hence, if our theorem hold for n quantities, it will hold for $n+1$. Now we have seen that $(a+b)/2 \nless (ab)^{\frac{1}{2}}$, that is, the theorem holds for 2 quantities; therefore it holds for 3; therefore for 4; and so on. Hence we have in general

$$(a+b+c+\ldots+k)/n \nless (abc\ldots k)^{1/n}.$$

It is, of course, obvious that the inequality becomes an equality when $a=b=c=\ldots=k$.

There is another proof of this theorem so interesting and fundamental in its character that it deserves mention here.*

Consider the geometric mean $(abc \ldots k)^{1/n}$. If $a, b, c, \ldots$ be not all equal, replace the greatest and least of them, say a and k, by $(a+k)/2$; then, since $\{(a+k)/2\}^2 > ak$, the result has been to increase the geometric mean, while the arithmetic mean of the n quantities $(a+k)/2, b, c, \ldots, (a+k)/2$ is evidently the same as the arithmetic mean of $a, b, c, \ldots, k$. If the new set of n quantities be not all equal, replace the greatest and least as before; and so on.

By repeating this process sufficiently often, we can make all the quantities as nearly equal as we please; and then the geometric mean becomes equal to the arithmetic mean.

But, since the latter has remained unaltered throughout, and the former has been increased at each step, it follows that the first geometric mean, namely, $(abc \ldots k)^{1/n}$, is less than the arithmetic mean, namely, $(a+b+c+\ldots+k)/n$.

As an illustration of this reasoning, we have $(1 \cdot 3 \cdot 5 \cdot 9)^{1/4} < (5 \cdot 3 \cdot 5 \cdot 5)^{\frac{1}{4}} < (5 \cdot 4 \cdot 4 \cdot 5)^{\frac{1}{4}} < (4{\cdot}5 \cdot 4{\cdot}5 \cdot 4{\cdot}5 \cdot 4{\cdot}5)^{\frac{1}{4}} \leq 4{\cdot}5 < (1+3+5+9)/4$.

Cor. *If $a, b, \ldots, k$ be n positive quantities, and $p, q, \ldots, t$ be n positive commensurable quantities, then*

$$\frac{pa+qb+\ldots+tk}{p+q+\ldots+t} \leqslant (a^p b^q \ldots k^t)^{1/(p+q+\ldots+t)}.$$

It is obvious that we are only concerned with the ratios $p:q:\ldots:t$. Hence we may replace $p, q, \ldots, t$ by positive integral numbers proportional to them. It is, therefore, sufficient to prove the theorem on the hypothesis that $p, q, \ldots, t$ are positive integers. It then becomes a mere particular case of the theorem of the present paragraph, namely, that the arithmetic mean of $p+q+\ldots+t$ positive quantities, p of which are equal to a, q to $b, \ldots, t$ to k, is not less than their geometric mean.

* See also the ingenious proof of the theorem given by Cauchy (*Analyse Algébrique*, p. 457), who seems to have been the first to state the theorem in its most general form.

Example 1. Show that, if $a, b, \ldots, k$ be n positive quantities,

$$\left(\frac{a^2+b^2+\ldots+k^2}{a+b+\ldots+k}\right)^{a+b+\ldots+k} \leqslant a^a b^b \ldots k^k \leqslant \left(\frac{a+b+\ldots+k}{n}\right)^{a+b+\ldots+k}.$$

The first part of the proposition follows from the above corollary by taking $p=a, q=b, \ldots, k=c$.

To prove the second part let us assume that

$$a^a b^b \ldots k^k \leqslant \{(a+b+\ldots+k)/n\}^{a+b+\ldots+k}.$$

Then $$a^a b^b \ldots k^k l^l \leqslant l^l \{(a+b+\ldots+k)/n\}^{a+b+\ldots+k}.$$

But

$$l^l \{(a+b+\ldots+k)/n\}^{a+b+\ldots+k} \leqslant \{(a+b+\ldots+k+l)/(n+1)\}^{a+b+\ldots+k+l},$$

provided $$1 \leqslant \{n(1+x)/(n+1)\}\ \{(1+1/x)/(n+1)\}^x,$$

that is, $$\{(1+1/x)/(n+1)\}^x \geqslant (n+1)/n(1+x),$$

where $x=l/(a+b+\ldots+k)$, which we may suppose $\geqslant 1$, since there is no loss of generality in supposing $a, b, \ldots, k, l$ arranged in descending order of magnitude.

Now, by § 7, since $x \geqslant 1$,

$$\{(1+1/x)/(n+1)\}^x \geqslant 1+x\{(1+1/x)/(n+1)-1\},$$
$$\geqslant (n+2-nx)/(n+1).$$

Also, $$(n+2-nx)/(n+1) \geqslant (n+1)/n(1+x),$$

provided $$n(n+2-nx)(1+x) \geqslant (n+1)^2,$$

$$n^2+2n+2nx-n^2x^2 \geqslant n^2+2n+1,$$

$$0 \geqslant (nx-1)^2,$$

which is true.

Hence, if the proposition hold for n quantities, it will hold for $n+1$. But, obviously, $a^a \leqslant (a/1)^a$, hence &c.

Example 2. Prove that $1.3 \ldots (2n-1) < n^n$.

We have $$\{1+3+\ldots+(2n-1)\}/n > \{1.3 \ldots (2n-1)\}^{1/n},$$

that is, $$n^2/n > \{1.3 \ldots (2n-1)\}^{1/n}.$$

Hence $$n^n > 1.3 \ldots (2n-1).$$

§ 9.] *If $a, b, \ldots, k$ be n positive quantities, and $p, q, \ldots, t$ be n positive quantities, then*

$$\frac{pa^m+qb^m+\ldots+tk^m}{p+q+\ldots+t} \lessgtr \left(\frac{pa+qb+\ldots+tk}{p+q+\ldots+t}\right)^m \quad (1),$$

according as m does not or does lie between 0 and +1.

If we denote

$$p/(p+q+\ldots+t), \quad q/(p+q+\ldots+t), \text{ &c.},$$

by $\lambda, \mu, \ldots, \tau$, and

$$a/(\lambda a+\mu b+\ldots+\tau k), \quad b/(\lambda a+\mu b+\ldots+\tau k), \text{ &c.},$$

by $x, y, \ldots, w$, so that

$$\lambda + \mu + \ldots + \tau = 1 \quad (2),$$

$$\lambda x + \mu y + \ldots + \tau w = 1 \quad (3),$$

then, dividing both sides of (1) by

$$\{(pa + qb + \ldots + tk)/(p + q + \ldots + t)\}^m,$$

we have to prove that

$$\lambda x^m + \mu y^m + \ldots + \tau w^m \lessgtr 1 \qquad (4),$$

according as m does not or does lie between 0 and + 1.

Now, by § 7, if m does not lie between 0 and + 1, $x^m - 1 \lessgtr m(x-1)$, $y^m - 1 \lessgtr m(y-1)$, &c. Therefore, since λ, μ, &c., are positive,

$$\begin{aligned}\Sigma\lambda(x^m - 1) &\lessgtr \Sigma\lambda m(x-1),\\ &\lessgtr m\{\Sigma\lambda x - \Sigma\lambda\},\\ &\lessgtr m(1-1),\end{aligned}$$

by (2) and (3), that is,

$$\Sigma\lambda x^m - \Sigma\lambda \lessgtr 0.$$

Hence

$$\Sigma\lambda x^m \lessgtr 1.$$

In like manner, we show that, if m lies between 0 and + 1,

$$\Sigma\lambda x^m \gtrless 1.$$

Cor. *If we make* $p = q = \ldots = t$, *we have*

$$\frac{a^m + b^m + \ldots + k^m}{n} \lessgtr \left(\frac{a + b + \ldots + k}{n}\right)^m \qquad (5),*$$

that is to say, the arithmetical mean of the m*th powers of* n *positive quantities is not less or not greater than the* m*th power of their arithmetical mean, according as* m *does not or does lie between* 0 *and* + 1.

Remark.—It is obvious that each of the inequalities (1), (4), (5) becomes an equality if $a = b = \ldots = k$, if $m = 0$, or if $m = 1$.

Example. Show that $\Sigma\lambda x^m$, considered as a function of m, increases as m increases when $m > +1$, and decreases as m increases when $m < -1$, λ, μ, ν, . . ., x, y, z, . . . being as above.

1st. Let $m > 1$. We have to show that $\Sigma\lambda x^{m+r} > \Sigma\lambda x^m$, where r is very small and positive, that is,

$$\Sigma\lambda x^m(x^r - 1) > 0.$$

Now,

$$\begin{aligned}\Sigma\lambda x^m(x^r - 1) &> \Sigma\lambda x^m r x^{r-1}(x-1),\\ &> r\Sigma\lambda x^{m+r-1}(x-1).\end{aligned}$$

---

* The earliest notice of this theorem with which we are acquainted is in Reynaud and Duhamel's *Problèmes et Développemens sur Diverses Parties des Mathématiques* (1823), p. 155. Its surroundings seem to indicate that it was suggested by Cauchy's theorem of § 8. The original proof rests on a maximum or minimum theorem, established by means of the Differential Calculus; and the elementary proofs hitherto given have usually involved the use of infinite series.

Since $m>1$, $m+r>1$, therefore $(m+r)x^{m+r-1}(x-1)>(m+r)(x-1)$, that is, $x^{m+r-1}(x-1)>(x-1)$.

Hence
$$\Sigma\lambda x^m(x^r-1)>r\Sigma\lambda(x-1),$$
$$>r(\Sigma\lambda x-\Sigma\lambda),$$
$$>0.$$
Therefore
$$\Sigma\lambda x^{m+r}>\Sigma\lambda x^m.$$

2nd. Let $m<-1$.
$$\Sigma\lambda x^m(x^r-1)<r\Sigma\lambda x^m(x-1).$$

Now $(m+1)x^m(x-1)>(m+1)(x-1)$, since $m+1$ is negative. Hence, dividing by the negative quantity $m+1$, we have
$$x^m(x-1)<(x-1).$$
Hence
$$\Sigma\lambda x^m(x^r-1)<r\Sigma\lambda(x-1),$$
$$<r(\Sigma\lambda x-\Sigma\lambda),$$
$$<0.$$
Therefore,
$$\Sigma\lambda x^{m+r}<\Sigma\lambda x^m.$$

Exercises V.*

(1.) For what values of x/y is $(a+b)xy/(ax+by)\gtrless(ax+by)/(a+b)$?

(2.) If x, y, z be any real quantities, and $x>y>z$, then $x^4y+y^4z+z^4x>xy^4+yz^4+zx^4$.

(3.) If x, y, z be any real quantities, then $\Sigma(y-z)(z-x)\gtrless 0$ and $\Sigma yz/\Sigma x^2\gtrless 1$.

(4.) If $x^2+y^2+z^2+2xyz=1$, then will all or none of the quantities x, y, z lie between -1 and $+1$.

(5.) If x and m be positive integers, show that
$$x^{2m+3}<x(x+1)(2x+1)(3x^3+3x+1)^m/2\,.\,3^m<(x+1)^{2m+3}.$$

(6.) $(a^2/b)^{\frac{1}{2}}+(b^2/a)^{\frac{1}{2}}\lessgtr a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

(7.) If $x_1, x_2, \ldots, x_n$ all have the same sign, and $1+x_1, 1+x_2, \ldots, 1+x_n$ be all positive, then
$$\Pi(1+x_1)>1+\Sigma x_1.$$

(8.) Prove that $8xyz\gtrless\Pi(y+z)\gtrless\frac{8}{9}\Sigma x^3$.

(9.) If $x, y, z, \ldots, a, b, c \ldots$ be two sets, each containing n real quantities positive or negative, show that
$$\Sigma a^2\Sigma x^2\lessgtr(\Sigma ax)^2;$$
also that, if all the quantities be positive,
$$\Sigma(x/a)/\Sigma x\lessgtr\Sigma x/\Sigma ax;$$
and, if $\Sigma x=1$,
$$\Sigma 1/x\lessgtr n^2.$$

(10.) If $x_1, x_2, \ldots, x_n$ and also $y_1, y_2, \ldots, y_n$ be positive and in ascending or in descending order of magnitude, then
$$\Sigma x_1^2y_1/\Sigma x_1y_1>\Sigma x_1^2/\Sigma x_1. \qquad (Laplace.)$$

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* Unless the contrary is stated, all letters in this set of exercises stand for real positive quantities.

(11.) If $a, b, \ldots, l$ be in A.P., show that

$$a^2b^2 \ldots l^2 > a^n l^n.$$

(12.) For what values of x is $(x-3)/(x^2+x+1) > (x-4)/(x^2-x+1)$?

(13.) Find the limits of x and y in order that

$$c > ax + by > d,$$
$$a > cx + dy > b;$$

where $$ad - bc \neq 0.$$

(14.) $x^6 - x^5y + 4x^4y^2 - 2x^3y^3 + 4x^2y^4 - xy^5 + y^6 > 0$, for all real values of x and y.

(15.) Is $10x^2+5y^2+13z^2 \geqq 8yz+2xy+18zx$?

(16.) If $p \leqslant 2-\sqrt{2}$, then $\sqrt{(x^2+y^2)}+p\sqrt{(xy)} > x+y$.

(17.) Is $\sqrt{(a^2+ab+b^2)} - \sqrt{(a^2-ab+b^2)} \geqq 2\sqrt{(ab)}$?

(18.) If x and a be positive, between what limits must x lie in order that $x+a > \sqrt{\{\frac{1}{2}(x^2+xa+a^2)\}} + \sqrt{\{\frac{1}{2}(x^2-xa+a^2)\}}$?

(19.) If $x<1$, then $\{x+\sqrt{(x^2-1)}\}^{\frac{1}{3}} + \{x-\sqrt{(x^2-1)}\}^{\frac{1}{3}} < 2$.

(20.) If all the three quantities $\sqrt{\{a(b+c-a)\}}$, $\sqrt{\{b(c+a-b)\}}$, $\sqrt{\{c(a+b-c)\}}$ be real, then the sum of any two is greater than the third.

(21.) If the sum of any two of the three x, y, z be greater than the third, then $\frac{1}{3}\Sigma x \Sigma x^2 > \Sigma x^3 + xyz$.

(22.) $\Sigma 1/x \leqslant \Sigma x^3/x^3y^3z^3$.

(23.) If p_r denote the sum of the products r at a time of a, b, c, d (each positive and <1), then $p_3 + 2p_4 > 2p_3$.

(24.) $\Sigma x^4 \leqslant xyz\Sigma x$.

(25.) If $s = a+b+c+\ldots$ n terms, then $\Sigma s/(s-a) \leqslant n^2/(n-1)$.

(26.) If $m>1$, $x<1$, and $mx<1+x$, then $1/(1\mp mx) > (1\pm x)^m > 1\pm mx$.

If $m<1$, $x<1$, $mx<1+x$, then $(1+x)/\{1\pm(1-m)x\} < (1\pm x)^m < 1\pm mx$.

(27.) If $z^n = x^n + y^n$, then $z^m \gtrless x^m + y^m$ according as $m \gtrless n$.

(28.) If x and y be unequal, and $x+y \leqslant 2a$, then $x^m + y^m > 2a^m$, m being a positive integer.

(29.) $n\{(n+1)^{1/n}-1\} < 1+1/2+\ldots+1/n < n\{1-1/(n+1)^{1/n}+1/(n+1)\}$.
(Schlömilch, *Zeitschr. f. Math.*, vol. iii. p. 25.)

(30.) If $x_1x_2 \ldots x_n = y^n$, $\Pi(1+x_1) \leqslant (1+y)^n$.

(31.) If $a, b, \ldots, k$ be n positive quantities arranged in ascending order of magnitude, and if $M_r = \{\Sigma a^r/n\}^{1/r}$, $N_r = \{\Sigma a^{1/r}\}^r/n$, then

$$(ab \ldots k)^{1/n} < M_1 < M_2 < \ldots < k,$$
$$(ab \ldots k)^{1/n} < \ldots < N_3 < N_2 < N_1.$$
(Schlömilch, *Zeitschr. f. Math.*, vol. iii. p. 301.)

(32.) If p, q, r be all unequal, and $x \neq 1$, then $\Sigma p x^{q-r} > \Sigma p$.

(33.) If n be integral, and x and n each >1, then

$$x^n - 1 > n(x^{(n+1)/2} - x^{(n-1)/2}).$$

(34.) Prove for x, y, z that $(2\Sigma yz - \Sigma x^2)^{\Sigma x} \leqslant (\Sigma x)^{\Sigma x} \Pi(\Sigma x - 2x)^x$.

(35.) If $s = a_1 + a_2 + \ldots + a_n$, then $\Pi(s/a_r - 1)^{a_r} \geqslant (n-1)^s$.

(36.) $3m(3m+1)^2 > 4(3m!)^{1/m}$.

(37.) If s_m be the sum of the nth powers of $a_1, a_2, \ldots, a_n$, and p_m the sum of their products m at a time, then $(n-1)!s_m \not< (n-m)!m!p_m$.

(38.) If $a_1 > a_2 > \ldots > a_n$, then

$$(a_1 - a_n)^{n-1} > (n-1)^{n-1}(a_1 - a_2)(a_2 - a_3) \ldots (a_{n-1} - a_n).$$

Hence, or otherwise, show that $\{(n-1)!\}^2 > n^{n-2}$.

(39.) Which is the greatest of the numbers $\sqrt[2]{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, . . . ?

(40.) If there be n positive quantities $x_1, x_2, \ldots, x_n$, each > 1, and if $\xi_1, \xi_2, \ldots, \xi_n$ be the arithmetic means, or the geometric means, of all but x_1, all but x_2, . . ., all but x_n, then $\Pi x_1^{\xi_1} \not> \Pi \xi_1^{x_1}$.

(41.) If a, b, c be such that the sum of any two is greater than the third, and x, y, z such that Σx is positive, then, if $\Sigma a^2/x = 0$, show that xyz is negative.

(42.) If $A = a_1 + a_2 + \ldots + a_n$, $B = b_1 + b_2 + \ldots + b_n$, then $\Sigma(a_r/A - b_r/B)(a_r/b_r)^n$ has the same sign as n for all finite values of n.

(Math. Trip., 1870.)

APPLICATIONS TO THE THEORY OF MAXIMA AND MINIMA.

§ 10.] The general nature of the connection between the theory of maxima and minima and the theory of inequalities may be illustrated as follows:—Let $\phi(x, y, z)$, $f(x, y, z)$ be any two functions of x, y, z, and suppose that for all values consistent with the condition

$$f(x, y, z) = A \qquad (1),$$

we have the inequality

$$\phi(x, y, z) \not> f(x, y, z) \qquad (2).$$

If we can find values of x, y, z, say a, b, c, which satisfy the equation (1) and at the same time make the inequality (2) an equality, then $\phi(a, b, c)$ is a maximum value of $\phi(x, y, z)$. For, by hypothesis, $\phi(a, b, c) = A$ and $\phi(x, y, z) \not> A$; therefore $\phi(x, y, z)$ cannot, for the values of x, y, z considered, be greater than A, that is, than $\phi(a, b, c)$.

Again, if we consider all values of x, y, z for which

$$\phi(x, y, z) = A \qquad (1'),$$

if we have

$$f(x, y, z) \not< \phi(x, y, z) \not< A \qquad (2'),$$

it follows in like manner that, if a, b, c be such that $\phi(a, b, c) = A$, $f(a, b, c) = A$, then $f(a, b, c)$ is a minimum value of $f(x, y, z)$.

The reasoning is, of course, not restricted to the case of three variables, although for the sake of brevity we have spoken of only three. The nature of this method for finding turning values may be described by saying that such values arise from exceptional or limiting cases of an inequality.

§ 11.] The reader cannot fail to be struck by the reciprocal character of the two theorems deduced in last section from the same inequality. The general character of this reciprocity will be made clear by the following useful general theorem:—

If for all values of x, y, z, *consistent with the condition*

$$f(x, y, z) = \mathrm{A},$$

$\phi(x, y, z)$ *have a maximum value* $\phi(a, b, c) = \mathrm{B}$ *say* (*where* B *depends, of course, upon* A), *and if when* A *increases* B *also increases, and* vice versa, *then for all values of* x, y, z, *consistent with the condition*

$$\phi(x, y, z) = \mathrm{B},$$

$f(x, y, z)$ *will have a minimum value* $f(a, b, c) = \mathrm{A}$.

Proof.—Let $\mathrm{A}' < \mathrm{A}$, then, by hypothesis, when $f(x, y, z) = \mathrm{A}'$, $\phi(x, y, z) \not> \mathrm{B}'$ where $\mathrm{B}' < \mathrm{B}$.

Hence, if $\phi(x, y, z) = \mathrm{B}$, $f(x, y, z) \not< \mathrm{A}$; for suppose if possible that $f(x, y, z) = \mathrm{A}' < \mathrm{A}$, then we should have $\phi(x, y, z) \not> \mathrm{B}'$, that is, since $\mathrm{B}' < \mathrm{B}$, $\phi(x, y, z)$ could not be equal to B as required. Hence, if a, b, c be such that $\phi(a, b, c) = \mathrm{B}$ and $f(a, b, c) = \mathrm{A}$, $f(a, b, c)$ is a minimum value of $f(x, y, z)$.

By means of the two general theorems just proved, we can deduce the solution of a large number of maximum and minimum problems from the inequalities established in the present chapter.

§ 12.] From the theorem of § 8 we deduce immediately the two following:—

I. *If* $x, y, z, \ldots$ *be* n *positive quantities subject to the condition*

$$\Sigma x = k,$$

then their product Πx *has a maximum value,* $(k/n)^n$, *when* $x = y = \ldots = k/n$.

II. *If* $x, y, z, \ldots$ *be* n *positive quantities subject to the condition*

$$\Pi x = k,$$

then their sum Σx *has a minimum value,* $nk^{1/n}$, *when* $x = y = \ldots = k^{1/n}$.

The second of these might be deduced from the first by the reciprocity-theorem.

From the corollary in § 8 we deduce the following :—

III. *If* $x, y, z, \ldots$ *be* n *positive quantities subject to the condition*

$$\Sigma px = k,$$

where $p, q, r, \ldots$ *are all positive constants, then* Πx^p *has a maximum value,* $\{k/\Sigma p\}^{\Sigma p}$, *when* $x = y = \ldots = k/\Sigma p$.

IV. *If* $x, y, z, \ldots$ *be* n *positive quantities subject to the restriction*

$$\Pi x^p = k,$$

where $p, q, r, \ldots$ *are all positive constants, then* Σpx *has a minimum value,* $(\Sigma p)k^{1/\Sigma p}$, *when* $x = y = \ldots = k^{1/\Sigma p}$.

From the last pair we can deduce the following, which are still more general :—

V. *If* $\lambda, \mu, \nu, \ldots, l, m, n, \ldots, p, q, r, \ldots$ *be all positive constants, and* $x, y, z, \ldots$ *be all positive, then if*

$$\Sigma \lambda x^l = k,$$

Πx^p *is a maximum when*

$$l\lambda x^l/p = m\mu y^m/q = n\nu z^n/r = \ldots$$

VI. *And if* $\Pi x^p = k,$

$\Sigma \lambda x^l$ *is a minimum when*

$$l\lambda x^l/p = m\mu y^m/q = n\nu z^n/r = \ldots$$

Proof.—Denote $p/l, q/m, r/n, \ldots$ by $\alpha, \beta, \gamma, \ldots$;

and let $\lambda x^l = a\xi, \quad \mu y^m = \beta\eta, \quad \nu z^n = \gamma\zeta$, &c.

So that $x = (a\xi/\lambda)^{1/l}$, &c.; $x^p = (a\xi/\lambda)^\alpha$, &c.

We then have in the first case

$$\Sigma a\xi = k \qquad (1),$$

$$\Pi x^p \equiv \Pi(a/\lambda)^\alpha \Pi\xi^\alpha \qquad (2).$$

Hence, since $(a/\lambda)^\alpha, (\beta/\mu)^\beta, \ldots$ are all constant and all positive, Πx^p is a maximum when $\Pi\xi^\alpha$ is a maximum. Now, under the condition (1), $\Pi\xi^\alpha$ is a maximum when $\xi = \eta = \ldots = k/\Sigma a$.

Hence Πx^p is a maximum when $\lambda x^l/a = \mu y^m/\beta = \ldots$, that is, when $l\lambda x^l/p = m\mu y^m/q = \ldots$

The maximum value of Πx^p is $\Pi(a/\lambda)^a(k/\Sigma a)^{\Sigma a}$, and the corresponding values of $x, y, z, \ldots$ are given by

$$x = (ak/\lambda\Sigma a)^{1/l} \ldots$$

Applying the reciprocity-theorem, we see that, if

$$\Pi x^p = \Pi(a/\lambda)^a(k/\Sigma a)^{\Sigma a},$$

the minimum value of $\Sigma\lambda x^l$ is k, corresponding to

$$x = (ak/\lambda\Sigma a)^{1/l} \ldots$$

Whence, putting $j = \Pi(a/\lambda)^a(k/\Sigma a)^{\Sigma a}$, we see that, if $\Pi x^p = j$, the minimum value of $\Sigma\lambda x^l$ is $\Sigma a\{j/\Pi(a/\lambda)^a\}^{1/\Sigma a}$, corresponding to

$$x = [a\{j/\Pi(a/\lambda)^a\}^{1/\Sigma a}/\lambda]^{1/l} \ldots$$

Cor. If we put $l = m = n = \ldots = 1$, $p = q = r = \ldots = 1$, we obtain the following particular cases, which are of frequent occurrence:—

If $\Sigma\lambda x = k$, Πx *is a maximum when* $\lambda x = \mu y = \ldots$;

If $\Pi x = k$, $\Sigma\lambda x$ *is a minimum when* $\lambda x = \mu y = \ldots$

Example 1. The cube is the rectangular parallelopiped of maximum volume for given surface, and of minimum surface for given volume.

If we denote the lengths of three adjacent edges of a rectangular parallelopiped by x, y, z, its surface is $2(yz+zx+xy)$ and its volume is xyz. If we put $\xi = yz$, $\eta = zx$, $\zeta = xy$, the surface is $2(\xi+\eta+\zeta)$ and the volume $\sqrt{(\xi\eta\zeta)}$. Hence, analytically considered, the problem is to make $\xi\eta\zeta$ a maximum when $\xi+\eta+\zeta$ is given, and to make $\xi+\eta+\zeta$ a minimum when $\xi\eta\zeta$ is given. This, by Th. I., is done in either case by making $\xi = \eta = \zeta$, that is, $yz = zx = xy$; whence $x = y = z$.

Example 2. The equilateral triangle has maximum area for given perimeter, and minimum perimeter for given area.

The area is $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. Let $x = s-a$, $y = s-b$, $z = s-c$; then $x+y+z = s$; and the area is $\sqrt{xyz}$. Since, in the first place, s is given, we have merely to make xyz a maximum subject to the condition $x+y+z = s$. This leads to $x = y = z$ (by Th. I.)

Next, let Δ be given.

Then
$$(x+y+z)xyz = \Delta^2 \quad (1);$$
$$s = \Delta^2/xyz \quad (2).$$

If we put $\xi = x^2yz$, $\eta = xy^2z$, $\zeta = xyz^2$, we have

$$\xi+\eta+\zeta = \Delta^2 \quad (1');$$
$$s = \Delta^2/(\xi\eta\zeta)^{1/4} \quad (2').$$

Hence, to make s a minimum when Δ is given, we have to make $\xi\eta\zeta$ a *maximum*, subject to the condition (1'). This leads to $\xi=\eta=\zeta$, that is, $x^2yz=xy^2z=xyz^2$; whence $x=y=z$.

Example 3. To construct a right circular cylinder of given volume and minimum total surface.

Let x be the radius of the ends, and y the height of the cylinder. The total surface is $2\pi(x^2+xy)$, and the volume is πx^2y.

We have, therefore, to make $u=x^2+xy$ a minimum, subject to the condition $x^2y=c$. We have

$$u=x^2+xy=c/y+c/x \quad (1);$$
$$x^2y=c \quad (2).$$

Let $1/x=2\xi, \quad 1/y=\eta$;

then
$$u=c(2\xi+\eta) \quad (1');$$
$$\xi^2\eta=1/4c \quad (2').$$

We have now to make $2\xi+\eta$ (that is, $\xi+\xi+\eta$) a minimum, subject to the condition $\xi^2\eta=$ constant. This, by Th. II., leads to $\xi=\xi=\eta$, which gives $2x=y$. Hence the height of the cylinder is equal to its diameter.

By the reciprocity theorem (applied to the problem as originally stated in terms of x and y), it is obvious that a cylinder of this shape also has maximum volume for given total surface.

§ 13.] From the inequality of § 9 we infer the following:—

VII. *If m do not lie between 0 and + 1, and if $p, q, r, \ldots$ be all constant and positive, then, for all positive values of $x, y, z, \ldots$ such that*

$$\Sigma px = k,$$

Σpx^m (m unchanged) has a minimum value when $x=y=z=\ldots$

If m lie between 0 and + 1, instead of a minimum we have a maximum.

In stating the reciprocal theorem it is necessary to notice that, in the inequality, Σpx occurs raised to the mth power; so that, *if m be negative*, a maximum of Σpx corresponds to a minimum of $(\Sigma px)^m$. Attending to this point, we see that—

VIII. *If $m > +1$, and if $p, q, r, \ldots$ be all constant and positive, then, for all positive values of $x, y, z, \ldots$ such that*

$$\Sigma px^m = k \ (m \text{ unchanged}),$$

Σpx has a maximum value when $x=y=z=\ldots$

If $m < +1$, we have a minimum instead of a maximum.

Theorem VIII. might also be deduced from Theorem VII. by the substitution $\xi=x^m$, $\eta=y^m$, $\zeta=z^m$, &c. . . .

§ 14.] Theorem VII. may be generalised by a slight transformation into the following:—

IX. *If m/n do not lie between 0 and + 1, and if $p, q, r, \ldots, \lambda, \mu, \nu, \ldots$ be all constant and positive, then, for all positive values of $x, y, z, \ldots$ such that*

$$\Sigma\lambda x^n = k \text{ (} n \text{ unchanged)},$$

Σpx^m (*m unchanged*) *has a minimum value when* $px^m/\lambda x^n = qy^m/\mu y^n = \ldots$

If m/n lie between 0 and + 1, instead of a minimum we have a maximum.

The transformation in question is as follows:—

Let
$$\lambda x^n = \rho\xi, \quad \mu y^n = \sigma\eta \qquad (1),$$
$$px^m = \rho\xi^f, \quad qy^m = \sigma\eta^f \qquad (2).$$

From the first two equations in (1) and (2) we deduce $\xi^{f-1} = px^{m-n}/\lambda$, $\rho^{f-1} = \lambda^f x^{fn-m}/p$, &c. Hence, if we take $fn = m$, that is, $f = m/n$, $\rho, \sigma, \ldots$ will be all constant and obviously all positive; we have, in fact,

$$\xi = (px^{m-n}/\lambda)^{1/(f-1)}, \quad \eta = (qy^{m-n}/\mu)^{1/(f-1)}, \ \ldots \ (3),$$
$$\rho = (\lambda^f/p)^{1/(f-1)}, \quad \sigma = (\mu^f/q)^{1/(f-1)}, \ \ldots \qquad (4);$$

and we have now to make $\Sigma\rho\xi^f$ a maximum or minimum, subject to the condition

$$\Sigma\rho\xi = k.$$

Now, by Th. VII., $\Sigma\rho\xi^f$ is a minimum or maximum, according as f does not or does lie between 0 and + 1, when $\xi = \eta = \ldots$

Thus the conditions for a turning value are

$$(px^{m-n}/\lambda)^{1/(f-1)} = (qy^{m-n}/\mu)^{1/(f-1)} = \ldots,$$

which lead at once to

$$px^m/\lambda x^n = qy^m/\mu y^n = \ldots$$

Cor. A very common case is that where $n = 1$, $\lambda = \mu = \ldots = 1$.

We then have, subject to the condition $\Sigma x = k$, Σpx^m, a minimum or maximum when $px^{m-1} = qy^{m-1} = \ldots$, according as m does not or does lie between 0 and + 1.

§ 15.] We have hitherto restricted $p, q, r, \ldots$ in the in-

equality of § 9 to be constant. This is unnecessary; they may be functions of the variables, provided they be such that they remain positive for all positive values of x, y, z.

We therefore have the following theorem and its reciprocal (the last omitted for brevity):—

X. *If $p, q, r, \ldots$ be functions of $x, y, z, \ldots$ which are real and positive for all real and positive values of $x, y, z, \ldots$, then, for all positive values of $x, y, z, \ldots$ which satisfy*

$$\Sigma px = k,$$

$(\Sigma px^m)(\Sigma p)^{m-1}$ *(m unchanged) has a minimum or maximum value when $x = y = \ldots$, according as m does not or does lie between* 0 *and* $+1$.

For example, we may obviously put $p = \lambda x^a$, $q = \mu y^b$, $\ldots$

We thus deduce that if $m > +1$ or < 0, then, for all positive values of $x, y, z, \ldots$ consistent with $\Sigma\lambda x^{a+1} = k$, $(\Sigma\lambda x^{m+a})(\Sigma\lambda x^a)^{m-1}$ is a minimum when $x = y = \ldots$

Theorem X. may again be transformed into others in appearance more general, by methods which the student will readily divine after the illustrations already given.

Also the inequalities of § 8 may be used to deduce maxima and minima theorems in the same way as those of § 9 were used in the proof of Theorem X.

Example 1. To find the minimum value of $u = x + y + z$, subject to the conditions $a/x + b/y + c/z = 1$, $x > 0$, $y > 0$, $z > 0$, a, b, c being positive constants.

Let $$x = \rho\xi^f, \quad y = \sigma\eta^f, \quad z = \tau\zeta^f;$$
$$a/x = \rho\xi, \quad b/y = \sigma\eta, \quad c/z = \tau\zeta.$$

Hence $\rho^{f-1} = a^f/x^{f+1}$. If we take $f = -1$, we therefore get
$$x = \sqrt{a}\xi^{-1}, \quad y = \sqrt{b}\eta^{-1}, \quad z = \sqrt{c}\zeta^{-1};$$
$$a/x = \sqrt{a}\xi, \quad by = \sqrt{b}\eta, \quad cz = \sqrt{c}\zeta.$$

The problem now is to make $u = \Sigma\sqrt{a}\xi^{-1}$ a minimum subject to the condition $\Sigma\sqrt{a}\xi = 1$. By Th. VII. this is accomplished by making $\xi = \eta = \zeta$. Hence $\xi = \eta = \zeta = 1/\Sigma\sqrt{a}$. The minimum value required is therefore $(\Sigma\sqrt{a})^2$; the corresponding values of x, y, z are $\sqrt{a}\Sigma\sqrt{a}$, $\sqrt{b}\Sigma\sqrt{a}$, $\sqrt{c}\Sigma\sqrt{a}$ respectively.

Example 2. To find a point within a triangle such that the sum of the mth powers of its distances from the sides shall be a minimum ($m > 1$).

Let a, b, c be the sides, x, y, z the three distances; then we have to make $u = \Sigma x^m$ a minimum, subject to the condition $\Sigma ax = 2\Delta$, where Δ is the area of the triangle.

If $\rho\xi^m = x^m$, $\rho\xi = ax$, then $\rho^{m-1} = a^m$, $\rho = a^{m/(m-1)}$.

Hence, if we put $ax = a^{m/(m-1)}\xi$, $by = b^{m/(m-1)}\eta$, $cz = c^{m/(m-1)}\zeta$, we have

$$u = \Sigma a^{m/(m-1)}\xi^m,$$
$$2\Delta = \Sigma a^{m/(m-1)}\xi.$$

The solution is therefore given by $\xi = \eta = \zeta = 2\Delta/\Sigma a^{m/(m-1)}$.

Whence $x = 2\Delta a^{1/(m-1)}/\Sigma a^{m/(m-1)}$, $y =$ &c., $z =$ &c.

Example 3. Show that, if $x^3 + y^4 + z^5 = 3$, then $(x^4 + y^5 + z^6)(x^2 + y^3 + z^4)$ has a minimum value for all positive values of x, y, z when $x = y = z = 1$.

This follows from Th. X., if we put $m = 2$, $p = x^2$, $q = y^3$, $r = z^4$, which is legitimate since x, y, z are all positive.

Example 4. If $x, y, z, \ldots$ be n positive quantities, and m do not lie between 0 and 1, show that the least possible value of $(\Sigma x^{m-1})(\Sigma 1/x)^{m-1}$ is n^m.

This follows at once from the inequality of § 9, if we put $p = 1/x$, $q = 1/y, \ldots$

§ 16.] The field of application of some of the foregoing theorems can be greatly extended by the use of undetermined multipliers in a manner indicated by Grillet.*

Suppose, for example, it were required to discuss the turning values of the function

$$u \equiv (ax + p)^l (bx + q)^m (cx + r)^n \qquad (1),$$

where l, m, n are all positive.

We may write

$$u \equiv (\lambda ax + \lambda p)^l (\mu bx + \mu q)^m (\nu cx + \nu r)^n / \lambda^l \mu^m \nu^n \qquad (2),$$

where λ, μ, ν are three arbitrary quantities, which we may subject to any three conditions we please.

Let the first condition be

$$l\lambda a + m\mu b + n\nu c = 0 \qquad (3);$$

then we have

$$l(\lambda ax + \lambda p) + m(\mu bx + \mu q) + n(\nu cx + \nu r)$$
$$= l\lambda p + m\mu q + n\nu r = k \qquad (4),$$

where k is an arbitrary positive constant.

This being so, we see by Th. III. that $\Pi(\lambda ax + \lambda p)^l$ is a maximum when

$$\lambda ax + \lambda p = \mu bx + \mu q = \nu cx + \nu r$$
$$= k/\Sigma l \qquad (5).$$

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* *Nouvelles Annales de Math.*, ser. i., tt. 9, 16.

The four equations (3) and (5) are not more than sufficient to exhaust the three conditions on λ, μ, ν, and to determine x.

We can easily determine x by itself. In fact, from (3) and (5) we deduce at once

$$la/(ax+p)+mb/(bx+q)+nc/(cx+r)=0 \qquad (6).$$

This quadratic gives two values for x, say x_1 and x_2; and the equations (5) give two corresponding sets of values for λ, μ, ν, in terms of k, say λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2.

If, then, $\lambda_1^l\mu_1^m\nu_1^n$ be positive, x_1 will correspond to a maximum value of u; if $\lambda_1^l\mu_1^m\nu_1^n$ be negative, x_1 will correspond to a minimum value of u; and the like for x_2.

Example 1. To discuss $u\equiv(x+3)^2(x-3)$.

We have $$u\equiv(\lambda x+3\lambda)^2(\mu x-3\mu)/\lambda^2\mu.$$

Now $$2(\lambda x+3\lambda)+(\mu x-3\mu)=k,$$

provided $$2\lambda+\mu=0 \qquad (1),$$

$$6\lambda-3\mu=k \qquad (2).$$

Therefore $(\lambda x+3\lambda)^2(\mu x-3\mu)$ will be a maximum, provided

$$\lambda x+3\lambda=\mu x-3\mu \qquad (3).$$

Hence, by (1),

$$2/(x+3)+1/(x-3)=0;$$

which gives $x=1$. From (2) and (3) we deduce $\lambda=k/12$, $\mu=-k/6$; so that $\lambda^2\mu$ is negative.

We therefore conclude that u is a minimum when $x=1$.

The student should trace the graph of the function u; he will thus find that it has also a maximum value, corresponding to $x=-3$, of which this method gives no account.

Example 2. For what values of x and y is

$$u=(a_1x+b_1y+c_1)^2+(a_2x+b_2y+c_2)^2+\ldots+(a_nx+b_ny+c_n)^2$$

a minimum?

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be undetermined multipliers. Then we may write

$$u=\Sigma\lambda_1^2\{(a_1x+b_1y+c_1)/\lambda_1\}^2 \qquad (1);$$

and

$$k=\Sigma\lambda_1^2\{(a_1x+b_1y+c_1)/\lambda_1\} \qquad (2),$$

where k is an arbitrary positive constant, that is, independent of x and y, provided

$$\Sigma a_1\lambda_1=0, \quad \Sigma b_1\lambda_1=0, \quad \Sigma c_1\lambda_1=k \qquad (3).$$

This being so, by Th. VII., u is a minimum when

$$(a_1x+b_1y+c_1)/\lambda_1=(a_2x+b_2y+c_2)/\lambda_2=\ldots=k/\Sigma\lambda_1^2 \qquad (4).$$

The $n+2$ equations, (3) and (4), just suffice for the determination of $\lambda_1, \lambda_2, \ldots, \lambda_n, x, y$.

From the first two of (3), and from (4), we deduce

$$\Sigma a_1(a_1x + b_1y + c_1) = 0,$$
$$\Sigma b_1(a_1x + b_1y + c_1) = 0.$$

Hence the values of x and y corresponding to the minimum value of n are given by the system

$$\Sigma a_1^2x + \Sigma a_1b_1y + \Sigma a_1c_1 = 0,$$
$$\Sigma a_1b_1x + \Sigma b_1^2y + \Sigma b_1c_1 = 0.$$

This is the solution of a well-known problem in the Theory of Errors of Observation.

§ 17. *Method of Increments.*—Following the method already exemplified in the case of a function of one variable, we may define

$$I = \phi(x + h,\ y + k,\ z + l) - \phi(x,\ y,\ z)$$

as the increment of $\phi(x, y, z)$. If, when $x = a$, $y = b$, $z = c$, the value of I be negative for all small values of h, k, l, then $\phi(a, b, c)$ is a maximum value of $\phi(x, y, z)$; and if, under like circumstances, I be positive, $\phi(a, b, c)$ is a minimum value of $\phi(x, y, z)$.

Owing to the greater manifoldness of the variation, the examination of the sign of the increment when there are more variables than one is often a matter of considerable difficulty; and any general theory of the subject can scarcely be established without the use of the infinitesimal calculus.

We may, however, illustrate the method by establishing a case of the following general theorem, which includes some of those stated above as particular cases.

Purkiss's Theorem.*—*If* $\phi(x, y, z, \ldots)$ $f(x, y, z, \ldots)$ *be symmetric functions of* $x, y, z, \ldots$, *and if* $x, y, z, \ldots$ *be subject to an equation of the form*

$$f(x, y, z, \ldots) = 0 \qquad (1),$$

then $\phi(x, y, z, \ldots)$ *has in general a turning value when* $x = y = z = \ldots$, *provided these conditions be not inconsistent with the equation* (1).

In our proof we shall suppose that there are only three variables; and so far as that is concerned it will be obvious that there is no loss of generality. But we shall also suppose both

* Given with inadequate demonstration in the Oxford, Cambridge, and Dublin *Messenger of Mathematics*, vol. i. (1862).

$\phi(x, y, z)$ and $f(x, y, z)$ to be integral functions, and this supposition, although it restricts the generality of the proof, renders it amenable to elementary treatment.

We remark, in the first place, that the conditions

$$x = y = z \text{ and } f(x, y, z) = 0$$

are in general just sufficient to determine a set of values for x, y, z. In fact, if the common value of x, y, z be a, then a will be a root of the equation $f(a, a, a) = 0$.

Consider the functions

$$\mathrm{I} = \phi(a + h,\ a + k,\ a + l) - \phi(a, a, a), \text{ and } f(a + h,\ a + k,\ a + l).$$

Each of them is evidently a symmetric function of h, k, l, and can therefore be expanded as an integral function of the elementary symmetric functions Σh, Σhk, hkl. We observe also that, since each of the functions vanishes when $h = 0$, $k = 0$, $l = 0$, there will be no term independent of h, k, l.

Let us now suppose h, k, l to be finite multiples of the same very small quantity r, say $h = \alpha r$, $k = \beta r$, $l = \gamma r$. Then $\Sigma h = r\Sigma\alpha = ru$ say, $\Sigma hk = r^2\Sigma\alpha\beta = r^2 v$, $hkl = r^3 w$. Expanding as above indicated, and remembering that by the conditions of our problem $f(a + h,\ a + k,\ a + l) = 0$, we have, if we arrange according to powers of r,

$$\mathrm{I} = \mathrm{A}ur + (\mathrm{B}u^2 + \mathrm{C}v)r^2 + \&\text{c.} \qquad (1),$$

$$0 = \mathrm{P}ur + (\mathrm{Q}u^2 + \mathrm{R}v)r^2 + \&\text{c.} \qquad (2),$$

where the &c. stands for terms involving r^3 and higher powers.

From (2) we have

$$ur = -(\mathrm{Q}u^2 + \mathrm{R}v)r^2/\mathrm{P} + \&\text{c.},$$

$$u^2r^2 = 0 + \&\text{c.},$$

$$2\Sigma\alpha\beta r^2 = -\Sigma\alpha^2 r^2 + \&\text{c.},$$

&c. as before including powers of r not under the 3rd.

Hence, substituting in (1) and writing out only such terms as contain no higher power of r than r^2, we have

$$\mathrm{I} = (\mathrm{C} - \mathrm{AR}/\mathrm{P})vr^2 + \&\text{c.},$$

$$= -\tfrac{1}{2}r^2(\mathrm{C} - \mathrm{AR}/\mathrm{P})\Sigma\alpha^2 + \&\text{c.}$$

Now (see chap. xv., § 10), by taking r sufficiently small, we may cause the first term on the right to dominate the sign of I.

Hence I will be negative or positive according as (CP – AR)/P is positive or negative; that is, $\phi(a, a, a)$ will be a maximum or minimum according as (CP – AR)/P is positive or negative.

Example. Discuss the turning values of $\phi(x, y, z) \equiv xyz + b(yz + zx + xy)$, subject to the condition $x^2 + y^2 + z^2 = 3a^2$.

The system

$$x = y = z, \quad x^2 + y^2 + z^2 - 3a^2 = 0$$

has the two solutions $x = y = z = \pm a$.

If we take $x = y = z = +a$, we find, after expanding as above indicated,

$$I \equiv (a^2 + 2ab)ur + (a + b)vr^2 + \&c.,$$
$$0 = 2aur + (u^2 - 2v)r^2.$$

In this case, therefore, $A = a^2 + 2ab$, $C = a + b$, $P = 2a$, $R = -2$; and (CP – AR)/P $= 2a + 3b$.

Hence, when $x = y = z = +a$, ϕ is a maximum or a minimum according as $2a + 3b$ is positive or negative.

In like manner, we see that, when $x = y = z = -a$, ϕ is a maximum or a minimum according as $-2a + 3b$ is positive or negative.

Exercises VI.*

(1.) Find the minimum value of $bcx + cay + abz$ when $xyz = abc$.

(2.) Find the maximum value of xyz when $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(3.) If $\Sigma x^2 = c$, Σlx is a maximum when $x : y : z : \ldots = l : m : n : \ldots$

(4.) Find the turning values of $\lambda x^{ma} + \mu y^{mb} + \nu z^{mc}$, subject to the condition $px^a + qy^b + rz^c = d$.

(5.) Find the turning values of $ax^p + by^q + cz^r$ when $xyz = d^3$.

(6.) If $xyz = a^2(x + y + z)$, then $yz + zx + xy$ is a minimum when $x = y = z = \sqrt{3}a$.

(7.) Find the turning values of $(x + l)(y + m)(z + n)$ where $a^x b^y c^z = d$.

(8.) Find the minimum value of $ax^m + b/x^n$.

(9.) Find the turning values of $(3x - 2)(x - 2)^2(x - 3)^2$.

(10.) If $cx(b - y) = ay(c - z) = bz(a - x)$, find the maximum value of each.

(11.) Find the turning values of x^m/y^n $(m > n)$, subject to the condition $x - y = c$. (Bonnet, *Nouv. Ann.*, ser. i., t. 2.)

(12.) If $x^p y^q + x^q y^p = a$, then $x^{p+q} + y^{p+q}$ has a minimum value when $x = y = (a/2)^{1/(p+q)}$; and, in general, if $\Sigma x^p y^q = a$, Σx^{p+q} has a minimum value, $a/(n - 1)$, when $x = y = z = \ldots = \{a/(n - 1)n\}^{1/(p+q)}$. Discuss specially the case where p and q have opposite signs.

(13.) If $x^p y^q + x^r y^s = c$, then $x^t y^u$ is a maximum when $x^{p-r}/(ru - st) = y^{s-q}/(qt - pu)$, the denominators, $ru - st$ and $qt - pu$, being assumed to have the same sign. (Desboves, *Questions d'Algèbre*, p. 455. Paris, 1878.)

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* Here, unless the contrary is indicated, all letters denote positive quantities.

(14.) If $p>q$, and $x^p+y^p=a^p$, then x^q+y^q is a minimum when $x=y=a/2^{1/p}$. State the reciprocal theorem.

(15.) Find the turning values of $(ax^2+by^2)/\sqrt{(a^2x^2+b^2y^2)}$ when $x^2+y^2=1$.

(16.) If $x_1, x_2, \ldots, x_n$ be each $>a$, and such that $(x_1-a)(x_2-a)\ldots(x_n-a)=b^n$, the least value of $x_1x_2\ldots x_n$ is $(a+b)^n$, a and b being both positive.

(17.) If $f(m)$ denote the greatest product that can be formed with n integers whose sum is m, show that $f(m+1)/f(m)=1+1/q$ where q is the integral part of m/n.

(18.) ABCD is a rectangle, APQ meets BC in P, and DC produced in Q. Find the position of APQ when the sum of the areas ABP, PCQ is a minimum.

(19.) O is a given point within a circle, and POQ and ROS are two perpendicular chords. Find the position of the chords when the area of the quadrilateral PRQS is a maximum or a minimum.

(20.) Two given circles meet orthogonally at A. PAQ meets the circles in P and Q respectively. Find the position of PAQ when PA,AQ is a maximum or minimum.

(21.) To inscribe in a given sphere the right circular cone of maximum volume.

(22.) To circumscribe about a given sphere the right circular cone of minimum volume.

(23.) Given one of the parallel sides and also the non-parallel sides of an isosceles trapezium, to find the fourth side in order that its area may be a maximum.

(24.) To draw a line through the vertex of a given triangle, such that the sum of the projections upon it of the two sides which meet in that vertex shall be a maximum.

CHAPTER XXV.

Limits.

§ 1.] In laying down the fundamental principles of algebra, it was necessary, at the very beginning, to admit certain limiting cases of the operations. Other cases of a similar kind appeared in the development of the science; and several of them were discussed in chap. xv. In most of these cases, however, there was little difficulty in arriving at an appropriate interpretation; others, in which a difficulty did arise, were postponed for future consideration. In the present chapter we propose to deal specially with these critical cases of algebraical operation, to which the generic name of "Indeterminate Forms" has been given. The subject is one of the highest importance, inasmuch as it forms the basis of two of the most extensive branches of modern mathematics—namely, the Differential Calculus and the Theory of Infinite Series (including from one point of view the Integral Calculus). It is too much the habit in English courses to postpone the thorough discussion of indeterminate forms until the student has mastered the notation of the differential calculus. This, for several reasons, is a mistake. In the first place, the definition of a differential coefficient involves the evaluation of an indeterminate form; and no one can make intelligent applications of the differential calculus who is not familiar beforehand with the notion of a limit. Again, the methods of the differential calculus for evaluating indeterminate forms are often less effective than the more elementary methods which we shall discuss below, and are always more powerful in combination with them.

§ 2.] The characteristic difficulty and the way of meeting it will be best explained by discussing a simple example. If in the function $(x^2-1)/(x-1)$ we put $x=2$, there is no difficulty in carrying out successively all the operations indicated by the synthesis of the function; the case is otherwise if we put $x=1$, for we have $1^2-1=0$, $1-1=0$, so that the last operation indicated is $0/0$—a case specially excluded from the fundamental laws; not included even under the case $a/0$ $(a \neq 0)$ already discussed in chap. xv., § 6. The first impulse of the learner is to assume that $0/0=1$, in analogy with $a/a=1$; but for this he has no warrant in the laws of algebra.

Strictly speaking, the function $(x^2-1)/(x-1)$ has no definite value when $x=1$; that is to say, it has no value that can be deduced from the principles hitherto laid down. This being so, and it being obviously desirable to make as general as possible the law that a function has a definite value corresponding to every value of its argument, we proceed to define the value of $(x^2-1)/(x-1)$ when $x=1$. In so doing we are naturally guided by the principle of continuity, which leads us to define the value of $(x^2-1)/(x-1)$ when $x=1$, so that it shall differ infinitely little from values of $(x^2-1)/(x-1)$, corresponding to values of x that differ infinitely little from 1. Now, so long as $x \neq 1$, no matter how little it differs from 1, we can perform the indicated division; and we have the identity $(x^2-1)/(x-1) \equiv x+1$. The evaluation of $x+1$ presents no difficulty; and we now see that for values of x differing infinitely little from 1, the value of $(x^2-1)/(x-1)$ differs infinitely little from 2. *We therefore define the value of* $(x^2-1)/(x-1)$ *when* $x=1$ *to be* 2; and we see that its value *is* 2 in the useful and perfectly intelligible sense that, *by bringing* x *sufficiently near to* 1, *we can cause* $(x^2-1)/(x-1)$ *to differ from* 2 *by as little as we please.** The value of $(x^2-1)/(x-1)$ thus specially defined is spoken of as the *limiting value*, or the *limit of* $(x^2-1)/(x-1)$ *for* $x=1$; and it is symbolised by writing

* The reader should observe that the definition of the critical value just given has another advantage, namely, it enables us to assert the truth of the identity $(x^2-1)/(x-1) \equiv x+1$ without exception in the case where $x=1$.

$$\mathrm{L}_{x=1}\frac{x^2-1}{x-1}=2,$$

where L is the initial of the word "limit." The subscript $x=1$ may be omitted when the value of the argument for which the limiting value is to be taken is otherwise sufficiently indicated.

We are thus led to construct the following definition of the value of a function, so as to cover the cases where the value indicated by its synthesis is indeterminate :—

When, by causing x to differ sufficiently little from a, we can make the value of $f(x)$ approach as near as we please to l, then l is said to be the limiting value, or limit, of $f(x)$ when $x=a$; and we write

$$\mathrm{L}_{x=a} f(x)=l.$$

Cor. 1. *A function is in general continuous in the neighbourhood of a limiting value; and, therefore, in obtaining that value we may subject the function to any transformation which is admissible on the hypothesis that the argument x has any value in the neighbourhood of the critical value a.*

We say "in general," because the statement will not be strictly true unless the phrase "differ infinitely little from" mean "differ *either in excess or in defect* infinitely little from." It may happen that we can only approach the limit from one side; or that we obtain two different limiting values according as we increase x *up to* the critical value, or diminish it *down to* the critical value. In this last case, the graph of the function in the neighbourhood of $x=a$ would have the peculiarity figured in chap. XV., Fig. 5; and the function would be discontinuous. The latter part of the corollary still applies, however, provided the proper restriction on the variation of x be attended to.

When it is necessary to distinguish the process of taking a limit by increasing x up to a from the process of taking a limit by decreasing x down to a, we may use the symbol $\mathrm{L}_{x=a-0}$ for the former, and the symbol $\mathrm{L}_{x=a+0}$ for the latter.

Cor. 2. *If $\mathrm{L}_{x=a} f(x)=l$, then $f(a+h)=l+d$, where d is a function of a and h, whose value may be made as small as we please by sufficiently diminishing h.*

This is simply a re-statement of the definition of a limit from another point of view.

Cor. 3. *Any ordinary value of a function satisfies the definition of a limiting value.*

For example, $\underset{x=2}{\mathrm{L}}\,(x^2-1)/(x-1) = (2^2-1)/(2-1) = 3$. This remark would be superfluous, were it not that attention to the point enables us to abbreviate demonstrations of limit theorems, by using the symbol L where there is no peculiarity in the evaluation of the function to which it is prefixed.

§ 3.] It may happen that the critical value a, instead of being a definite finite quantity, is merely a quantity greater than any finite quantity, however great. We symbolise the process of taking the limit in this case by writing $\underset{x=+\infty}{\mathrm{L}}\,f(x)$, or $\underset{x=-\infty}{\mathrm{L}}\,f(x)$, according as the quantity in question is positive or negative. For example,

$$\underset{x=\infty}{\mathrm{L}}\,(x+1)/x = \underset{x=\infty}{\mathrm{L}}\,(1+1/x) = 1.$$

In this case, we can, strictly speaking, approach the limit from one side only; and the question of continuity on both sides of the limit does not arise. If, however, we, as it were, join the series of algebraical quantity $-\infty \ldots -1 \ldots 0 \ldots +1 \ldots +\infty$ through infinity, by considering $+\infty$ and $-\infty$ as consecutive values; then we say that $f(x)$ is, or is not, continuous for the critical value $x=\infty$, according as $\underset{x=\infty}{\mathrm{L}}\,f(x)$ and $\underset{x=-\infty}{\mathrm{L}}\,f(x)$ have, or have not, the same value. For example, $(x+1)/x$ is continuous for $x=\infty$, for we have $\underset{x=\infty}{\mathrm{L}}\,(x+1)/x = 1 = \underset{x=-\infty}{\mathrm{L}}\,(x+1)/x$; but $(x^2+1)/x$ is not continuous for $x=\infty$.

§ 4.] The value 0 may of course occur as a limiting value; for example, $\underset{x=1}{\mathrm{L}}\,x(x-1)^2/(x^2-1) = 0$. It may also happen, even for a finite value of a, that $f(x)$ can be made greater than any finite quantity, however great, by bringing x sufficiently near to a. In this case we write $\underset{x=a}{\mathrm{L}}\,f(x) = \infty$. In thus admitting 0 and ∞ as limiting values, the student must not forget that the general rules for evaluating limits are, as will be shown presently, subject in certain cases to exception when these particular limits occur.

ENUMERATION OF THE ELEMENTARY INDETERMINATE FORMS.

§ 5.] Let u and v be any two functions of x. We have already seen, in chap. XV., that $u+v$ becomes indeterminate when u and v are infinite but of opposite sign; that $u \times v$ becomes indeterminate if one of the factors become zero and the other infinite; and that $u \div v$ becomes indeterminate if u and v become both zero, or both infinite. We thus have the indeterminate forms—(I.) $\infty - \infty$, (II.) $0 \times \infty$, (III.) $0 \div 0$, (IV.) $\infty \div \infty$.

It is interesting to observe that all these really reduce to (III.). Take $\infty - \infty$ for example. Since $u+v \equiv (1+v/u)/(1/u)$, and $\mathrm{L}1/u = 1/\infty = 0$, this function will not be really indeterminate unless $\mathrm{L}v/u = -1$. The evaluation of the form $\infty - \infty$ therefore reduces to a consideration of cases (IV.) and (III.) at most. Now, since $u \div v \equiv (1/v) \div (1/u)$, case (IV.) can be reduced to (III.); and finally, since $u \times v = u \div (1/v)$, case (II.) can be reduced to (III.).

To exhaust the category of elementary algebraical operations we have to discuss the critical values of u^v. This is most simply done by writing $u^v = a^{v \log_a u}$ where a is positive and >1. We thus see that u^v is determinate so long as $v \log_a u$ is determinate. The only cases where $v \log_a u$ ceases to be determinate are those where—(V.) $v = 0$, $\log_a u = +\infty$, that is $v = 0$, $u = \infty$; (VI.) $v = 0$, $\log_a u = -\infty$, that is $v = 0$, $u = 0$; (VII.) $v = \pm\infty$, $\log_a u = 0$, that is $v = \pm\infty$, $u = 1$. There thus arise the indeterminate forms—(V.) ∞^0, (VI.) 0^0, (VII.) $1^{\pm\infty}$.*

All these depend on $a^{0 \times \infty}$; or, if we choose, upon $a^{0/0}$; so that it may be said that there is really only one fundamental case of indetermination, namely, $0 \div 0$.

EXTENSION OF THE FUNDAMENTAL OPERATIONS TO LIMITING VALUES.

§ 6.] We now proceed to show that limiting values as above defined may, under some restrictions, be dealt with in algebraical

* The reader is already aware that 1^0 gives 1; and he may easily convince himself that $0^{+\infty}$, $0^{-\infty}$, $\infty^{+\infty}$, $\infty^{-\infty}$ give 0, $\pm\infty$, $\pm\infty$, 0 respectively, no matter what their origin.

operations exactly like ordinary operands. This is established by means of the following theorems :—

I. *The limit of a sum of functions of x is the sum of their limits, provided the latter does not take the indeterminate form $\infty - \infty$.*

Consider the sum $f(x) - \phi(x) + \chi(x)$ for the critical value $x = a$; and let $\mathrm{L}f(x) = f'$, $\mathrm{L}\phi(x) = \phi'$, $\mathrm{L}\chi(x) = \chi'$. Then, by § 2, Cor. 2,

$$f(x) = f' + \alpha, \quad \phi(x) = \phi' + \beta, \quad \chi(x) = \chi' + \gamma,$$

where α, β, γ can each be made as small as we please by bringing x sufficiently near to a.

Now, $$f(x) - \phi(x) + \chi(x) = f' - \phi' + \chi' + (\alpha - \beta + \gamma).$$

But, obviously, $\alpha - \beta + \gamma$ can be made as small as we please by bringing x sufficiently near to a. Hence

$$\mathrm{L}\{f(x) - \phi(x) + \chi(x)\} = f' - \phi' + \chi',$$

that is, $$= \mathrm{L}f(x) - \mathrm{L}\phi(x) + \mathrm{L}\chi(x) \quad (1).$$

This reasoning supposes f', ϕ', χ' to be each finite; but it is obvious that if one or more of them, all having the same sign, become infinite, then $f' - \phi' + \chi'$ and $\mathrm{L}\{f(x) - \phi(x) + \chi(x)\}$ are both infinite, and the theorem will still be true in the peculiar sense, at least, that both sides of the equality are infinite. If, however, some of the infinities have one sign and some the opposite, $f' - \phi' + \chi'$ ceases to be interpretable in any definite sense; and the proposition becomes meaningless.

II. *The limit of a product of functions of x is the product of their limits, provided the latter does not take the indeterminate form $0 \times \infty$.*

Using the same notation as before, we have

$$\begin{aligned} f(x)\,\phi(x)\,\chi(x) &= (f' + \alpha)(\phi' + \beta)(\chi' + \gamma) \\ &= f'\phi'\chi' + \Sigma\alpha\phi'\chi' + \Sigma\alpha\beta\chi' + \alpha\beta\gamma. \end{aligned}$$

Now, provided none of the limits f', ϕ', χ' be infinite, since α, β, γ can all be made as small as we please by bringing x sufficiently near to a, the same is true of $\Sigma\alpha\phi'\chi'$, $\Sigma\alpha\beta\chi'$, and $\alpha\beta\gamma$. Hence

$$\mathrm{L}f(x)\,\phi(x)\,\chi(x) = f'\phi'\chi' = \mathrm{L}f(x)\,\mathrm{L}\phi(x)\,\mathrm{L}\chi(x) \quad (2).$$

If one or more of the limits f', ϕ', χ' be infinite, provided none of the rest be zero, the two sides of (2) will still be equal in the

sense that both are infinite; but, if there occur at the same time a zero and an infinite value, then the right-hand side assumes the indeterminate form $0 \times \infty$; and the equation (2) ceases to have any meaning.

III. *The limit of the quotient of two functions of x is the quotient of their limits, provided the latter does not take one of the indeterminate forms $0/0$ or ∞/∞.* We have

$$\frac{f(x)}{\phi(x)} = \frac{f' + \alpha}{\phi' + \beta} = \frac{f'}{\phi'} + \frac{f' + \alpha}{\phi' + \beta} - \frac{f'}{\phi'} = \frac{f'}{\phi'} + \frac{\alpha\phi' - \beta f'}{\phi'(\phi' + \beta)}.$$

From this equation, reasoning as above, we see at once that, if neither f' nor ϕ' be infinite, and ϕ' be not zero,

$$\mathrm{L}\frac{f(x)}{\phi(x)} = \frac{f'}{\phi'} = \frac{\mathrm{L}f(x)}{\mathrm{L}\phi(x)}, \qquad (3).$$

It is further obvious that if $f' = \infty$, $\phi' \neq \infty$, both sides of (3) will be infinite; if $\phi' = \infty$, $f' \neq \infty$, both sides will be zero; and if $\phi' = 0$, $f' \neq 0$, both sides will be infinite. In all these cases, therefore, the theorem may be asserted in a definite sense. If, however, we have simultaneously $f' = 0$, $\phi' = 0$, the right hand of (3) takes the form $0/0$; if $f' = \infty$, $\phi' = \infty$, the form ∞/∞; and then the theorem becomes meaningless.

§ 7.] If the reader will compare the demonstrations of last paragraph with those of § 8, chap. xv., he will see that (except in the cases where infinities are involved) the conclusions rest merely on the continuity of the sum, product, and quotient. This remark immediately suggests the following general theorem, which includes those of last paragraph as particular cases:—

If $\mathrm{F}(u, v, w, \ldots)$ *be any function of* $u, v, w, \ldots$, *which is determinate, and finite in value, and also continuous when*

$$u = \mathrm{L}f(x), \quad v = \mathrm{L}\phi(x), \quad w = \mathrm{L}\chi(x), \ldots,$$

then

$$\mathrm{LF}\{f(x), \phi(x), \chi(x), \ldots\} = \mathrm{F}\{\mathrm{L}f(x), \mathrm{L}\phi(x), \mathrm{L}\chi(x), \ldots\}.$$

The reader will easily prove this theorem by combining § 2, Cor. 2, with the definition of a continuous function given in chap. xv., §§ 5, 14.

The most important case of this proposition which we shall have occasion to use is that where we have a function of a single function. For example,

$$\mathop{\mathrm{L}}_{x=1}\{(x^2-1)/(x-1)\}^2=\{\mathop{\mathrm{L}}_{x=1}(x^2-1)/(x-1)\}^2=4.$$

$$\mathop{\mathrm{L}}_{x=1}\log\{(x^2-1)/(x-1)\}=\log\{\mathop{\mathrm{L}}_{x=1}(x^2-1)/(x-1)\}=\log 2.$$

ON THE FORMS $0/0$ AND ∞/∞ IN CONNECTION WITH RATIONAL FUNCTIONS.

§ 8.] The form $0/0$ will occur with a rational function for the value $x=0$ if the absolute terms in the numerator and denominator vanish. The rule for evaluating in this case is to arrange the terms in the numerator and denominator in order of ascending degree, divide by the lowest power of x that occurs in numerator or denominator, and then put $x=0$. The limit will be finite, and $\neq 0$, if the lowest terms in numerator and denominator be of the same degree; 0 if the term of lowest degree come from the denominator; ∞ if the term of lowest degree come from the numerator. All this will be best seen from the following examples:—

Example 1.

$$\mathop{\mathrm{L}}_{x=0}\frac{2x^2+3x^3+x^4}{3x^3+x^4+x^6}=\mathop{\mathrm{L}}_{x=0}\frac{2+3x+x^2}{3+x^2+x^4}=\frac{2}{3}.$$

Example 2.

$$\mathop{\mathrm{L}}_{x=0}\frac{2x^3+3x^4+x^5}{3x^2+x^4+x^6}=\mathop{\mathrm{L}}_{x=0}\frac{2x+3x^2+x^3}{3+x^2+x^4}=\frac{0}{3}=0.$$

Example 3.

$$\mathop{\mathrm{L}}_{x=0}\frac{2x^4+x^6}{x^6+x^8}=\mathop{\mathrm{L}}_{x=0}\frac{2+x^2}{x^2+x^4}=\frac{2}{0}=\infty.$$

§ 9.] The form ∞/∞ can arise from a rational function when, and only when, $x=\infty$. The limit can be found by dividing numerator and denominator by the highest power of x that occurs in either. If this highest power occur in both, the limit is finite; if it come from the denominator alone, the limit is 0; if from the numerator alone, the limit is ∞.

Example 1.

$$\mathop{\mathrm{L}}_{x=\infty}\frac{3x^3+x^4}{2x^2+x^3+3x^4}=\mathop{\mathrm{L}}_{x=\infty}\frac{3/x+1}{2/x^2+1/x+3}=\frac{0+1}{0+0+3}=\frac{1}{3}.$$

Example 2.

$$\mathop{\mathrm{L}}_{x=\infty} \frac{x^2+3x^3+4x^4}{2x+x^3+6x^5} = \mathop{\mathrm{L}}_{x=\infty} \frac{1/x^4+3/x^3+4/x^2}{2/x^5+1/x^3+6} = \frac{0}{6} = 0.$$

Example 3.

$$\mathop{\mathrm{L}}_{x=\infty} \frac{x^2+3x^3+4x^5}{2x+3x^2+x^3} = \mathop{\mathrm{L}}_{x=\infty} \frac{1/x^4+3/x^3+4}{2/x^5+3/x^4+1/x^3} = \frac{4}{0} = \infty.$$

§ 10.] If the rational function $f(x)/\phi(x)$ take the form $0/0$ for a finite value of x, $\neq 0$, say for $x = a$, then, since $f(a) = 0$, $\phi(a) = 0$, it follows from the remainder-theorem that $x - a$ is a common factor in $f(x)$ and $\phi(x)$. If we transform the function by removing this factor, the result of putting $x = a$ in the transformed function will in general be determinate; if not, it must be of the form $0/0$, and $x - a$ will again be a common factor, and must be removed. By proceeding in this way, we shall obviously in the end arrive at a determinate value, which will be the limit of $f(x)/\phi(x)$ when $x = a$.

Example. Evaluate $(3x^4 - 10x^3 + 3x^2 + 12x - 4)/(x^4 + 2x^3 - 22x^2 + 32x - 8)$ when $x = 2$. The value is, in the first instance, indeterminate, and of the form $0/0$; hence $x - 2$ is a common factor. If we divide out this factor, we find that the value is still of the form $0/0$; hence we must divide again. We then have a determinate result. The work may be arranged thus (see chap. v., § 13):—

	3 − 10 + 3 + 12 − 4		1 + 2 − 22 + 32 − 8
2	0 + 6 − 8 − 10 + 4	2	0 + 2 + 8 − 28 + 8
	3 − 4 − 5 + 2 | + 0		1 + 4 − 14 + 4 | + 0
	0 + 6 + 4 − 2 |		0 + 2 + 12 − 4 |
	3 + 2 − 1 | + 0		1 + 6 − 2 + 0
	0 + 6 + 16 |		0 + 2 + 16
	3 + 8 | + 15		1 + 8 | + 14

The process of division is to be continued until we have two remainders which are not both zero. The quotient of these, 15/14 in the present case, is the limit required.

The evaluation of the limit in the present case may also be effected by *changing the variable*, an artifice which is frequently of use in the theory of limits. If we put $x = a + z$, then we have to evaluate $\mathrm{L} f(a + z)/\phi(a + z)$ when $z = 0$. Since $f(a + z)$ and $\phi(a + z)$ are obviously integral functions of z, we can now apply the rule of § 8. It will save trouble in applying this method if it be remembered—1st, that in arranging $f(a + z)$ and $\phi(a + z)$ according to powers of z we need not calculate the absolute

terms, since they must, if the form to be evaluated be 0/0, be zero in each case; 2nd, that we are only concerned with the lowest powers of z that occur in the numerator and denominator respectively.

$$\underset{x=2}{\mathrm{L}} \frac{3x^4-10x^3+3x^2+12x-4}{x^4+2x^3-22x^2+32x-8} = \underset{z=0}{\mathrm{L}} \frac{3(2+z)^4-10(2+z)^3+3(2+z)^2+12(2+z)-4}{(2+z)^4+2(2+z)^3-22(2+z)^2+32(2+z)-8}$$

$$= \underset{z=0}{\mathrm{L}} \frac{15z^3+Pz^3+\&c.}{14z^3+Qz^3+\&c.},$$

$$= \underset{z=0}{\mathrm{L}} \frac{15+Pz+\&c.}{14+Qz+\&c.},$$

$$= \frac{15}{14}.$$

This method is of course at bottom identical with the former; for, since $z=x-a$, the division by z^3 corresponds to the rejection of the factor $(x-a)^3$.

§ 11.] The methods which are applicable to the quotient of two integral functions apply to the quotient of two algebraic sums of constant multiples of fractional powers of x. Each of the two sums might, in fact, be transformed into an integral function of y by putting $x=y^d$, where d is the L.C.M. of the denominators of all the fractional indices. It is, however, in general simpler to operate directly.

Example. Evaluate

$$l = \underset{x=0}{\mathrm{L}} \frac{x^{\frac{1}{2}}+x^{\frac{3}{4}}+3x^{\frac{3}{2}}}{x^{\frac{1}{2}}+2x^{\frac{1}{4}}+x}.$$

If we divide by $x^{\frac{1}{2}}$, the lowest power of x that occurs, we have

$$l = \underset{x=0}{\mathrm{L}} \frac{x^{0}+x^{\frac{1}{4}}+3x^{\frac{1}{2}}}{1+2x^{\frac{1}{4}}+x^{\frac{1}{2}}},$$

$$= \frac{0}{1} = 0.$$

§ 12.] The following theorem, although partly a special case under the present head, is of great importance, because it gives the fundamental limit on which depends the "differentiation" of algebraic functions:—

If m be any real commensurable quantity, positive or negative,

$$\underset{x=1}{\mathrm{L}} (x^m-1)/(x-1) = m \qquad (1).$$

First, let m be a positive integer. Then we have

$$(x^m - 1)/(x - 1) \equiv x^{m-1} + x^{m-2} + \ldots + x + 1.$$

Hence

$$\mathop{\mathrm{L}}_{x=1} (x^m - 1)/(x - 1) = 1 + 1 + \ldots + 1 + 1 \ (m \text{ terms}),$$
$$= m.$$

Next, let m be a positive fraction, say p/q, where p and q are positive integers. Then the limit to be evaluated is $\mathop{\mathrm{L}}_{x=1} (x^{p/q} - 1)/(x - 1)$.* If we put $x = z^q$, and observe that to $x = 1$ corresponds $z = 1$, the limit to be evaluated becomes $\mathop{\mathrm{L}}_{z=1} (z^p - 1)/(z^q - 1)$. This may be evaluated by removing the common factor $z - 1$; or thus

$$\mathop{\mathrm{L}}_{z=1} (z^p - 1)/(z^q - 1) = \mathop{\mathrm{L}}_{z=1} \left(\frac{z^p - 1}{z - 1}\right) \Big/ \left(\frac{z^q - 1}{z - 1}\right),$$
$$= \mathop{\mathrm{L}}_{z=1} \left(\frac{z^p - 1}{z - 1}\right) \Big/ \mathop{\mathrm{L}}_{z=1} \left(\frac{z^q - 1}{z - 1}\right),$$
$$= p/q = m.$$

Finally, suppose m to have any negative value, say $-n$, where n is positive. Then

$$\mathop{\mathrm{L}}_{x=1} (x^{-n} - 1)/(x - 1) = \mathop{\mathrm{L}}_{x=1} (1 - x^n)/x^n(x - 1),$$
$$= -\mathop{\mathrm{L}}_{x=1} (x^n - 1)/(x - 1)x^n,$$
$$= -\{\mathop{\mathrm{L}}_{x=1} (x^n - 1)/(x - 1)\} \times \mathop{\mathrm{L}}_{x=1} 1/x^n.$$

Now, by the last two cases, since n is positive, $\mathop{\mathrm{L}}_{x=1} (x^n - 1)/(x - 1) = n$. Also $\mathop{\mathrm{L}}_{x=1} 1/x^n = 1$. Hence

$$\mathop{\mathrm{L}}_{x=1} (x^{-n} - 1)/(x - 1) = -n ;$$

that is, in this case also,

$$\mathop{\mathrm{L}}_{x=1} (x^m - 1)/(x - 1) = m.$$

Second Demonstration.—The above theorem might also be deduced at once from the inequality of chap. xxiv., § 7, as follows :—For all positive values of x, and all positive or negative values of m, $x^m - 1$ lies between $mx^{m-1}(x - 1)$ and $m(x - 1)$. Hence $(x^m - 1)/(x - 1)$ lies between mx^{m-1} and m. Now, by

* There is here of course the usual understanding (see chap. x., § 2) as to the meaning of $x^{p/q}$.

bringing x sufficiently near to 1, mx^{m-1} can be made to differ as little from m as we please. The same is therefore true of $(x^m-1)/(x-1)$; that is to say,

$$\mathrm{L}(x^m-1)/(x-1)=m$$

for all real values of m.

Example 1. Find the limit of $(x^p-a^p)/(x^q-a^q)$ when $x=a$. We have

$$\mathop{\mathrm{L}}_{x=a}(x^p-a^p)/(x^q-a^q)=\mathop{\mathrm{L}}_{x=a}a^{p-q}\{(x/a)^p-1\}/\{(x/a)^q-1\},$$

$$=a^{p-q}\mathop{\mathrm{L}}_{y=1}\left(\frac{y^p-1}{y-1}\right)\Big/\left(\frac{y^q-1}{y-1}\right),$$

where $y=x/a$. Hence we have, by the theorem of the present paragraph

$$\mathop{\mathrm{L}}_{x=a}(x^p-a^p)/(x^q-a^q)=a^{p-q}p/q.$$

Example 2. Evaluate $\log(x^{\frac{3}{2}}-1)-\log(x^{\frac{1}{2}}-1)$ when $x=1$.

$$\mathrm{L}\{\log(x^{\frac{3}{2}}-1)-\log(x^{\frac{1}{2}}-1)\}=\mathrm{L}\log\{(x^{\frac{3}{2}}-1)/(x^{\frac{1}{2}}-1)\},$$

$$=\log\{\mathrm{L}(x^{\frac{3}{2}}-1)/(x^{\frac{1}{2}}-1)\},\qquad\text{by § 7,}$$

$$=\log\left\{\mathrm{L}\left(\frac{x^{\frac{3}{2}}-1}{x-1}\right)\Big/\mathrm{L}\left(\frac{x^{\frac{1}{2}}-1}{x-1}\right)\right\},$$

$$=\log\{\tfrac{3}{2}/\tfrac{1}{2}\},$$

$$=\log 3.$$

Example 3. If lx, l^2x, . . . denote $\log x$, $\log(\log x)$, . . . respectively, then, when $x=\infty$, $\mathrm{L}l^rx/l^rx=1$.

In the first place, we have

$$l(x+1)/lx=\{l(x+1)-lx+lx\}/lx,$$

$$=l(1+1/x)/lx+1.$$

Now, when $x=\infty$, $l(1+1/x)=l1=0$ and $lx=\infty$. Hence $\mathrm{L}l(x+1)/lx=1$.

If we assume that $\mathrm{L}l^r(x+1)/l^rx=1$, we have

$$l^{r+1}(x+1)/l^{r+1}x=\{l^{r+1}(x+1)-l^{r+1}x+l^{r+1}x\}/l^{r+1}x,$$

$$=l\{l^r(x+1)/l^rx\}/l^{r+1}x+1.$$

Hence

$$\mathrm{L}l^{r+1}(x+1)/l^{r+1}x=l1/\infty+1,$$

$$=1;$$

that is, the theorem holds for $r+1$ if it holds for r. But it holds for $r=1$, as we have seen, therefore for $r=2$, &c. It is obvious that this theorem holds for any logarithmic base for which $l\infty=\infty$.

Example 4. If l have the same meaning as before, and λ have a similar meaning for the base a, then

$$\mathop{\mathrm{L}}_{x=\infty}\lambda^rx/l^rx=1/\log a.$$

Let $\mu=1/\log a$. Since $\lambda x=\mu lx$, the theorem clearly holds when $r=1$. It is therefore sufficient to show that, if it is true for r, it is true for $r+1$. Now

$$\lambda^{r+1}x/l^{r+1}x=\lambda(\lambda^rx)/l^{r+1}x,$$

$$=\mu l(\lambda^rx)/l^{r+1}x,$$

$$=\mu\{l(\lambda^rx)-l^{r+1}x+l^{r+1}x\}/l^{r+1}x,$$

$$=\mu\{l(\lambda^rx/l^rx)/l^{r+1}x+1\},$$

Hence, if we assume that $\mathrm{L}\lambda^rx/l^rx=\mu$, we have

$$\mathrm{L}\lambda^{r+1}x/l^{r+1}x=\mu\{l\mu/\infty+1\},$$

$$=\mu.$$

EXPONENTIAL LIMITS.

§ 13.] The most important theorem in this part of the subject is the following, on which is founded the differentiation of exponential functions generally :—

The limit of $(1+1/x)^x$ *when* x *is increased without limit either positively or negatively is a finite number* (*denoted by* e) *lying between* 2 *and* 3.

The following proof is due to Fort.*

We have seen (chap. xxiv., § 7) that, if a and b be positive quantities, and m any positive quantity numerically greater than 1, then

$$ma^{m-1}(a-b) > a^m - b^m > mb^{m-1}(a-b) \qquad (1).$$

In this inequality we may put $a = (y+1)/y$, $b = 1$, $m = y/x$, where $y > x > 1$. We thus have

$$\left(\frac{y+1}{y}\right)^{y/x} - 1 > \frac{1}{x}.$$

Hence
$$\left(1+\frac{1}{y}\right)^{y/x} > 1 + \frac{1}{x},$$

that is,
$$\left(1+\frac{1}{y}\right)^{y} > \left(1+\frac{1}{x}\right)^{x} \qquad (2),$$

where $y > x$.

Again, if in (1) we put $a = 1$, $b = (y-1)/y$, (m, y, x being as before), we have

$$\frac{1}{x} > 1 - \left(\frac{y-1}{y}\right)^{y/x}.$$

Hence
$$\left(1-\frac{1}{y}\right)^{y/x} > 1 - \frac{1}{x},$$

$$\left(1-\frac{1}{y}\right)^{y} > \left(1-\frac{1}{x}\right)^{x},$$

and therefore
$$\left(1-\frac{1}{y}\right)^{-y} < \left(1-\frac{1}{x}\right)^{-x} \qquad (3),$$

where $y > x$.

We see from (2) and (3) that, if we give a series of in-

* *Zeitschrift für Mathematik*, vii., p. 46 (1862).

creasing positive values to x, the function $(1+1/x)^x$ continually increases, and the function $(1-1/x)^{-x}$ continually decreases. Moreover, since $x^3 > x^3 - 1$, we have

$$\frac{x}{x-1} > \frac{x+1}{x},$$

that is,

$$\left(1-\frac{1}{x}\right)^{-1} > 1+\frac{1}{x}.$$

Hence

$$\left(1-\frac{1}{x}\right)^{-x} > \left(1+\frac{1}{x}\right)^{x} \qquad (4).$$

The values of $(1-1/x)^{-x}$ and $(1+1/x)^x$ cannot, therefore, pass each other. Hence, when x is increased without limit, $(1-1/x)^{-x}$ must diminish down to a finite limit A, and $(1+1/x)^x$ must increase up to a finite limit B. The two limits A and B must be equal, for the difference $(1-1/x)^{-x} - (1+1/x)^x$ may be written $\{x/(x-1)\}^x - \{(x+1)/x\}^x$; and by (1) we have

$$\frac{1}{x}\left(\frac{x}{x-1}\right)^x > \left(\frac{x}{x-1}\right)^x - \left(\frac{x+1}{x}\right)^x > \frac{1}{x(1-1/x^2)}\left(\frac{x+1}{x}\right)^x \qquad (5).$$

But, since, as has already been shown, $\{x/(x-1)\}^x$ and $\{(x+1)/x\}^x$ remain finite when $x=\infty$, the upper and lower limits in (5) approach zero when x is increased without limit; the same is therefore true of the middle term of the inequality.

It has therefore been shown that $\mathrm{L}_{x=\infty}(1+1/x)^x$ and $\mathrm{L}_{x=\infty}(1-1/x)^{-x}$ have a common finite limit, which we may denote by the letter e.

Since $(1+1/6)^6 = 2{\cdot}521\ldots$ and $(1-1/6)^{-6} = 2{\cdot}985\ldots$, e lies between 2·5 and 2·9. A closer approximation might be obtained by using a larger value of x; but a better method of calculating this important constant will be given hereafter, by which it is found that

$$e = 2{\cdot}7182818285\ldots$$

The constant e is usually called Napier's Base *; and it is the logarithmic or exponential base used in most analytical calculations. In future, when no base is indicated, and mere arith-

* In honour of Napier, and not because he explicitly used this or indeed any other base.

metical computations are not in question, the base of a logarithmic or exponential function is understood to be e; thus $\log x$ and $\exp x$ are in general understood to mean $\log_e x$ and $\exp_e x$ (that is, e^x) respectively.

Cor. 1. $\underset{x=0}{\mathrm{L}}\,(1+x)^{1/x} = e.$

For $\underset{z=\infty}{\mathrm{L}}\,(1+1/z)^z = e$; and if we put $z = 1/x$, so that $x = 0$ corresponds to $z = \infty$, we have $\underset{x=0}{\mathrm{L}}\,(1+x)^{1/x} = e.$

Cor. 2. $\underset{z=\infty}{\mathrm{L}} \log_a \{(1+1/x)^x\} = \underset{x=0}{\mathrm{L}} \log_a \{(1+x)^{1/x}\} = \log_a e.$

For, since $\log_a y$ is a continuous function of y for finite values of y, we have, by § 7,

$$\underset{z=\infty}{\mathrm{L}} \log_a \{(1+1/x)^x\} = \log_a \{\underset{z=\infty}{\mathrm{L}}\,(1+1/x)^x\},$$
$$= \log_a e.$$

The other part of the corollary follows in like manner.

Cor. 3. $\underset{z=\infty}{\mathrm{L}}\,(1+y/x)^x = \underset{x=0}{\mathrm{L}}\,(1+xy)^{1/x} = e^y.$

If we put $1/z = y/x$, then to $x = \infty$ corresponds $z = \infty$; hence

$$\underset{x=\infty}{\mathrm{L}}\,(1+y/x)^x = \underset{z=\infty}{\mathrm{L}}\,(1+1/z)^{zy},$$
$$= \underset{z=\infty}{\mathrm{L}}\,\{(1+1/z)^z\}^y,$$
$$= \{\underset{z=\infty}{\mathrm{L}}\,(1+1/z)^z\}^y, \qquad \text{by § 7,}$$
$$= e^y.$$

Cor. 4. $\underset{x=0}{\mathrm{L}}\,(a^x - 1)/x = \log a.$

If we put $y = a^x - 1$, so that $x = \log_a(1+y)$, and to $x = 0$ corresponds $y = 0$, we have

$$\underset{x=0}{\mathrm{L}}\,(a^x - 1)/x = \underset{y=0}{\mathrm{L}}\, y/\log_a(1+y),$$
$$= \underset{y=0}{\mathrm{L}}\, 1/\log_a(1+y)^{1/y},$$
$$= 1/\log_a \{\underset{y=0}{\mathrm{L}}\,(1+y)^{1/y}\},$$
$$= 1/\log_a e = \log a.$$

It will be an excellent exercise for the student to deduce directly from the fundamental inequality (1) above, the important result that $\underset{x=0}{\mathrm{L}}\,(a^x - 1)/x$ is

finite; and thence, by transformation, to prove the leading theorem of this paragraph.*

Cor. 5. *If x be any positive quantity,*

$$e^x > 1 + x, \qquad \log(1 + x) < x;$$

and, if x be positive and less than 1,

$$e^{-x} > 1 - x, \quad -\log(1 - x) > x.$$

Since $e > (1 + 1/n)^n$, when n may be as great as we please,

$$e^x - 1 > (1 + 1/n)^{nx} - 1,$$
$$> nx\{(1 + 1/n) - 1\} > x, \qquad \text{by chap. xxiv., § 7,}$$

for, however small x, we can by sufficiently increasing n make $nx > 1$.

Hence $$e^x > 1 + x.$$

It follows at once that $\log e^x > \log(1 + x)$, that is, $x > \log(1 + x)$.

Again, since $e < (1 - 1/n)^{-n}$ and $e^{-1} > (1 - 1/n)^n$,

$$e^{-x} - 1 > \{(n - 1)/n\}^{nx} - 1,$$
$$> nx\{(n - 1)/n - 1\},$$
$$> -x.$$

Hence $e^{-x} > 1 - x$, and therefore $1/(1 - x) > e^x$.

It follows at once that $\log\{1/(1 - x)\}$, that is, $-\log(1 - x) > x$.

Cor. 6.† *If lx, l^2x, . . . denote $\log x$, $\log(\log x)$, . . . respectively, x be positive and* > 1, *and r any positive integer, then*

$$1/xlxl^2x \ldots l^rx > l^{r+1}(x + 1) - l^{r+1}x > 1/(x + 1)l(x + 1)l^2(x + 1) \ldots$$
$$\ldots l^r(x + 1).$$

For, by Cor. 5, $$l(x + 1) - lx = l(1 + 1/x),$$
$$< 1/x.$$

This proves the first inequality when $r = 0$. It remains to show that, if the inequality holds for r, it holds also for $r + 1$. We have

$$l^{r+2}(x + 1) - l^{r+2}x = l\{l^{r+1}(x + 1)/l^{r+1}x\},$$
$$= l[1 + \{l^{r+1}(x + 1) - l^{r+1}x\}/l^{r+1}x],$$
$$< \{l^{r+1}(x + 1) - l^{r+1}x\}/l^{r+1}x, \qquad \text{by Cor. 5.}$$

Hence, if we assume $l^{r+1}(x + 1) - l^{r+1}x < 1/xlx \ldots l^rx$, it follows that

$$l^{r+2}(x + 1) - l^{r+2}x < 1/xlx \ldots l^rxl^{r+1}x.$$

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* See Schlömilch, *Zeitschrift für Mathematik*, vol. iii., p. 387 (1858).

† Malmsten, *Grunert's Archiv.*, viii. (1846).

Again, by Cor. 5, we have

$$lx - l(x-1) = -l(1-1/x),$$
$$> 1/x.$$

Therefore $$l(x+1) - lx > 1/(x+1).$$

This proves the second inequality when $r = 0$. If we suppose it to hold for r, we have

$$\begin{aligned} l^{r+2}(x+1) - l^{r+2}x &= -l[l^{r+1}x/l^{r+1}(x+1)], \\ &= -l[1 - \{l^{r+1}(x+1) - l^{r+1}x\}/l^{r+1}(x+1)], \\ &> \{l^{r+1}(x+1) - l^{r+1}x\}/l^{r+1}(x+1), \quad \text{by Cor. 5,} \\ &> 1/(x+1)l(x+1) \ . \ . \ . \ l^r(x+1)l^{r+1}(x+1). \end{aligned}$$

Hence the induction is complete.

Cor. 7. From the inequality of Cor. 5, combined with the result of Example 3, § 12, we deduce at once the following important limits :—

$$\lim_{x=\infty} \{l^r(x+1) - l^r x\} = 0,$$

$$\lim_{x=\infty} \{l^{r+1}(x+1) - l^{r+1}x\}\, xlxl^2x \ . \ . \ . \ l^r x = 1.$$

Example 1. Show that the limit when n is infinite of $1+1/2+\ldots+1/n-\log n$ is a finite quantity, usually denoted by γ, lying between 0 and 1. (Euler, *Comm. Ac. Pet.* (1734-5).)

Since, by Cor. 5,

$$-\log(1-1/n) > 1/n \qquad > \log(1+1/n).$$

We have

$$\log\{n/(n-1)\} > 1/n \qquad > \log\{(n+1)/n\},$$
$$\log\{(n-1)/(n-2)\} > 1/(n-1) > \log\{n/(n-1)\},$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\log\{3/2\} > 1/3 > \log\{4/3\},$$
$$\log\{2/1\} > 1/2 > \log\{3/2\},$$
$$1 = 1 > \log\{2/1\}.$$

Hence $$1 + \log n > \Sigma 1/n > \log(n+1).$$

Therefore $$1 > \Sigma 1/n - \log n > \log(1+1/n).$$

Now, when $n = \infty$, $\log(1+1/n) = 0$. Thus, for all values of n, however great, $\Sigma 1/n - \log n$ lies between 0 and 1.

The important constant γ was first introduced into analysis by Euler, and is therefore usually called Euler's Constant. Its value was given by Euler himself to 16 places, namely, $\gamma = \cdot 577215664901532(5)$. (See *Inst. Calc. Diff.*, chap. vi.)*

* Euler's Constant was calculated to 32 places by Mascheroni in his *Adnotationes ad Euleri Calculum Integralem.* It is therefore sometimes called Mascheroni's Constant. His calculation, which was erroneous in the 20th place, was verified and corrected by Gauss and Nicolai. See Gauss, *Werke*, Bd. iii., p. 154. For an interesting historical account of the whole matter, see Glaisher, *Mess. Math.*, vol. i. (1872).

Example 2. Show that $\underset{n=\infty}{\mathrm{L}} \{1/1+1/2+\ldots+1/n\}/\log n = 1$.

This follows at once from the inequality of last example.

From this result, or from Example 1, we see that $\underset{n=\infty}{\mathrm{L}} \{1/1+1/2+\ldots+1/n\} = \infty$; and also that $\underset{n=\infty}{\mathrm{L}} \{1/k+1/(k+1)+\ldots+1/n\} = \infty$, where k is any finite positive integer.

GENERAL THEOREMS.

§ 14.] Before proceeding further with the theory of the limits of exponential forms, it will be convenient to introduce a few general theorems, chiefly due to Cauchy. Although these theorems are not indispensable in an elementary treatment of limits, the student will find that occasional reference to them will tend to introduce brevity and coherence into the subject.

I. *For any critical value of* x, $\mathrm{L}\{f(x)\}^{\phi(x)} = \{\mathrm{L}f(x)\}^{\mathrm{L}\phi(x)}$, *provided the latter form be not indeterminate.*

This is in reality a particular case of the general theorem of § 7. The only question that arises is as to the continuity of the functions of the limits. We may write

$$\{f(x)\}^{\phi(x)} = e^{\phi(x)\log f(x)}.$$

Now $w = \log u$ is a continuous function of u, so long, at least, as u lies between $+1$ and $+\infty$; and e^{vw} is a continuous function of v and w. Hence, so long as $\mathrm{L}\phi(x)$ and $\mathrm{L}\log f(x)$ are neither of them infinite, we have

$$\begin{aligned} \mathrm{L}\{f(x)\}^{\phi(x)} &= \mathrm{L}e^{\phi(x)\log f(x)}, \\ &= e^{\mathrm{L}\phi(x)\mathrm{L}\log f(x)}, \\ &= e^{\mathrm{L}\phi(x)\log \mathrm{L}f(x)}. \end{aligned}$$

Hence
$$\mathrm{L}\{f(x)\}^{\phi(x)} = \{\mathrm{L}f(x)\}^{\mathrm{L}\phi(x)} \quad . \qquad (1).$$

An examination of the special cases where either $\mathrm{L}\phi(x)$ or $\mathrm{L}\log f(x)$, or both, become infinite, shows that, so long as $\{\mathrm{L}f(x)\}^{\mathrm{L}\phi(x)}$ does not assume one of the indeterminate forms 0^0, ∞^0, $1^{\pm\infty}$, both sides of (1) become 0, or both ∞; so that the theorem may be stated as true for all cases where its sense is determinate.

II. $\underset{x=\infty}{\mathrm{L}}\{f(x+1)-f(x)\} = \underset{x=\infty}{\mathrm{L}}\, f(x)/x$, *provided* $\underset{x=\infty}{\mathrm{L}}\{f(x+1)-f(x)\}$ *be not indeterminate.** (Cauchy's Theorem.)

Since x is ultimately to be made as large as we please, we may put $x = h+n$, where h is a number not necessarily an integer, but as large as we please, and n is an integer as large as we please.

First, suppose that $\mathrm{L}\{f(x+1)-f(x)\}$ is not infinite, $= k$ say.

Since $\mathrm{L}\{f(x+1)-f(x)\} = k$, we can always choose for h a definite value, so large that for $x = h$ and all greater values $f(x+1)-f(x)-k$ is numerically less than a given quantity α, no matter how small α may be. Hence we have numerically

$$\begin{aligned} f(h+1)-f(h)-k &< \alpha, \\ f(h+2)-f(h+1)-k &< \alpha, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ f(h+n)-f(h+n-1)-k &< \alpha; \end{aligned}$$

and, by addition, $f(h+n)-f(h)-nk < n\alpha$;

that is, $f(x)-f(h)-(x-h)k < (x-h)\alpha$.

Hence $$\frac{f(x)}{x}-\frac{f(h)}{x}-\left(1-\frac{h}{x}\right)k < \left(1-\frac{h}{x}\right)\alpha,$$

$$\frac{f(x)}{x}-k < \alpha+\frac{f(h)}{x}-\frac{h(k+\alpha)}{x}.$$

Since $f(h)$, h, k, and α are, for the present, fixed, it results that, by making x sufficiently large, we can make $f(x)/x-k$ numerically less than α. Now α can be made as small as we please by properly choosing h; hence the theorem follows.

Next, suppose that $\mathrm{L}\{f(x+1)-f(x)\} = +\infty$; then, by taking h sufficiently large, we can assume that

$$\begin{aligned} f(h+1)-f(h) &> l, \\ f(h+2)-f(h+1) &> l, \\ \cdots\cdots\cdots\cdots & \\ f(h+n)-f(h+n-1) &> l, \end{aligned}$$

where l is a definite quantity as large as we please.

* Theorems II. and III. are given by Cauchy in his *Analyse Algébrique* (which is Part I. of his *Cours d'Analyse de l'École Royale Polytechnique*). Paris, 1821.

Hence $f(h+n)-f(h)>nl,$

that is $f(x)-f(h)>(x-h)l.$

Hence $$\frac{f(x)}{x}>l+\frac{f(h)}{x}-\frac{hl}{x}.$$

Since $f(h)$, h, l are all definite, we can, by sufficiently increasing x, render $f(h)/x - hl/x$ as small as we please, therefore $f(x)/x>l$. Now, by properly choosing h, l can be made as large as we please; hence $\mathrm{L}f(x)/x = \infty$.

The case where $\mathrm{L}\{f(x+1)-f(x)\} = -\infty$ can be included in the last by observing that $(-f(x+1))-(-f(x))$ has in this case $+\infty$ for its limiting value.

III. $\underset{x=\infty}{\mathrm{L}} f(x+1)/f(x) = \underset{x=\infty}{\mathrm{L}} \{f(x)\}^{1/x}$, *provided* $\underset{x=\infty}{\mathrm{L}} f(x+1)/f(x)$ *be not indeterminate.*

This theorem can be deduced from the last by transformation, as follows:—*

We have $$\underset{x=\infty}{\mathrm{L}} \{\psi(x+1)-\psi(x)\} = \underset{x=\infty}{\mathrm{L}} \frac{\psi(x)}{x},$$

where $\psi(x)$ is any function such that $\underset{x=\infty}{\mathrm{L}} \{\psi(x+1)-\psi(x)\}$ is not indeterminate. Let now $\psi(x) = \log f(x)$; so that $\psi(x+1) - \psi(x) = \log f(x+1) - \log f(x) = \log \{f(x+1)/f(x)\}$; and $\psi(x)/x = \{\log f(x)\}/x = \log \{f(x)\}^{1/x}$. Then we have

$$\underset{x=\infty}{\mathrm{L}} \log \left\{\frac{f(x+1)}{f(x)}\right\} = \underset{x=\infty}{\mathrm{L}} \log \{f(x)\}^{1/x},$$

Hence $$\log \left\{\underset{x=\infty}{\mathrm{L}} \frac{f(x+1)}{f(x)}\right\} = \log \left[\underset{x=\infty}{\mathrm{L}} \{f(x)\}^{1/x}\right],$$

provided $\mathrm{L}f(x+1)/f(x)$ be not indeterminate. Hence, finally,

$$\underset{x=\infty}{\mathrm{L}} \frac{f(x+1)}{f(x)} = \underset{x=\infty}{\mathrm{L}} \{f(x)\}^{1/x}.$$

Cauchy makes the important remark that the demonstrations of his two theorems evidently apply to functions of an integral variable such as $x!$, where only positive integral values of x are admissible.

* The reader will find it a good exercise to establish this theorem directly from first principles, as Cauchy does.

For example, we have $\underset{x=\infty}{\mathrm{L}}\,(x+1)!/x! = \underset{x=\infty}{\mathrm{L}}\,(x+1) = \infty$. Hence $\underset{x=\infty}{\mathrm{L}}\,(x!)^{1/x} = \infty$, and consequently $\underset{x=\infty}{\mathrm{L}}\,(1/x!)^{1/x} = 0$.

EXPONENTIAL LIMITS RESUMED.

§ 15.] *If* $a>1$, *then* $\underset{x=\infty}{\mathrm{L}}\, a^x/x = \infty$; $\underset{x=\infty}{\mathrm{L}}\log_a x/x = 0$; $\underset{x=+0}{\mathrm{L}}\, x\log_a x = 0$.

The first of these follows at once from Cauchy's Theorem (§ 14, II.) for we have

$$\mathrm{L}(a^{x+1} - a^x) = \mathrm{L}a^x(a-1) = \infty.$$

Hence $$\mathrm{L}a^x/x = \infty.$$

As the theorem is fundamental, it may be well to give an independent proof from first principles.

First, we observe that it is sufficient to prove it for integral values of x alone, for, however large x may be, we can always put $x = f + z$ where f is a positive proper fraction and z a positive integer. Then we have

$$\begin{aligned}
\underset{x=\infty}{\mathrm{L}}\,\frac{a^x}{x} &= \underset{z=\infty}{\mathrm{L}}\,\frac{a^{f+z}}{f+z},\\
&= \underset{z=\infty}{\mathrm{L}}\, a^f \cdot \frac{z}{f+z}\cdot\frac{a^z}{z},\\
&= a^f \underset{z=\infty}{\mathrm{L}}\,\frac{1}{f/z+1}\,\underset{z=\infty}{\mathrm{L}}\,\frac{a^z}{z},\\
&= a^f \underset{z=\infty}{\mathrm{L}}\,\frac{a^z}{z}, \qquad (1),
\end{aligned}$$

where we have to deal merely with $\mathrm{L}a^z/z$, z being a positive integer.

Let $u_z = a^z/z$, then $u_{z+1}/u_z = az/(z+1) = a/(1+1/z)$. Now, since $\underset{z=\infty}{\mathrm{L}}\, a/(1+1/z) = a > 1$, we can always assign an integral value of z, say $z = r$, such that, for that and all greater values of z, $u_{z+1}/u_z > b$, where $b>1$. We therefore have

$$\begin{aligned}
u_r &= u_r,\\
u_{r+1}/u_r &> b,\\
u_{r+2}/u_{r+1} &> b,\\
&\cdots\\
u_z/u_{z-1} &> b.
\end{aligned}$$

Hence, by multiplying all these inequalities together, we deduce

$$u_z > b^{z-r} u_r > b^z u_r / b^r.$$

Now u_r/b^r is finite, and, since $b > 1$, b^z can be made as great as we please by sufficiently increasing z. Hence $\mathrm{L}_{z=\infty} u_z = \infty$, on the supposition that z is always integral. But, since a^f is finite, it follows at once from (1) that $\mathrm{L}_{x=\infty} a^x/x = \infty$, when x is unrestricted.

The latter parts of the theorem follow by transformation.

If we put $a^x = y$, so that $x = \log_a y$, and to $x = \infty$ corresponds $y = \infty$, we have

$$\infty = \mathrm{L}_{x=\infty} a^x/x = \mathrm{L}_{y=\infty} y/\log_a y.$$

Hence

$$\mathrm{L}_{y=\infty} \log_a y/y = 1/\infty = 0.$$

If we put $a^x = 1/y$, so that $x = -\log_a y$, and to $x = \infty$ corresponds $y = 0$, we have

$$\infty = \mathrm{L}_{x=\infty} a^x/x = -\mathrm{L}_{y=+0} 1/y \log_a y.$$

Hence

$$\mathrm{L}_{y=+0} y \log_a y = -1/\infty = 0.$$

Example 1. Show that, if $a > 1$ and n be positive, then $\mathrm{L}_{x=\infty} a^x/x^n = \infty$; $\mathrm{L}_{x=\infty} \log_a x/x^n = 0$; $\mathrm{L}_{x=+0} x^n \log_a x = 0$.

$$\mathrm{L}_{x=\infty} a^x/x^n = \mathrm{L}_{x=\infty} \{a^{x/n}/x\}^n,$$
$$= \{\mathrm{L}_{x=\infty} (a^{1/n})^x/x\}^n,$$
$$= \infty^n = \infty;$$

for, since $a > 1$ and n is positive, we have $a^{1/n} > 1$, so that $\mathrm{L}(a^{1/n})^x/x = \infty$ and $\infty^n = \infty$.

The two remaining results can be established in like manner, if we put $y = \log_a x$ in the one case, and $y = -\log_a x$ in the other.

It should be noticed that if n be negative we see at once that $\mathrm{L}_{x=\infty} a^x/x^n = \infty$; $\mathrm{L}_{x=\infty} \log_a x/x^n = \infty$; $\mathrm{L}_{x=0} x^n \log_a x = -\infty$.

Example 2. If x be any fixed finite quantity, $\mathrm{L}_{n=\infty} x^n/n! = 0$.

Since n is to be made infinite, and x is finite, we may select some finite positive integer k such that $x < k < n$. Then we have

$$\frac{x^n}{n!} = \frac{x^{k-1}}{(k-1)!} \cdot \frac{x}{k} \cdot \frac{x}{k+1} \cdots \frac{x}{n},$$
$$< \frac{x^{k-1}}{(k-1)!} \left(\frac{x}{k}\right)^{n-k+1}.$$

Now, since $x < k$, $\mathrm{L}(x/k)^{n-k+1} = 0$, hence the theorem.

Example 3. $\mathrm{L}m(m-1)\ldots(m-n+1)/n!=0$ or ∞, according as $m>$ or <-1.

First, let $m>-1$, then $m+1$ is positive. We can always find a finite positive integer k such that $m+1<k<n$. Therefore we may write

$$ {}_mC_n=\frac{m(m-1)\ldots(m-n+1)}{n!}=(-)^{n-k+1}{}_mC_{k-1}\left(1-\frac{m+1}{k}\right)\left(1-\frac{m+1}{k+1}\right)\ldots \ldots\left(1-\frac{m+1}{n}\right), $$

$$ =(-)^{n-k+1}{}_mC_{k-1}P, \text{ say.} $$

Now

$$ \log 1/P=-\log\left(1-\frac{m+1}{k}\right)-\log\left(1-\frac{m+1}{k+1}\right)-\ldots-\log\left(1-\frac{m+1}{n}\right), $$

$$ >(m+1)/k+(m+1)/(k+1)+\ldots+(m+1)/n, $$

by § 13, Cor. 5. Also, by § 13, Example 2, the limit of $(m+1)/k+(m+1)/(k+1)+\ldots+(m+1)/n$ is infinite when $n=\infty$. It follows, therefore, that $LP=0$, and therefore that $L_mC_n=0$.

Next, let $m<-1$, say $m=-(1+\alpha)$, where α is a positive finite quantity. We may now write

$$ {}_mC_n=(-)^n\frac{(1+\alpha)(2+\alpha)\ldots(n+\alpha)}{1.2.\ldots n}=(-)^nP, \text{ say.} $$

Now

$$ \log P=-\log\left(1-\frac{\alpha}{1+\alpha}\right)-\log\left(1-\frac{\alpha}{2+\alpha}\right)-\ldots-\log\left(1-\frac{\alpha}{n+\alpha}\right), $$

$$ >\alpha/(1+\alpha)+\alpha/(2+\alpha)+\ldots+\alpha/(n+\alpha), $$

$$ >\alpha/(1+p)+\alpha/(2+p)+\ldots+\alpha/(n+p), $$

where p is the least integer which exceeds α. But the limit of $\alpha/(1+p)+\alpha/(2+p)+\ldots+\alpha/(n+p)$ is infinite. Hence $LP=\infty$.

When $m=-1$, ${}_mC_n=(-1)^n$, and the question regarding the limiting value does not arise.

§ 16.] *The fundamental theorem for the form* 0^0 *is that* $\mathrm{L}_{x=+0}\, x^x=1$.

This follows at once from last paragraph; for we have

$$ \mathrm{L}x^x=\mathrm{L}e^{x\log x}=e^{\mathrm{L}x\log x}=e^0=1. $$

Example 1. $\mathrm{L}_{x=+0}\,(x^n)^x=1$.

For $$ \mathrm{L}(x^n)^x=\mathrm{L}x^{nx}=\mathrm{L}(x^x)^n=(\mathrm{L}x^x)^n=1^n=1. $$

Example 2. $\mathrm{L}_{x=+0}\, x^{x^n}=1$ (n positive).

For $$ \mathrm{L}x^{x^n}=\mathrm{L}e^{x^n\log x}=e^{\mathrm{L}x^n\log x}=e^0=1, \text{ by § 15, Example 1.} $$

N.B.—If n be negative, $\mathrm{L}_{x=+0}\, x^{x^n}=0^\infty=0$.

§ 17.] *If u and v be functions of x, both of which vanish when $x=a$, and are such that* $\underset{x=a}{\mathrm{L}}\, v/u^n = l$, *where n is positive and neither* 0 *nor* ∞, *and l is not infinite, then* $\underset{x=a}{\mathrm{L}}\, u^v = 1$, *provided the limit be so approached that u is positive.**

For $$\mathrm{L}u^v = \mathrm{L}(u^n)^{v/u^n} = (\mathrm{L}u^n)^{\mathrm{L}v/u^n}.$$

Now, by § 16, Example 2, since n is positive $\underset{u=+0}{\mathrm{L}}\, u^{u^n} = 1$. Hence $\mathrm{L}u^v = 1^l = 1$.

If $\underset{x=a}{\mathrm{L}}\, v/u^n = \infty$, this transformation leads to the form 1^∞; and therefore becomes illusory.

The above theorem includes a very large number of particular cases. We see, for example, that, *if* $\mathrm{L}v/u$ *be determinate and not infinite, then* $\mathrm{L}u^v = 1$. Again, since, as we shall prove in chapter xxx., every algebraic function vanishes in a finite ratio to a positive finite power of $x-a$, it follows that every such function vanishes in a finite ratio to a positive finite power of every other such function. Hence $\mathrm{L}u^v = 1$ whenever u and v are algebraic functions of x.†

Example. Evaluate $\mathrm{L}\{x-1+\sqrt{(x^2-1)}\}^{\sqrt[3]{(x-1)}}$ when $x=1$.

Here $u \equiv \sqrt{(x-1)}\{\sqrt{(x-1)}+\sqrt{(x^2+x+1)}\}$, $v \equiv \sqrt[3]{(x-1)}$, $u^{2/3}/v \equiv \{\sqrt{(x-1)}+\sqrt{(x^2+x+1)}\}^{2/3}$.

Hence $\mathrm{L}u^{2/3}/v = \sqrt[3]{3}$. Therefore $\mathrm{L}u^v = \mathrm{L}_{(u^{u^{2/3}})^{v/u^{2/3}}} 1^{1/\sqrt[3]{3}} = 1$.

§ 18.] In cases where the last theorem does not apply, the evaluation of the limit can very often be effected by writing u^v in the form $e^{v \log u}$, and then seeking by transformation to deduce the limit of $v \log u$ from some combination of standard cases.‡

Example. Evaluate $x^{1/\log(e^x-1)}$ when $x=0$.

It is obviously suggested to attempt to make this depend on $\underset{x=0}{\mathrm{L}}\, \{(e^x-1)/x\} = 1$. This may be effected as follows. We have

$$x^{1/\log(e^x-1)} = e^{\log x/\log(e^x-1)}.$$

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* See Franklin, *American Journal of Mathematics*, 1878.

† See Sprague, *Proc. Edinb. Math. Soc.*, vol. iii., p. 71 (1885).

‡ At one time an erroneous impression prevailed that the indeterminate form 0^0 has always the value 1.—See *Crelle's Jour.*, Bd. xii.

Now
$$\frac{\log x}{\log(e^x-1)} = \frac{\log x}{\log\{(e^x-1)/x\}+\log x},$$
$$= \frac{1}{\log\{(e^x-1)/x\}/\log x+1}.$$
Since $\mathrm{L}\log\{(e^x-1)/x\}=0$, by § 13, Cor. 4, and $\mathrm{L}\log x=-\infty$, we see that $\mathrm{L}\log x/\log(e^x-1)=1$.
Hence
$$\mathrm{L}x^{1/\log(e^x-1)}=e.$$

§ 19.] Since $u^v = 1/(1/u)^v$, indeterminates of the form ∞^0 can always be made to depend on others of the form 0^0, and treated by the methods already explained.

Example. Evaluate $(1+x)^{1/x}$ when $x=\infty$.

Let $1+x=1/y$, so that $y=0$ when $x=\infty$; then we have
$$\mathop{\mathrm{L}}_{x=\infty}(1+x)^{1/x} = \mathop{\mathrm{L}}_{y=0}\{1/y^{y/(1-y)}\} = 1/\mathrm{L}(y^y)^{1/(1-y)}.$$
Now $\mathrm{L}y^y=1$ and $\mathrm{L}1/(1-y)=1$; hence $\mathop{\mathrm{L}}_{x=\infty}(1+x)^{1/x}=1$.

§ 20.] The fundamental case for the form 1^∞ is $\mathop{\mathrm{L}}_{x=\infty}(1+1/x)^x = \mathop{\mathrm{L}}_{x=0}(1+x)^{1/x} = e$, already discussed in § 13. A great variety of other cases can be reduced to this by means of the following theorem.

If u and v be functions of x such that $u=1$ and $v=\infty$ when $x=a$, then $\mathrm{L}u^v = e^{\mathrm{L}v(u-1)}$, provided $\mathrm{L}v(u-1)$ be determinate.

We have in fact
$$u^v = \{(1+\overline{u-1})^{1/(u-1)}\}^{v(u-1)}.$$
Hence, by § 7,
$$\mathrm{L}u^v = \mathrm{L}\{(1+\overline{u-1})^{1/(u-1)}\}^{\mathrm{L}v(u-1)},$$
$$= e^{\mathrm{L}v(u-1)}$$
provided $\mathrm{L}v(u-1)$ be determinate.

Example 1. $\mathop{\mathrm{L}}_{x=1}x^{1/(x-1)} = \mathop{\mathrm{L}}_{x=1}(1+\overline{x-1})^{1/(x-1)} = e$.

Example 2. Evaluate $(1+\log x)^{1/(x-1)}$ when $x=1$.

We have
$$l = \mathrm{L}(1+\log x)^{1/(x-1)} = \mathrm{L}\{(1+\log x)^{1/\log x}\}^{\log x/(x-1)},$$
$$= e^{\mathrm{L}\log x/(x-1)}.$$
Now $\mathrm{L}\log x/(x-1) = \mathrm{L}\log x^{1/(x-1)} = \log \mathrm{L}x^{1/(x-1)} = \log e = 1$. Hence $l=e$.

TRIGONOMETRICAL LIMITS.

§ 21.] We deal with this part of the subject only in so far as it is necessary for the analytical treatment of the Trigonometrical Functions in the following chapters.

We shall require the following inequality theorems:—

If x be the number of radians (circular units) in any positive angle less than a right angle, then

I. $\tan x > x > \sin x$;

II. $x > \sin x > x - \frac{1}{4}x^3$;

III. $1 > \cos x > 1 - \frac{1}{2}x^2$.

If PQ be the arc of a circle of radius r, which subtends the central angle $2x$, and if PT QT be the tangents at P and Q, then we assume as an axiom that

$$\text{PT} + \text{TQ} > \text{arc PQ} > \text{chord PQ}.$$

Hence, as the reader will easily see from the geometric definition of the trigonometrical functions, we have

$$2r \tan x > 2rx > 2r \sin x;$$

that is,
$$\tan x > \quad x > \quad \sin x,$$

which is I.

To prove II., we remark that $\sin x = 2 \sin \frac{1}{2}x \cos \frac{1}{2}x = 2 \tan \frac{1}{2}x \cos^2 \frac{1}{2}x = 2 \tan \frac{1}{2}x (1 - \sin^2 \frac{1}{2}x)$. Hence, since, by I., $\tan \frac{1}{2}x > \frac{1}{2}x$ and $\sin \frac{1}{2}x < \frac{1}{2}x$, we have

$$\begin{aligned} \sin x &> 2 . \tfrac{1}{2}x \{1 - (\tfrac{1}{2}x)^2\}, \\ &> x - \tfrac{1}{4}x^3. \end{aligned}$$

The first part of III. is obvious from the geometric definition of $\cos x$. To prove the latter part, we notice that $\cos x = 1 - 2 \sin^2 \frac{1}{2}x$; hence, by I.,

$$\begin{aligned} \cos x &> 1 - 2(\tfrac{1}{2}x)^2, \\ &> 1 - \tfrac{1}{2}x^2. \end{aligned}$$

§ 22.] The fundamental theorem regarding trigonometrical limits is as follows:—

If x be the radian measure of an angle, then* $\lim_{x=0} (\sin x/x) = 1$.

This follows at once from the first inequality of last paragraph. For, if $x < \frac{1}{2}\pi$, we have

$$\tan x > x > \sin x;$$

therefore
$$\sec x > x/\sin x > 1.$$

* In all that follows, and, in fact, in all analytical treatment of the trigonometrical functions, the argument is assumed to denote radian measure.

If we diminish x sufficiently, sec x can be made to differ from 1 by as little as we please. Hence, by making x sufficiently small, we can make $x/\sin x$ lie between 1 and a quantity differing from 1 as little as we please. Therefore

$$\mathrm{L}\, x/\sin x = 1.$$

Hence also

$$\mathrm{L}\, \sin x/x = 1.$$

Cor. 1.

$$\mathop{\mathrm{L}}_{x=0} \tan x/x = 1.$$

For $\mathrm{L} \tan x/x = \mathrm{L}(\sin x/x)/\cos x = \mathrm{L} \sin x/x \times \mathrm{L}\, 1/\cos x = 1 \times 1 = 1$.

Cor. 2. $\mathop{\mathrm{L}}_{x=\infty} \sin \frac{a}{x} \Big/ \frac{a}{x} = \mathop{\mathrm{L}}_{x=\infty} \tan \frac{a}{x} \Big/ \frac{a}{x} = 1$ *provided* a *is either a constant, or a function of* x *which does not become infinite when* $x = \infty$.

This is merely a transformation of the preceding theorems. It should also be remarked that

$$\mathop{\mathrm{L}}_{x=\infty} \left(\sin \frac{a}{x} \Big/ \frac{a}{x} \right)^{\beta} = \mathop{\mathrm{L}}_{x=\infty} \left(\tan \frac{a}{x} \Big/ \frac{a}{x} \right)^{\beta} = 1,$$

provided a and β are constants, or else functions of x which do not become infinite when $x = \infty$.

If, however, a were constant, and β a function of x which becomes infinite when $x = \infty$, then each of the two limits would take the form 1^{∞}, and would require further examination.

§ 23.] Many of the cases excepted at the end of last paragraph can be dealt with by means of the following results, which we shall have occasion to use later on :—

If a *be constant, or a function of* x *which is not infinite when* $x \pm \infty$, *then*

$$\mathop{\mathrm{L}}_{x=\infty} \left(\sin \frac{a}{x} \Big/ \frac{a}{x} \right)^{x} = 1;$$

$$\mathop{\mathrm{L}}_{x=\infty} \left(\cos \frac{a}{x} \right)^{x} = 1;$$

$$\mathop{\mathrm{L}}_{x=\infty} \left(\tan \frac{a}{x} \Big/ \frac{a}{x} \right)^{x} = 1.$$

To prove the first of these, we observe that for all values of a/x less than $\frac{1}{2}\pi$ we have, by § 21, II.,

$$1 > \left(\sin \frac{a}{x} \Big/ \frac{a}{x} \right)^{x} > \left\{ 1 - \tfrac{1}{4} \left(\frac{a}{x} \right)^{2} \right\}^{x}.$$

Now

$$\begin{aligned}\mathop{\mathrm{L}}_{x=\infty}(1-a^2/4x^2)^x &= \mathop{\mathrm{L}}_{x=\infty}\{(1-a^2/4x^2)^{-4x^2/a^2}\}^{-a^2/4x},\\ &= \{\mathop{\mathrm{L}}_{x=\infty}(1-a^2/4x^2)^{-4x^2/a^2}\}^{-\mathrm{L}a^2/4x},\\ &= e^0 = 1, \text{ by §§ 7 and 13.}\end{aligned}$$

Hence $$\mathop{\mathrm{L}}_{x=\infty}\left(\sin\frac{a}{x}\Big/\frac{a}{x}\right)^x = 1.$$

In exactly the same way we can prove that $\mathop{\mathrm{L}}_{x=\infty}\left(\cos\frac{a}{x}\right)^x = 1.$

Finally, since

$$\mathrm{L}\left(\tan\frac{a}{x}\Big/\frac{a}{x}\right)^x = \mathrm{L}\left(\sin\frac{a}{x}\Big/\frac{a}{x}\right)^x \times \mathrm{L}\,1\Big/\left(\cos\frac{a}{x}\right)^x,$$

the third result follows as a combination of the first two.

Example. Evaluate $(\cos x)^{1/x^2}$ when $x=0$. By § 20, we have $\mathrm{L}(\cos x)^{1/x^2} = e^{\mathrm{L}(\cos x-1)/x^2}$. Now $(\cos x-1)/x^2 = -2\sin^2\frac{1}{2}x/x^2 = -\frac{1}{2}(\sin\frac{1}{2}x/\frac{1}{2}x)^2$. Hence $\mathrm{L}(\cos x-1)/x^2 = -\frac{1}{2}$.

We therefore have $\mathrm{L}(\cos x)^{1/x^2} = e^{-\frac{1}{2}}$.

SUM OF AN INFINITE NUMBER OF INFINITELY SMALL TERMS.

§ 24.] If we consider the sum of n terms, say, $u_1 + u_2 + \ldots + u_n$, each of which depends on n in such a way that it becomes infinitely small when n becomes infinitely great, it is obvious that we cannot predict beforehand whether the sum will be finite or infinite. Such a sum partakes of the nature of the form $0 \times \infty$; for we cannot tell *a priori* whether the smallness of the individual terms, or the infiniteness of their number, will ultimately predominate. We shall have more to do with such cases in our next chapter; but the following instance is so famous in the history of the Infinitesimal Calculus before Newton and Leibnitz that it deserves a place here.

If $r+1$ be positive, then

$$\mathop{\mathrm{L}}_{n=\infty}(1^r + 2^r + \ldots + n^r)/n^{r+1} = 1/(r+1).$$

In the case where r is an integer this theorem may be deduced from the formula of chap. xx., § 9.

The proofs usually given for the other cases are not very rigorous; but a satisfactory proof may be obtained by means of the inequality

$$(r+1)x^r(x-y) \gtrless x^{r+1} - y^{r+1} \gtrless (r+1)y^r(x-y) \qquad (1),$$

which we have already used so often.

If we put first $x = p$, $y = p-1$, and then $x = p+1$, $y = p$, we deduce

$$(p+1)^{r+1} - p^{r+1} \gtrless (r+1)p^r \gtrless p^{r+1} - (p-1)^{r+1} \qquad (2),$$

where the upper or the lower signs of inequality are to be taken according as the positive number $r+1$ is $>$ or <1.

If in (2) we put for p in succession 1, 2, 3, . . . , n and add all the resulting inequalities we deduce

$$(n+1)^{r+1} - 1 \gtrless (r+1)(1^r + 2^r + \ldots + n^r) \gtrless n^{r+1}.$$

Hence

$$\{(1+1/n)^{r+1} - 1/n^{r+1}\}/(r+1) \gtrless (1^r + 2^r + \ldots + n^r)/n^{r+1} \gtrless 1/(r+1).$$

That is to say, $(1^r + 2^r + \ldots + n^r)/n^{r+1}$ always lies between $1/(r+1)$ and $\{(1+1/n)^{r+1} - 1/n^{r+1}\}/(r+1)$. But $\mathrm{L}_{n=\infty} (1+1/n)^{r+1} = 1$; and $\mathrm{L}_{n=\infty} 1/n^{r+1} = 0$, *since* $r+1$ *is positive*. Hence the second of the two enclosing values ultimately coincides with the first, and our theorem follows.

It may be observed that, if $r+1$ were negative, the proof would fail, simply because in this case $\mathrm{L}\, 1/n^{r+1} = \infty$.

Cor. 1. *If* s *be any finite integer, and* $r+1$ *be positive,*

$$\mathrm{L}_{n=\infty} \{1^r + 2^r + \ldots + (n-s)^r\}/n^{r+1} = 1/(r+1).$$

This is obvious, since $\mathrm{L}\{1^r + 2^r + \ldots + (n-s)^r\}/n^{r+1}$ differs from $\mathrm{L}(1^r + 2^r + \ldots + n^r)/n^{r+1}$ by a *finite* number of infinitely small terms.

Cor. 2. *If* a *be any constant, and* $r+1$ *be positive,*

$$\mathrm{L}_{n=\infty} \{(a+1)^r + (a+2)^r + \ldots + (a+n)^r\}/n^{r+1} = 1/(r+1).$$

This may be proved by a slight generalisation of the method used in the proof of the original theorem.

Cor. 3. *If a and c be constants, and $r+1 \neq 0$,*

$$\mathop{\mathrm{L}}_{n=\infty} \{(na+c)^r + (na+2c)^r + \ldots + (na+nc)^r\}/n^{r+1} = \{(a+c)^{r+1} - a^{r+1}\}/c(r+1).$$

This also may be proved in the same way, the only fresh point being the inclusion of cases where $r+1$ is negative.

§ 25.] Closely connected with the results of the foregoing paragraph is the following Limit Theorem, to which attention has been drawn by the researches of Dirichlet:—

If a, b, ρ be all positive, the limit, when $n=\infty$, of the sum of n terms of the series

$$\frac{1}{a^{1+\rho}} + \frac{1}{(a+b)^{1+\rho}} + \frac{1}{(a+2b)^{1+\rho}} + \ldots + \frac{1}{(a+nb)^{1+\rho}} + \ldots \quad (1),$$

is finite for all finite values of ρ, however small; and, if $\sum_{n=1}^{\infty} 1/(a+nb)^{1+\rho}$ denote this limit, then

$$\mathop{\mathrm{L}}_{\rho=0} \rho \sum_{n=1}^{\infty} 1/(a+nb)^{1+\rho} = 1/b \quad (2).$$

By means of the inequality (1) of last paragraph, we readily establish that

$$\{a+(p-1)b\}^{-\rho} - \{a+pb\}^{-\rho} > \rho b\{a+pb\}^{-\rho-1} > \{a+pb\}^{-\rho} - \{a+(p+1)b\}^{-\rho} \quad (3).$$

Putting, in (3), $0, 1, 2, \ldots, n$ successively in place of p, adding the resulting inequalities, and dividing by $b\rho$, we deduce

$$\frac{1}{b\rho}\left\{\frac{1}{\{a-b\}^{\rho}} - \frac{1}{\{a+nb\}^{\rho}}\right\} > \sum_{p=0}^{n} \frac{1}{\{a+pb\}^{1+\rho}} > \frac{1}{b\rho}\left\{\frac{1}{a^{\rho}} - \frac{1}{\{a+(n+1)b\}^{\rho}}\right\} \quad (4).$$

Since $\mathrm{L}1/\{a+nb\}^{\rho} = 0$, and $\mathrm{L}1/\{a+(n+1)b\}^{\rho} = 0$, when $n=\infty$, we deduce from (4),

$$\frac{1}{\rho b(a-b)^{\rho}} > \sum_{p=0}^{\infty} \frac{1}{(a+pb)^{1+\rho}} > \frac{1}{\rho b a^{\rho}} \quad (5).$$

From (5) the first part of the above theorem follows at once; and we see that $1/\rho b(a-b)^{\rho}$ and $1/\rho b^{\rho+1}$ are finite upper and lower limits for the sum in question.

We also have

$$\frac{1}{b(a-b)^{\rho}} > \rho \sum_{p=0}^{\infty} \frac{1}{(a+pb)^{1+\rho}} > \frac{1}{ba^{\rho}};$$

whence it follows, since $\mathrm{L}\,1/b(a-b)^{\rho} = \mathrm{L}\,1/ba^{\rho} = 1/b$, when $\rho = 0$, that

$$\mathrm{L}_{\rho=0}\, \rho \sum_{p=0}^{\infty} \frac{1}{(a+pb)^{1+\rho}} = \frac{1}{b}.$$

From the theorem thus proved it is not difficult to deduce the following more general one, also given by Dirichlet :—

If $k_1, k_2, \ldots, k_n, \ldots$ be a series of positive quantities, no one of which is less than any following one, and if they be such that $\mathrm{L}_{t=\infty}\, \mathrm{T}/t = a$, *where* T *is the number of the k's that do not exceed t, then* $\sum_1^{\infty} 1/k_n^{1+\rho}$ *is finite for all positive finite values of* ρ, *however small; and* $\mathrm{L}_{\rho=0}\, \rho \sum_1^{\infty} 1/k_n^{1+\rho} = a$.*

Cor. It follows from (5) that

$$\frac{1}{\rho(a-1)^{\rho}} > \mathrm{L}_{n=\infty} \left\{ \frac{1}{a^{1+\rho}} + \frac{1}{(a+1)^{1+\rho}} + \ldots + \frac{1}{(a+n)^{1+\rho}} \right\} > \frac{1}{\rho a^{\rho}} \quad (6),$$

an inequality which we shall have occasion to use hereafter.

GEOMETRICAL APPLICATIONS OF THE THEORY OF LIMITS.

§ 26.] The reader will find that there is no better way of strengthening his grasp of the Analytical Theory of Limits than by applying it to the solution of geometrical problems. We may point out that the problem of drawing a tangent at any point of the graph of the function $y = f(x)$ can be solved by evaluating the limit when $h = 0$ of $\{f(x+h) - f(x)\}/h$; for, as will readily be seen by drawing a figure, the expression just written is the tangent of the inclination to the axis of x of the secant drawn through the two points on the graph whose abscissae are x and $x+h$; and the tangent at the former point is the limit of the

* See Dirichlet, *Crelle's Jour.*, Bd. 19 (1839) and 53 (1857); also Heine, *ibid.*, Bd. 31.

secant when the latter point is made to approach infinitely close to the former.*

Example. To find the inclination of the tangent to the graph of $y=e^x$ at the point where this graph crosses the axis of y.

If θ be the inclination of the tangent to the x-axis, we have

$$\begin{aligned}\tan\theta &= \mathrm{L}(e^{0+h}-e^0)/h,\\ &= \mathrm{L}(e^h-1)/h,\\ &= \log e = 1.\end{aligned}$$

Hence $\theta=\frac{1}{4}\pi$.

§ 27.] The limit investigated in § 24 enables us to solve a problem in quadratures; and thus to illustrate in an elementary way the fundamental idea of the Calculus of Definite Integrals. We may in fact deduce from it an expression for the area included between the graph of the function $y=x^r/l^{r-1}$, the axis of x, and any two ordinates.

Let A and B be the feet of the two ordinates, a, b the corresponding abscissae, and $b-a=c$.† Divide AB into n equal parts; draw the ordinates through A, B, and the $n-1$ points of division; and construct—1st, the series of rectangles whose bases are the n parts, and whose altitudes are the 1st, 2nd, . . . , nth ordinates respectively; 2nd, the series of rectangles whose bases are as before, but whose altitudes are the 2nd, 3rd, . . . , $(n+1)$th ordinates. If I_n and J_n be the sums of the areas of the first and second series of rectangles, and A the area enclosed between the curve, the axis of x and the ordinates through A and B, then obviously $\mathrm{I}_n<\mathrm{A}<\mathrm{J}_n$.

Now

$$\mathrm{I}_n = c\{a^r+(a+c/n)^r+(a+2c/n)^r+\ldots+(a+\overline{n-1}c/n)^r\}/nl^{r-1};$$
$$\mathrm{J}_n = c\{(a+c/n)^r+(a+2c/n)^r+\ldots+(a+nc/n)^r\}/nl^{r-1}.$$

Since $\mathrm{J}_n-\mathrm{I}_n=c(b^r-a^r)/nl^{r-1}$, which vanishes when $n=\infty$, $\mathrm{LI}_n=\mathrm{LJ}_n$ and therefore $\mathrm{A}=\mathrm{LJ}_n$, when $n=\infty$. Hence

$$\mathrm{A}=\frac{c}{l^{r-1}}\mathrm{L}\frac{(na+1c)^r+(na+2c)^r+\ldots+(na+nc)^r}{n^{r+1}},$$

$$=\frac{c}{l^{r-1}}\left\{\frac{(a+c)^{r+1}-a^{r+1}}{c(r+1)}\right\}, \text{ by § 24, Cor. 3.}$$

Hence $\mathrm{A}=(b^{r+1}-a^{r+1})/(r+1)l^{r-1}$.

This gives, when $r=\frac{1}{2}$, and $a=0$, the Archimedian rule for the quadrature of a parabolic segment.

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* We would earnestly recommend the learner at this stage to begin (if he has not already done so) the study of Frost's *Curve Tracing*, a work which should be in the hands of every one who aims at becoming a mathematician, either practical or scientific.

† The reader should draw the figure for himself.

Exercises VII.

Limits.

Find the limiting values of the following functions for the given values of the variables:—

(1.) $(3x^{\frac{1}{2}}+2x^{\frac{2}{3}}+3x^{\frac{1}{3}})/(x^{\frac{1}{2}}+x^{\frac{1}{3}}+x^{\frac{1}{4}})$, $x=0$, and $x=\infty$.

(2.) $(x^4-x^3-9x^2+16x-4)/(x^3-2x^2-4x+8)$, $x=2$.

(3.) $\log(x^3-2x^2-2x-3)-\log(x^3-4x^2+4x-3)$, $x=3$.

(4.) $\{x-(n+1)x^{n+1}+nx^{n+2}\}/(1-x)^2$, $x=1$ (n a positive integer). (Euler, *Diff. Calc.*)

(5.) $\{\sqrt{(x-1)}-(x-1)\}/\{\sqrt[3]{(x-1)}-\sqrt{(x-1)}\}$, $x=1$.

(6.) $(x^{m+n}-a^mx^n)/(x^{p+q}-a^px^q)$, $x=a$.

(7.) $\{(a+x)^m-(a-x)^m\}/\{(a+x)^n-(a-x)^n\}$, $x=0$.

(8.) $\{(x^m-1)^p-(x^n-1)^q\}/\{(x-1)^p-(x-1)^q\}$, $x=1$.

(9.) $\dfrac{(x^m-1)^2-(x^m-1)(x^n-1)+(x^n-1)^2}{(x^m-1)^2+(x^m-1)(x^n-1)+(x^n-1)^2}$, $x=1$.

(10.) $\{a-\sqrt{(a^2-x^2)}\}/x^2$, $x=0$. (Euler, *Diff. Calc.*)

(11.) $\{\sqrt[p]{(a+x)}-\sqrt[p]{(a-x)}\}/\{\sqrt[q]{(a+x)}-\sqrt[q]{(a-x)}\}$, $x=0$.

(12.) $\{(a^2+ax+x^2)^{\frac{1}{2}}-(a^2-ax+x^2)^{\frac{1}{2}}\}/\{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}\}$, $x=0$. (Euler, *Diff. Calc.*)

(13.) $\{(2a^2x-x^4)^{\frac{1}{2}}-a(a^2x)^{\frac{1}{2}}\}/\{a-(ax^3)^{\frac{1}{4}}\}$, $x=a$. (Gregory, *Examples in Diff. Calc.*)

(14.) $\{a+\sqrt{(2a^2-2ax)}-\sqrt{(2ax-x^2)}\}/\{a-x+\sqrt{(a^2-x^2)}\}$, $x=a$. (Euler, *Diff. Calc.*)

(15.) $x-\sqrt{(x^2-y^2)}$, when $x=\infty$, $y=\infty$, but y^2/x finite $=2p$.

(16.) $\Sigma x^n(y-z)/\Pi(y-z)$, $x=y=z$.

(17.) $\Sigma x^m(y^n-z^n)/\Sigma x^p(y^q-z^q)$, $x=y=z=a$.

(18.) $nx^{n-1}/(x^n-a^n)-1/(x-a)$, $x=a$.

(19.) $2^x(a^{1/2^x}-1)$, $x=\infty$.

(20.) $x^{1/x}$, $x=\infty$.

(21.) $(1+1/x^2)^x$, $x=\infty$.

(22.) $x^{2x}/(1+x^2)^x$, $x=\infty$.

(23.) $(1+1/x)^x$, $x=0$.

(24.) $(1+1/x)^{x^2}$, $x=\infty$.

(25.) $x^{1/(x-1)^2}$, $x=1$.

(26.) $x^{1/(x^2-1)}$, $x=1$.

(27.) a^{x^n}/x, $x=\infty$.

(28.) $(\log x)^{1/x}$, $x=\infty$.

(29.) $(\log x/x)^{1/x}$, $x=\infty$.

(30.) $\log^m x/\log^n x$, $x=\infty$.

(31.) $a^xf(x)$, $x=\infty$, where $f(x)$ is a rational function of x, and a a constant.

(32.) $(ax^n+bx^{n-1}+\ldots)^{1/x}$, $x=\infty$. (Cauchy.)

(33.) $x^{1/(1+2\log x)}$, $x=0$.

(34.) $\{(x^2+x+1)/(x^2-x+1)\}^x$, $x=\infty$.

(35.) $\{\frac{1}{2}(a^x+b^x)\}^{1/x}$, $x=0$.

(36.) $\{1+2/\sqrt{(x^2+1)}\}^{\sqrt[3]{(3x^3+1)}}$, $x=\infty$. (Longchamps.)

(37.) $\left(\dfrac{a_0+a_1x+\ldots+a_rx^r}{b_0+b_1x+\ldots+b_rx^r}\right)^{\lambda_0+\lambda_1x}$, $x=\infty$. (Math. Trip., 1886.)

(38.) $\{1/(e^x-1)\}^{1/x}$, $x=\infty$.

(39.) $\{\log(1+x)\}^{\log(1+x^2)}$, $x=0$.

(40.) $\log(1+ax)/\log(1+bx)$, $x=0$.

(41.) $(e^x-e^{-x})/\log(1+x)$, $x=0$. (Euler, *Diff. Calc.*)

(42.) $(\frac{1}{2}\pi-x)\tan x$, $x=\frac{\pi}{2}$.

(43.) $\tan^{-1}x/x$, $x=0$.

(44.) $(1-\sin x+\cos x)/(\sin x+\cos x-1)$, $x=\frac{1}{2}\pi$. (Euler, *Diff. Calc.*)

(45.) $\sin x/(1-x^2/\pi^2)$, $x=\pi$.

(46.) $x\{\cos(a/x)-1\}$, $x=\infty$.

(47.) $(\sin x-\sin a)/(x-a)$, $x=a$.

(48.) $\sec x-\tan x$, $x=\frac{1}{2}\pi$.

(49.) $(\sin^4 x-\tan^4 x)/(1+\cos x)(1-\cos x)^3$, $x=0$.

(50.)* $\sinh x/x$, $x=0$.

(51.) $(\cosh x-1)/x^2$, $x=0$.

(52.) $\tanh^{-1}x/x$, $x=0$.

(53.) $\sin x/\log(1+x)$, $x=0$.

(54.) $\sin x \log x$, $x=0$.

(55.) $\cos x \log\tan x$, $x=\frac{1}{2}\pi$.

(56.) $\log\tan mx/\log\tan nx$, $x=0$.

(57.) $(\log\sin mx-\log x)/(\log\sin nx-\log x)$, $x=0$.

(58.) $\sin x^{\sin x}$, $x=0$.

(59.) $\sin x^{\tan x}$, $x=0$.

(60.) $(\sinh x)^{\tan x}$, $x=0$.

(61.) $\{(x/a)\sin(a/x)\}^{x^m}(m<2)$, $x=\infty$.

(62.) $(\cos mx)^{n/x^2}$, $x=0$.

(63.) $(\cos mx)^{\operatorname{cosec}^2 nx}$, $x=0$.

(64.) $(2-x/a)^{\tan \pi x/2a}$, $x=a$.

(65.) $\log_e(\log_e x)/\cos\frac{\pi e}{2x}$, $x=e$.

(66.) Show that $\sin x\cot(a/x)\log(1+\tan(a/x))$ has no determinate limit when $x=\infty$.

(67.) If $l_a{}^2x$ stand for $\log_a(\log_a x)$, $l_a{}^3x$ for $\log_a(\log_a(\log_a x))$, &c., show that $\mathrm{L}_{x=\infty}[1-\{l_a{}^p x/l_a{}^p(x+1)\}^m]xl_a x l_a{}^2x \ldots l_a{}^p x = m(l_a e)^p$. (Schlömilch, *Algebraische Analysis*, chap. ii.)

(68.) Show that $\mathrm{L}_{n=\infty}\sum_{s=1}^{s=n}(a+s)^{1/n}/n=1$.

(69.) Show that $\mathrm{L}_{n=\infty}\sum_{s=1}^{s=n}\{(a+s)/n\}^n$ lies between e^a and e^{a+1}.

(70.) Show that $\mathrm{L}_{n=\infty}\sum_{s=1}^{s=n}\{(a+sc/n)/(a+c)\}^n$ is finite if $a+c$ be numerically greater than a, and that $\mathrm{L}_{n=\infty}\sum_{s=1}^{s=n}\{(a+sc/n)/a\}^n$ is finite if $a+c$ be numerically less than a.

(71.) Trace the graph of $y=(a^x-1)/x$, when $a>1$, and when $a<1$.

(72.) Trace the graph of $y=x^{1/x}$ for positive values of x; and find the direction in which the graph approaches the origin.

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* For the definition and elementary properties of the hyperbolic functions $\cosh x$, $\sinh x$, $\tanh x$, &c., see chap. xxix. All that is really wanted here is $\cosh x=\frac{1}{2}(e^x+e^{-x})$, $\sinh x=\frac{1}{2}(e^x-e^{-x})$.

(73.) Trace the graph of $y=(1+1/x)^x$; and find the angle at which it crosses the axis of y.

(74.) Find the orders of the zero and infinity values of y when determined as a function of x by the following equations:—*

(α) $x(x^2-ay)^2-y^5=0$. (Frost's *Curve Tracing*, § 155, Ex. 3).

(β) $x^2y^6+a^3y^5-x^5y^3+ax^6y-a^6x^2=0$. (*Ib.*, Ex. 7.)

(γ) $(x-1)y^3+(x^2-1)y^2-(x-2)^2y+x(x-2)=0$.

(75.) If u and v be functions of the integral variable n determined by the equations $u_n=u_{n-1}+v_{n-1}$, $v_n=u_{n-1}$, show that $\mathrm{L}_{n=\infty} u_n/v_n=(1\pm\sqrt{5})/2$. How ought the ambiguous sign to be settled when u_0 and u_1 are both positive?

(76.) Show that

$$2^{n(n+1)}>(n+1)^{n+1}\left(\frac{n}{1}\right)^n\left(\frac{n-1}{2}\right)^{n-1}\cdots\left(\frac{2}{n-1}\right)^2\left(\frac{1}{n}\right)^1.$$

(77.) Show that $\mathrm{L}_{n=\infty}\left\{\dfrac{(m+1)(m+2)\ldots(m+n)}{1\,.\,2\ldots n}\right\}^{1/n}=1$.

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* For a general method for dealing with such problems, see chap. XXX.

CHAPTER XXVI.

Convergence of Infinite Series and of Infinite Products.

§ 1.] The notion of the repetition of an algebraical operation upon a series of operands formed according to a given law presents two fundamental difficulties when the frequency of the repetition may exceed any number, however great, or, as it is shortly expressed, become infinite. Since the mind cannot overlook the totality of an infinite series of operations, some definition must be given of what is to be understood as the result of such a series of operations; and there also arises the further question whether the series of operations, even when its meaning is defined, can, consistently with its definition, be subjected to the laws of algebra which are in the first instance laid down for chains of operations wherein the number of links is finite. That the two difficulties thus raised are not imaginary the student will presently see, by studying actual instances in the theory of sums and products involving an infinite number of summands and multiplicands.

§ 2.] One very simple case of an infinite series, namely, a geometric series, has already been discussed in chap. xx., § 15. The fact that the geometric series can be summed considerably simplifies the first of the two difficulties just mentioned;* nevertheless the leading features of the problem of infinite series are all present in the geometric series; and it will be found that most questions regarding the convergence of infinite series are ultimately referred to this standard case.

* The second was not considered.

The consideration of the infinite geometric series suggests the following definitions.

Consider a succession of finite real summands $u_1, u_2, u_3, \ldots, u_n, \ldots$, unlimited in number, formed according to a given law, so that the nth term u_n is a finite one-valued function of n; and consider the successive sums

$$S_1 = u_1, \quad S_2 = u_1 + u_2, \quad S_3 = u_1 + u_2 + u_3$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$S_n = u_1 + u_2 + \ldots + u_n.$$

When n is increased more and more, one of three things must happen :—

1st. S_n *may approach a fixed finite quantity* S *in such a way that by increasing n sufficiently we can make* S_n *differ from* S *by as little as we please; that is, in the notation of last chapter,* $\mathrm{L}_{n=\infty} S_n = S$. *In this case the series*

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is said to be CONVERGENT, *and to converge to the value* S, *which is spoken of as the sum to infinity.*

Example. $1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} + \ldots$ Here $S = \mathrm{L}_{n=\infty} S_n = 2$.

2nd. S_n *may increase with n in such a way that by increasing n sufficiently we can make the numerical value of* S_n *exceed any quantity, however large; that is,* $\mathrm{L}_{n=\infty} S_n = \infty$. *In this case the series is said to be* DIVERGENT.

Example. $1 + 2 + 3 + \ldots$ Here $\mathrm{L}_{n=\infty} S_n = \infty$.

3rd. S_n *may neither become infinite nor approach a definite limit, but oscillate between a number of finite values the selection among which is determined by the integral character of n, that is, by such considerations as whether n is odd or even; of the form* $3m$, $3m+1$, $3m+2$, *&c. In this case the series is said to* OSCILLATE.

Example. $3 - 1 - 2 + 3 - 1 - 2 + 3 - 1 - 2 + \ldots$ Here $\mathrm{L}_{n=\infty} S_n = 0$, 3, or 2, according as n is of the form $3m$, $3m+1$, or $3m+2$.

In cases 2 and 3 the series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is also said to be *non-convergent.* In many important senses

non-convergent series cannot be said to have a sum; and it is obvious that infinite series of this description cannot, except in special cases, and under special precautions, be employed in mathematical reasoning.

Series are said to be *more or less rapidly convergent* according as the number of terms which it is necessary to take in order to get a given degree of approximation to the sum is smaller or larger. Thus a geometric series is more rapidly convergent the smaller its common ratio. Rapid convergency is obviously a valuable quality in a series from the arithmetical point of view.

It should be carefully noticed that the definition of the convergency of the series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

involves the supposition that the terms are taken successively *in a given order*. In other words, the sum to infinity of a convergent series may be, so far as the definition is concerned, dependent upon the order in which the terms are written. As a matter of fact there is, as was first pointed out by Dirichlet, a class of series which may converge to one value, or to any other, or even become divergent, according to the order in which the terms are written.

§ 3.] Two essential conditions are involved in the definition of a convergent series—1st, that S_n shall not become infinite for any value of n, however great; 2nd, that, as n increases, there shall be continual approach to a definite limit S. If we introduce the symbol ${}_mR_n$ to denote $u_{n+1} + u_{n+2} + \ldots + u_{n+m}$, that is, the sum of m terms following the nth, we may state the following criterion:—

The necessary and sufficient conditions for the convergency of a series are that S_n *be finite for all values of* n*; and that, by taking* n *sufficiently great, it be possible to make* ${}_mR_n$ *as small as we please, no matter what the value of* m *may be.*

That S_n must be finite is obvious from the definition of convergency. Since $\underset{n=\infty}{L}\, S_n = S$ (where S is finite), therefore

$\underset{n=\infty}{\mathrm{L}} S_{n+m} = S$. Hence $\underset{n=\infty}{\mathrm{L}} (S_{n+m} - S_n) = S - S = 0$; that is, $\underset{n=\infty}{\mathrm{L}} {}_mR_n = 0$.

The two conditions are sufficient. For, since S_n is finite for all values of n, the limit of S_n cannot be infinite. Also the limit of S_n cannot have one finite value when n has any particular integral character, and another value when n has a different integral character; for any such result would involve that $\underset{n=\infty}{\mathrm{L}} S_n$ and $\underset{n=\infty}{\mathrm{L}} S_{n+m}$ should have different values; but such cannot be the case, since $\mathrm{L}(S_{n+m} - S_n) = \mathrm{L}_m R_n = 0$.

The above criterion is often stated with the omission of the first part regarding the finiteness of S_n, it being implied that the second condition $\underset{n=\infty}{\mathrm{L}} {}_mR_n = 0$ involves the first. Cauchy, the originator of the modern theory of convergence, states the matter in this way. The discussion of this subtle point need not be taken up here, because in most cases a slight alteration of the demonstration which proves that $\underset{n=\infty}{\mathrm{L}} {}_mR_n = 0$ shows that S_n is always finite.

Cor. 1. *In any convergent series* $\underset{n=\infty}{\mathrm{L}} u_n = 0$.

For $u_n = S_n - S_{n-1} = {}_1R_{n-1}$, and, by the criterion for convergency, we must have $\underset{n=\infty}{\mathrm{L}} {}_1R_{n-1} = 0$. This condition, although necessary, is not of itself sufficient, as will presently appear in many examples.

Cor. 2. *If* $R_n = \underset{m=\infty}{\mathrm{L}} {}_mR_n$, *and* S *and* S_n *have the meanings above assigned to them, then* $S_n = S - R_n$.

For $S_{n+m} = S_n + {}_mR_n$, therefore $\underset{m=\infty}{\mathrm{L}} S_{n+m} = S_n + \underset{m=\infty}{\mathrm{L}} {}_mR_n$; and $\underset{m=\infty}{\mathrm{L}} S_{n+m} = S$, hence the theorem. R_n is usually called the *residue* of the series, and ${}_mR_n$ a *partial residue*. Obviously, the smaller R_n/S_n is for a given value of n, the more convergent is the series; for R_n is the difference between S_n and the limit of S_n when n is infinitely great. R_n is, of course, the sum of the infinite series $u_{n+1} + u_{n+2} + u_{n+3} + \ldots$; and it is an obvious

remark that *the residue of a convergent series is itself a convergent series.*

Cor. 3. *The convergency or divergency of a series is not affected by neglecting a finite number of its terms.*

For the sum of a finite number of terms is finite and definite; and the neglect of that sum alters $\underset{n=\infty}{\mathrm{L}}\, S_n$ merely by a finite determinate quantity; so that, if the series was originally convergent, it will remain so; if originally oscillating or divergent, it will remain so.

Example 1. Consider the series $1/1+1/2+1/3+\ldots+1/n+\ldots$

Here $$\begin{aligned} {}_mR_n &= 1/(n+1)+1/(n+2)+\ldots+1/(n+m), \\ &> 1/(n+m)+1/(n+m)+\ldots+1/(n+m), \\ &> m/(n+m), \\ &> 1/(n/m+1). \end{aligned}$$

Now, however great n may be, we can always choose m so much greater that n/m shall be less than any quantity, however small. Hence we cannot cause ${}_mR_n$ to vanish *for all values* of m by sufficiently increasing n. We therefore conclude that the series is not convergent, notwithstanding the fact that the terms ultimately become infinitely small. We shall give below a direct proof that $LS_n=\infty$.

Example 2.

$$\frac{1}{1}\log\frac{2^2}{1.3}+\frac{1}{2}\log\frac{3^2}{2.4}+\ldots+\frac{1}{n}\log\frac{(n+1)^2}{n(n+2)}.$$

Since $(n+1)^2/n(n+2)=(1+1/n)/\{1+1/(n+1)\}$, we have

$$\begin{aligned} {}_mR_n &= \frac{1}{n+1}\log\frac{1+1/(n+1)}{1+1/(n+2)}+\frac{1}{n+2}\log\frac{1+1/(n+2)}{1+1/(n+3)} \\ &\quad +\ldots+\frac{1}{n+m}\log\frac{1+1/(n+m)}{1+1/(n+m+1)}, \\ &< \frac{1}{n+1}\left\{\log\frac{1+1/(n+1)}{1+1/(n+2)}+\log\frac{1+1/(n+2)}{1+1/(n+3)}+\ldots+\log\frac{1+1/(n+m)}{1+1/(n+m+1)}\right\}, \\ &< \frac{1}{n+1}\log\frac{1+1/(n+1)}{1+1/(n+m+1)} \qquad (1). \end{aligned}$$

Now, whatever m may be, by making n large enough we can make $1/(n+1)$, and, *a fortiori*, $1/(n+m+1)$, as small as we please, therefore $\underset{n=\infty}{\mathrm{L}}\, {}_mR_n=0$ for all values of m.

If in (1) we put 0 in place of n, and n in place of m, and observe that $S_n={}_nR_0$, we see that

$$S_n<\log\frac{1+1/1}{1+1/(n+1)};$$

so that S_n can never exceed $\log 2$ whatever n may be.

Both conditions of convergency are therefore satisfied.

Putting $m=\infty$ in (1), we find for the residue of the series

$$R_n<[\log\{1+1/(n+1)\}]/(n+1);$$

a result which would enable us to estimate the rapidity of the convergency, and to settle how many terms of the series we ought to take to get an approximation to its limit accurate to a given place of decimals.

§ 4.] The following theorems follow at once from the criterion for convergency given in last paragraph. Some of them will be found very useful in discussing questions regarding convergence. We shall use Σu_n as an abbreviation for $u_1+u_2+\ldots+u_n+\ldots$, that is, "the series whose nth term is u_n."

I. *If u_n and v_n be positive, $u_n<v_n$ for all values of n, and Σv_n convergent, then Σu_n is convergent.*

If u_n and v_n be positive, $u_n>v_n$ for all values of n, and Σv_n divergent, then Σu_n is divergent.

For, under the first set of conditions, the values of S_n and ${}_mR_n$ belonging to Σu_n are less than the values of the corresponding functions S'_n and ${}_mR'_n$ belonging to Σv_n. Hence we have $0<S_n<S'_n$, $0<{}_mR_n<{}_mR'_n$. But, by hypothesis, S'_n is finite for all values of n, and $\lim_{n=\infty} {}_mR'_n=0$; hence S_n is finite for all values of n, and $\lim_{n=\infty} {}_mR_n=0$; that is, Σu_n is convergent.

Under the second set of conditions, $S_n>S'_n$. Hence, since $\lim_{n=\infty} S'_n=\infty$, we must also have $\lim_{n=\infty} S_n=\infty$; that is, Σu_n is divergent.

II. *If, for all values of n, $v_n>0$, and u_n/v_n is finite, then Σu_n is convergent if Σv_n is convergent, and divergent if Σv_n is divergent.*

By chap. xxiv., § 5, if A be the least, and B the greatest of the fractions, u_{n+1}/v_{n+1}, u_{n+2}/v_{n+2}, . . ., u_{n+m}/v_{n+m}, then

$$A<\frac{u_{n+1}+u_{n+2}+\ldots+u_{n+m}}{v_{n+1}+v_{n+2}+\ldots+v_{n+m}}<B.$$

Now, since u_n/v_n is finite for all values of n, A and B are finite. Hence we must have in all cases ${}_mR_n=C_mR'_n$, where C is a finite quantity whatever values we assign to m and n.

Hence S_n (that is, ${}_nR_0$) will be finite or infinite according as S'_n is finite or infinite; and if $\underset{n=\infty}{L}\ {}_mR'_n = 0$, we must also have $\underset{n=\infty}{L}\ {}_mR_n = 0$.

III. *If u_n and v_n be positive, and if, for all values of n, $u_{n+1}/u_n < v_{n+1}/v_n$, and Σv_n is convergent, then Σu_n is convergent; and if $u_{n+1}/u_n > v_{n+1}/v_n$, and Σv_n is divergent, then Σu_n is divergent.*

We have, if $u_{n+1}/u_n < v_{n+1}/v_n$,

$$S_n = u_1\left\{1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\cdot\frac{u_2}{u_1} + \ldots\right\},$$

$$< u_1\left\{1 + \frac{v_2}{v_1} + \frac{v_3}{v_2}\cdot\frac{v_2}{v_1} + \ldots\right\},$$

$$< \frac{u_1}{v_1} S'_n.$$

Now, by hypothesis, LS'_n is finite: hence LS_n must be finite. Also, since all the terms of Σu_n are positive, the series cannot oscillate, therefore Σu_n must be convergent.

In like manner, we can show that, if $u_{n+1}/u_n > v_{n+1}/v_n$, and Σv_n be divergent, then Σu_n is divergent.

N.B.—In Theorems I., II., III. we have, for simplicity, stated that the conditions must hold *for all values of n*; but we see from § 3, Cor. 3, that it is sufficient if they hold for *all values of n exceeding a certain finite value r*; for all the terms up to the rth in both series may be neglected.

It is convenient to speak of u_{n+1}/u_n as the *Ratio of Convergence* of Σu_n. Thus we might express Theorem III. as follows:—Any series is convergent (divergent) if its ratio of convergence is always less (greater) than the ratio of convergence of a convergent (divergent) series.

IV. *If a series which contains negative terms be convergent when all the negative terms have their signs changed, it will be convergent as it stood originally.*

For the effect of restoring the negative signs will be to diminish the numerical value both of S_n and of ${}_mR_n$.

Definition.—*A series which is convergent when all its terms are taken positively is said to be* ABSOLUTELY CONVERGENT.

It will be seen immediately that there are series whose convergency depends on the presence of negative signs, and which become divergent when all the terms are taken positively. Such series are said to be *semi-convergent.* In §§ 5 and 6, unless the contrary is indicated, we suppose any series of real terms to consist of positive terms only, and convergence to mean absolute convergence.

SPECIAL TESTS OF CONVERGENCY FOR SERIES WHOSE TERMS ARE ULTIMATELY ALL POSITIVE.

§ 5.] If we take for standard series a geometric progression, say Σr^n, which will be convergent or divergent according as $r <$ or > 1, and apply § 4, Th. I., we see that Σu_n will be convergent if, on and after a certain finite value of n, $u_n < r^n$, where $r < 1$; divergent if, on and after a certain finite value of n, $u_n > r^n$, where $r > 1$. Hence

I. Σu_n *is convergent or divergent according as* $u_n^{1/n}$ *is ultimately less or greater than unity.*

This test settles nothing in the case where $u_n^{1/n}$ *is ultimately unity.*

Example. $\Sigma 1/(1+1/n)^{n^2}$ is a convergent series; for

$$\mathrm{L}_{n=\infty}\, u_n^{1/n} = 1/\mathrm{L}(1+1/n)^n = 1/e,$$

by chap. xxv., § 13, where $e > 2$, and therefore $1/e < 1$.

If, with the series Σr^n for standard of comparison, we apply § 4, Th. III., we see that Σu_n is convergent or divergent according as u_{n+1}/u_n is, on and after a certain finite value of n, always < 1 or always > 1. Hence

II. Σu_n *is convergent or divergent according as its ratio of convergency is ultimately* $<$ *or* > 1.

Nothing is settled in the case where the ratio of convergency is ultimately equal to 1.

The examination of the ratio u_{n+1}/u_n is the most useful of all the tests of convergence.* It is sufficient for all the series that occur in elementary mathematics, except in certain extreme cases where these series are rarely used. In fact, this test, along with the Condensation Test of § 6, will suffice for the reader who is not concerned with more than the simpler applications of infinite series.

Notwithstanding their outward difference, Tests I. and II. are fundamentally the same. This will be readily seen by recalling the theorem of Cauchy, given in chap. xxv., § 14, which shows that $\mathrm{L}_{n=\infty} u_{n+1}/u_n = \mathrm{L}_{n=\infty} u_n^{1/n}$. It is useful to have the two forms of test, because in certain cases I. is more easily applied than II.

Example 1. To test the convergence of $\Sigma n^r x^n$, where r and x are constants. We have in this case

$$u_{n+1}/u_n = (n+1)^r x^{n+1}/n^r x^n,$$
$$= (1+1/n)^r x.$$

Hence $\mathrm{L} u_{n+1}/u_n = x$. The series is therefore convergent if $x<1$, and divergent if $x>1$.

If $x=1$, we cannot settle the question by means of the present test.

Example 2. If $\phi(n)$ be any algebraical function of n, $\Sigma\phi(n)x^n$ is convergent if $x<1$, divergent if $x>1$.

This hardly needs proof if $\mathrm{L}_{n=\infty} \phi(n)$ be finite. If $\mathrm{L}_{n=\infty} \phi(n)$ be infinite, we know (see chap. xxx.) that we can always find a positive value of r, such that $\mathrm{L}_{n=\infty} \phi(n)/n^r$ is finite, $=\mathrm{A}$ say. We therefore have

$$\mathrm{L} u_{n+1}/u_n = x\mathrm{L}\phi(n+1)/\phi(n),$$
$$= x\left\{ \mathrm{L}\frac{\phi(n+1)}{(n+1)^r} \Big/ \mathrm{L}\frac{\phi(n)}{n^r} \right\} \mathrm{L}\frac{(n+1)^r}{n^r},$$
$$= x\{\mathrm{A}/\mathrm{A}\} \times 1,$$
$$= x.$$

This very general theorem includes, among other important cases, the integro-geometric series

$$\phi(1)x + \phi(2)x^2 + \ldots + \phi(n)x^n + \ldots$$

where $\phi(n)$ is an integral function of n; and the series

$$\frac{x}{1} + \frac{x^2}{2} + \ldots + \frac{x^n}{n} + \ldots \qquad (1),$$

which, as we shall see in chap. xxviii., represents (when it is convergent)

* We here use (as is often convenient) "convergence" to mean "the quality of the series as regards convergency or divergency."

$-\log(1-x)$. It follows, by § 4, Th. IV., that since the series (1) is convergent when $x<1$, the series

$$\frac{x}{1}-\frac{x^2}{2}+\ldots(-)^{n-1}\frac{x^n}{n}+\ldots \qquad (2)$$

is also convergent when $x<1$.

When (2) is convergent, it represents $\log(1+x)$.

Example 3. $\Sigma x^n/n!$ (the Exponential Series) is convergent for all values of x.

$$u_{n+1}/u_n=\{x^{n+1}/(n+1)!\}/\{x^n/n!\},$$
$$=x/(n+1).$$

Hence, however great x may be, since it is independent of n, we may always choose r so great that, for all values of $n>r$, $x/(n+1)<1$. Since the limit of the ratio of convergence is zero in this case, we should expect the convergency for moderate values of x to be very rapid; and this is so, as we shall show by examining the residue in a later chapter.

Example 4. $\Sigma(-)^n m(m-1)\ldots(m-n+1)x^n/n!$ (x positive), where m has any real value,* is convergent if $x<1$, divergent if $x>1$.

For
$$\mathrm{L}u_{n+1}/u_n=-x\mathrm{L}\frac{m-n}{n+1}$$
$$=-x\mathrm{L}\frac{m/n-1}{1+1/n}$$
$$=x.$$

Hence the theorem.

The series just examined is the expansion of $(1-x)^m$ when $x<1$. It follows, by § 4, Th. IV., that the series $\Sigma m(m-1)\ldots(m-n+1)x^n/n!$, whose terms are ultimately alternately positive and negative, is convergent if $x<1$; this series is, as we shall see hereafter, the expansion of $(1+x)^m$ when $x<1$.

§ 6.] *Cauchy's Condensation Test.*—The general principle of this method, upon which many of the more delicate tests of convergence are founded, will be easily understood from the following considerations:—

Let Σu_n be a series of positive terms which constantly decrease in value from the first onwards. Without altering the order of these, we may associate them in groups according to some law. If $v_1, v_2, \ldots v_m, \ldots$ be the 1st, 2nd, ... mth, ... of these groups, the series Σv_m will contain all the terms of Σu_n; and it is obvious from the definition of convergency that Σu_n is convergent or divergent according as Σv_m is convergent or

* If m were a positive integer, the series would terminate, and the question of convergency would not arise.

divergent; we have in fact $\underset{n=\infty}{\mathrm{L}}\, \Sigma u_n = \underset{m=\infty}{\mathrm{L}}\, \Sigma v_m$. It is clear that the convergency or divergency of Σv_m will be more apparent than that of Σu_n, because in Σv_m we proceed by longer steps towards the limit, the sum of n terms of Σv_m being nearer the common limit than the sum of n terms of Σu_n. Finally, if $\Sigma v'_n$ be a new series such that $v'_n \gtrless v_n$, then obviously Σu_n is $\begin{matrix}\text{convergent}\\ \text{divergent}\end{matrix}$ if $\Sigma v'_n$ is $\begin{matrix}\text{convergent}\\ \text{divergent}\end{matrix}$.

We shall first apply this process of reasoning to the following case :—

Example. The series $1/1+1/2+\ldots+1/n+\ldots$ is divergent.

Arrange the given series in groups, the initial terms in which are of the following orders, $1, 2, 2^2, \ldots 2^m, 2^{m+1}, \ldots$. The numbers of terms in the successive groups will be $2-1, 2^2-2, 2^3-2^2, \ldots 2^{m+1}-2^m, 2^{m+2}-2^{m+1}, \ldots$ respectively. Since the terms constantly decrease in value, if 2^{m+1} be the greatest power of 2 which does not exceed n, then

$$\begin{aligned} S_n &< \frac{1}{1}+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2^2}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\ldots+\left(\frac{1}{2^m}+\frac{1}{2^m+1}+\ldots+\frac{1}{2^{m+1}-1}\right), \\ &> 1+(2^2-2)\frac{1}{2^2}+(2^3-2^2)\frac{1}{2^3}+\ldots+(2^{m+1}-2^m)\frac{1}{2^{m+1}}, \\ &> 1+\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}, \\ &> 1+\frac{m}{2}. \end{aligned}$$

Hence, by making n sufficiently great, we can make S_n as large as we please. The series $1/1+1/2+1/3+\ldots$ is therefore divergent. This might also be deduced from the inequality (6) of chap. XXV., § 25.

Cauchy's Condensation Test, of which the example just discussed is a particular case, is as follows :—

If $f(n)$ be positive for all values of n, and constantly decrease as n increases, then $\Sigma f(n)$ is convergent or divergent according as $\Sigma a^n f(a^n)$ is convergent or divergent, where a is any positive integer $\not< 2$.

The series $\Sigma f(n)$ may be arranged as follows :—

$$\begin{aligned} &[f(1)+\ldots+f(a-1)]+\{f(a)+f(a+1)+\ldots+f(a^2-1)\} \\ &\qquad +\{f(a^2)+f(a^2+1)+\ldots+f(a^3-1)\} \\ &\qquad \cdots\cdots\cdots\cdots \\ &\qquad +\{f(a^m)+f(a^m+1)+\ldots+f(a^{m+1}-1)\} \\ &\qquad \cdots\cdots\cdots\cdots \end{aligned}$$

Hence, neglecting the finite number of terms in the square brackets, we see that $\Sigma f(n)$ is convergent or divergent according as

$$\Sigma \{f(a^m) + f(a^m + 1) + \ldots + f(a^{m+1} - 1)\} \qquad (1)$$

is convergent or divergent. Now, since $f(a^m) > f(a^m + 1) > \ldots > f(a^{m+1} - 1) > f(a^{m+1})$, we have

$$(a^{m+1} - a^m) f(a^m) > f(a^m) + f(a^m + 1) + \ldots + f(a^{m+1} - 1) > (a^{m+1} - a^m) f(a^{m+1}),$$

that is,

$$(a - 1) a^m f(a^m) > f(a^m) + f(a^m + 1) + \ldots + f(a^{m+1} - 1) > \{(a - 1)/a\}\, a^{m+1} f(a^{m+1}).$$

Hence, by § 4, Th. I., the series (1) is convergent if $\Sigma(a - 1)\, a^m f(a^m)$ is convergent, divergent if $\Sigma\{(a - 1)/a\} a^{m+1} f(a^{m+1})$ is divergent. Now, by § 4, Th. II., $\Sigma(a - 1) a^m f(a^m)$ is convergent if $\Sigma a^m f(a^m)$ is convergent, and $\Sigma\{(a - 1)/a\} a^{m+1} f(a^{m+1})$ is divergent if $\Sigma a^{m+1} f(a^{m+1})$ is divergent; and for our present purpose $\Sigma a^m f(a^m)$ and $\Sigma a^{m+1} f(a^{m+1})$ are practically the same series, say $\Sigma a^n f(a^n)$. Hence Cauchy's Theorem is established.

N.B.—*It is obviously sufficient that the function $f(n)$ be positive and constantly decrease for all values of n greater than a certain finite value r.*

Cor 1. *The theorem will still hold if a have any positive value not less than 2.*

Let a lie between the positive integers b and $b + 1$, $(b \nless 2)$. If $\Sigma a^n f(a^n)$ be convergent, then $\underset{n=\infty}{\mathrm{L}}\, a^n f(a^n) = 0$, that is, $\underset{x=\infty}{\mathrm{L}}\, x f(x) = 0$. Hence, on and after some finite value of x, the function $xf(x)$ will begin to decrease constantly* as x increases. We must therefore have $(b + 1)^n f\{(b + 1)^n\} < a^n f(a^n)$, on and after some finite value of n. If, therefore, $\Sigma a^n f(a^n)$ is convergent, *a fortiori*, will $\Sigma(b + 1)^n f\{(b + 1)^n\}$ be convergent, and therefore, by Cauchy's Theorem, $\Sigma f(n)$ will be convergent.

If $\Sigma a^n f(a^n)$ be divergent, $xf(x)$ 1° may, or 2° may not decrease as x increases.

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* This assumes that $xf(x)$ has not an infinite number of turning values; so that we can take x so great that we are past the last turning value, which must be a maximum.

In case 1°, $b^n f(b^n) > a^n f(a^n)$. Hence the divergence of $\Sigma a^n f(a^n)$ involves the divergence of $\Sigma b^n f(b^n)$; and the divergence of $\Sigma f(n)$ follows by the main theorem.

In case 2°, the divergence of $\Sigma f(n)$ is at once obvious; for, if $\mathrm{L}_{x=\infty} xf(x) \neq 0$, then ultimately $xf(x) > \mathrm{A}$, where $\mathrm{A} > 0$. Hence $f(x) > \mathrm{A}/x$. Now $\Sigma \mathrm{A}/n$ is divergent, since $\Sigma 1/n$ is divergent; therefore $\Sigma f(n)$ is divergent.

In what follows we shall use ϵx, $\epsilon^2 x$, . . . to denote a^x, a^{a^x}, . . ., a being any positive quantity $\not< 2$; and λx, $\lambda^2 x$, . . . lx, $l^2 x$, . . . to denote $\log_a x$, $\log_a(\log_a x)$, . . . $\log_e x$, $\log_e(\log_e x)$, . . ., where e is Napier's Base.

Cor. 2. *$\Sigma f(n)$ is convergent or divergent according as $\Sigma \epsilon n \epsilon^2 n \ldots \epsilon^r n f(\epsilon^r n)$ is convergent or divergent.*

This follows, for integral values of the base a, by repeated application of Cauchy's Condensation Test.; and, for non-integral values of a, by repeated applications of Cor. 1. Thus $\Sigma f(n)$ is convergent or divergent according as $\Sigma \epsilon n f(\epsilon n)$ is convergent or divergent. Again, $\Sigma \epsilon n f(\epsilon n)$ is convergent or divergent according as $\Sigma \epsilon n \epsilon(\epsilon n) f\{\epsilon(\epsilon n)\}$, that is $\Sigma \epsilon n \epsilon^2 n f(\epsilon^2 n)$, is convergent or divergent; and so on.

Cor. 3. *$\Sigma f(n)$ is convergent or divergent according as the first of the functions*

$$\begin{aligned}
\mathrm{T}_0 &= \lambda f(x)/x, \\
\mathrm{T}_1 &= \lambda\{xf(x)\}/\lambda x, \\
\mathrm{T}_2 &= \lambda\{x\lambda x f(x)\}/\lambda^2 x, \\
\mathrm{T}_r &= \lambda\{x\lambda x \lambda^2 x \ldots \lambda^{r-1} x f(x)\}/\lambda^r x, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

which does not vanish when $x = \infty$, has a negative or a positive limit.

By Cor. 2, $\Sigma f(n)$ is convergent or divergent according as $\Sigma \epsilon n \epsilon^2 n \ldots \epsilon^r n f(\epsilon^r n)$ is convergent or divergent.

Now the latter series is (by § 5, Th. I.) convergent or divergent according as

$$\mathrm{L}_{n=\infty} \{\epsilon n \epsilon^2 n \ldots \epsilon^r n f(\epsilon^r n)\}^{1/n} < \text{or} > 1;$$

that is, according as

$$\mathrm{L}_{n=\infty} \log_a \{\epsilon n \epsilon^2 n \ldots \epsilon^r n f(\epsilon^r n)\}^{1/n} <> 0;$$

that is, $\underset{n=\infty}{\mathrm{L}} \log_a \{ \epsilon n \epsilon^2 n \ . \ . \ . \ \epsilon^r n f(\epsilon^r n) \} / n <> 0.$

If we put $x = \epsilon^r n$, so that $\lambda x = \epsilon^{r-1} n$, $\lambda^2 x = \epsilon^{r-2} n$, . . . $\lambda^{r-1} x = \epsilon n$, $\lambda^r x = n$, and $x = \infty$ when $n = \infty$, the condition for convergency or divergency becomes

$$\underset{x=\infty}{\mathrm{L}} \lambda \{ x \lambda x \lambda^2 x \ . \ . \ . \ \lambda^{r-1} x f(x) \} / \lambda^r x <> 0 \qquad (1).$$

If, on the strength of Cor. 1, we take e for the exponential base, the condition may be written

$$\underset{x=\infty}{\mathrm{L}} l \{ x l x l^2 x \ . \ . \ . \ l^{r-1} x f(x) \} / l^r x <> 0 \qquad (2),$$

where all the logs involved are Napierian logs.

We could establish the criterion (2) without the intervention of Cor. 1 by first establishing (1) for integral values of a, and then using the theorem of chap. xxv., § 12, Example 4, that $\underset{x=\infty}{\mathrm{L}} \lambda^r x / l^r x = 1/la$.

Cor. 4. *Each of the series*

$$\Sigma 1/n^{1+\alpha} \qquad (1),$$
$$\Sigma 1/n \{ ln \}^{1+\alpha} \qquad (2),$$
$$\Sigma 1/nln \{ l^2 n \}^{1+\alpha} \qquad (3),$$
. . . .
$$\Sigma 1/nlnl^2 n \ . \ . \ . \ l^{r-1} n \{ l^r n \}^{1+\alpha} \qquad (r+1),$$
. . . . . .

is convergent if $\alpha > 0$, *and divergent if* $\alpha =$ *or* < 0.

As the function $nlnl^2 n \ . \ . \ . \ l^r n$ frequently occurs in what follows, we shall denote it by $\mathrm{P}_r(n)$; so that $\mathrm{P}_0(n) = n$, $\mathrm{P}_1(n) = nln$, &c.

1*st Proof.*—Apply the criterion that $\Sigma f(n)$ is convergent or divergent according as $\mathrm{L} l \{ \mathrm{P}_r(x) f(x) \} / l^{r+1} x <> 0$. In the present case, $f(x) = 1/\mathrm{P}_r(x) (l^r x)^\alpha$. Hence

$$l \{ \mathrm{P}_r(x) f(x) \} / l^{r+1} x = l \{ 1/(l^r x)^\alpha \} / l^{r+1} x,$$
$$= -\alpha.$$

It follows that $(r+1)$ is convergent if $\alpha > 0$, and divergent if $\alpha < 0$. If $\alpha = 0$, the question is not decided. In this case, we must use the test function one order higher, namely, $l \{ \mathrm{P}_{r+1}(x) f(x) \} / l^{r+2} x$. Since $f(x) = 1/\mathrm{P}_r(x)$, we have

$$l\{P_{r+1}(x)f(x)\}/l^{r+2}x = l\{l^{r+1}x\}/l^{r+2}x,$$
$$= 1 > 0.$$

Hence, when $a = 0$, $(r+1)$ is divergent.

2nd Proof.—By the direct application of Cauchy's Condensation Test, the convergence of (1) is the same as the convergence of $\Sigma a^n/(a^n)^{1+\alpha}$, that is, $\Sigma(1/a^\alpha)^n$. Now the last series is a geometrical progression whose common ratio is $1/a^\alpha$; it is therefore convergent if $\alpha > 0$, and divergent if $\alpha =$ or < 0. Hence (1) is convergent if $\alpha > 0$, and divergent if $\alpha =$ or < 0.

Again, the convergence of (2) is by Cauchy's rule the same as the convergence of $\Sigma a^n/a^n\{la^n\}^{1+\alpha}$, that is, $\Sigma 1/(la)^{1+\alpha}n^{1+\alpha}$; and the convergence of this last the same as that of $\Sigma 1/n^{1+\alpha}$. Hence our theorem is proved for (2).

Let us now assume that the theorem holds up to the series (r). We can then show that it holds for $(r+1)$. In fact, the convergence of $(r+1)$ is the same as that of $\Sigma a^n/a^n la^n l^2 a^n \ldots l^{r-1}a^n\{l^r a^n\}^{1+\alpha}$, that is, $\Sigma 1/(nla)l(nla)\ldots l^{r-2}(nla)\{l^{r-1}(nla)\}^{1+\alpha}$.

First suppose $\alpha > 0$, and $a > e$. Then $la > 1$, $nla > n$. Hence

$$1/(nla)l(nla)\ldots l^{r-2}(nla)\{l^{r-1}(nla)\}^{1+\alpha}$$
$$< 1/nln\ldots l^{r-2}n\{l^{r-1}n\}^{1+\alpha}.$$

But, since $\alpha > 0$, $\Sigma 1/P_{r-1}(n)\{l^{r-1}n\}^\alpha$ is convergent, *a fortiori*, $\Sigma 1/P_r(n)\{l^r n\}^\alpha$ is convergent.

Next suppose $\alpha \not> 0$, and $2 < a < e$. Then $nla < n$; and, proceeding as before, we prove $\Sigma 1/P_r(n)\{l^r n\}^\alpha$ more divergent than the divergent series $\Sigma 1/P_{r-1}(n)\{l^{r-1}n\}^\alpha$.

Logarithmic Scale of Convergency.—The series just discussed are of great importance, inasmuch as they form a scale with which we can compare series whose ratio of convergence is ultimately unity. The scale is a descending one; for the least convergent of the convergent series of the rth order is more convergent than the most convergent of the convergent series of the $(r+1)$th order. This will be seen by comparing the nth terms, u_n and u'_n, of the rth and $(r+1)$th series. We have $u'_n/u_n = \{l^{r-1}n\}^\alpha/\{l^r n\}^{1+\alpha'}$, where α is very small but > 0, and α' is very large.

If we put $x = l^{r-1}n$, we may write $\underset{n=\infty}{\mathrm{L}}\, u'_n/u_n = \underset{x=\infty}{\mathrm{L}}\, \{x^{\alpha/(1+\alpha')}/lx\}^{1+\alpha'}$. Hence, however small α, so long as it is greater than 0, and however large α', $\mathrm{L}u'_n/u_n = \infty$.

If we suppose the character of the logarithmic scale established by means of the second demonstration given above, we may, by comparing Σu_n with the various series in the scale, and using § 4, Th. I., obtain a fresh demonstration of the criterion of Cor. 3. We leave the details as an exercise for the student. This is perhaps the best demonstration, because, apart from the criterion itself, nothing is presupposed regarding $f(x)$, except that it is positive when x is greater than a certain finite value.

By following the same course, and using § 4, Th. III., we can establish a new criterion for series whose ratio of convergence is ultimately unity, as follows, where $\rho_x = f(x+1)/f(x)$.

Cor. 5. *If $f(x)$ be always positive when x exceeds a certain finite value, $\Sigma f(n)$ is convergent or divergent according as the first of the following functions—*

$$\begin{aligned}
\tau_0 &= \rho_x - 1 ; \\
\tau_1 &= \mathrm{P}_0(x+1)\rho_x - \mathrm{P}_0(x) ; \\
\tau_2 &= \mathrm{P}_1(x+1)\rho_x - \mathrm{P}_1(x) ; \\
&\cdots\cdots\cdots\cdots \\
\tau_r &= \mathrm{P}_{r-1}(x+1)\rho_x - \mathrm{P}_{r-1}(x) ; \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

which does not vanish when $x = \infty$ has a negative or a positive limit.

Comparing $\Sigma f(n)$ with $\Sigma 1/\mathrm{P}_r(n)\{l^r n\}^\alpha$, we see that $\Sigma f(n)$ will be convergent if, for all values of x greater than a certain finite value,

$$\rho_x < \mathrm{P}_r(x)\{l^r x\}^\alpha / \mathrm{P}_r(x+1)\{l^r(x+1)\}^\alpha \qquad (1),$$

where $\alpha > 0$.

Now (1) is equivalent to

$$\mathrm{P}_r(x+1)\rho_x - \mathrm{P}_r(x) < \mathrm{P}_r(x)[\{l^r x/l^r(x+1)\}^\alpha - 1].$$

Also

$$\mathrm{LP}_r(x)[\{l^r x/l^r(x+1)\}^\alpha - 1]$$

$$= -\mathrm{LP}_{r-1}(x)\{l^r(x+1) - l^r x\} \cdot \frac{l^r x}{l^r(x+1)} \cdot \frac{\{l^r x/l^r(x+1)\}^\alpha - 1}{\{l^r x/l^r(x+1)\} - 1},$$

$$= -1 \times 1 \times \alpha = -\alpha,$$

by chap. xxv., §§ 12 and 13.

Hence a sufficient condition for the convergency of $\Sigma f(n)$ is

$$\underset{x=\infty}{\mathrm{L}} \{ \mathrm{P}_r(x+1)\rho_x - \mathrm{P}_r(x) \} < -a \ (a \text{ positive}),$$
$$< 0.$$

In like manner, the condition for divergency is shown to be

$$\underset{x=\infty}{\mathrm{L}} \{ \mathrm{P}_r(x+1)\rho_x - \mathrm{P}_r(x) \} > -a \ (a \text{ negative}),$$
$$> 0.$$

Example 1. Discuss the convergence of $\Sigma e^{-1-1/2-\ldots-1/n}/n^r$.

Here
$$\mathrm{T}_0 = l\{f(n)\}/n,$$
$$= -\frac{1+1/2+\ldots+1/n+rln}{n}.$$

Now, by chap. xxv., § 13, Example 1,

$$1+(r+1)ln > 1+1/2+\ldots+1/n+rln > rln+l(n+1).$$

Hence $\mathrm{LT}_0=0$. We must therefore examine T_1. Now

$$\mathrm{T}_1 = l\{nf(n)\}/ln,$$
$$= -\{1+1/2+\ldots+1/n+(r-1)ln\}/ln,$$
$$= -\{1+1/2+\ldots+1/n\}/ln-(r-1).$$

By chap. xxv., § 13, Example 2, $\mathrm{L}(1+1/2+\ldots+1/n)/ln=1$. Hence $\mathrm{LT}_1=-1-r+1=-r$. The given series is therefore convergent or divergent according as $r>$ or <0.

If $r=0$, $\mathrm{LT}_0=0$, and $\mathrm{LT}_1=0$. But we have

$$\mathrm{T}_2 = l\{nlnf(n)\}/l^2n,$$
$$= 1-\{1+1/2+\ldots+1/n-ln\}/l^2n.$$

Now, when n is very large, the value of $1+1/2+\ldots+1/n-ln$ approaches Euler's Constant. Hence $\mathrm{LT}_2=1>0$. In this case, therefore, the series under discussion is divergent.

Example 2. To discuss the convergence of the hypergeometric series,

$$1+\frac{\alpha.\beta}{\gamma.\delta}x+\frac{\alpha(\alpha+1).\beta(\beta+1)}{\gamma(\gamma+1).\delta(\delta+1)}x^2+\ldots$$

The general term of this series is

$$f(n)=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1).\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1).\delta(\delta+1)\ldots(\delta+n-1)}x^n.$$

The form of $f(n)$ renders the application of the first form of criterion somewhat troublesome. We shall therefore use the second. We have

$$\rho_n=\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(\delta+n)}x,$$
$$r_0=\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(\delta+n)}x-1,$$
$$\mathrm{L}r_0=x-1.$$

Hence the series is convergent if $x<1$, divergent if $x>1$.

If $x=1$, $\mathrm{L}r_0=0$, and we have

$$r_1=\frac{(n+1)(\alpha+n)(\beta+n)}{(\gamma+n)(\delta+n)}-n,$$
$$=\frac{(\alpha+\beta-\gamma-\delta+1)n^2+An+B}{n^2+Cn+D};$$
$$Lr_1=\alpha+\beta-\gamma-\delta+1.$$

If, therefore, $x=1$, the hypergeometric series is convergent or divergent according as $\alpha+\beta-\gamma-\delta+1<$ or >0.

If $\alpha+\beta-\gamma-\delta+1=0$, $Lr_1=0$. But we have

$$r_2=(n+1)l(n+1)\frac{(\alpha+n)(\beta+n)}{(\gamma+n)(\delta+n)}-nln,$$
$$=[n\{l(n+1)-ln\}+(\alpha+\beta+1)\{l(n+1)-ln\}+\{Al(n+1)+Bln\}/n$$
$$+Cl(n+1)/n^2]/[1+E/n+F/n^2],$$

Hence, since $L\,n\{l(n+1)-ln\}=1$, $L\{l(n+1)-ln\}=0$, $L\,l(n+1)/n^s=0$, $L\,ln/n^s=0(s>0)$, &c., we have

$$Lr_2=1>0.$$

In this case, therefore, the series is divergent.

Example 3. Consider the series

$$1-\frac{m}{1}+\frac{m(m-1)}{1.2}+\ldots+(-1)^n\frac{m(m-1)\ldots(m-n+1)}{1.2\ldots n}+\ldots$$

This may be written

$$1+\frac{-m}{1}+\frac{(-m)(-m+1)}{1.2}+\ldots+\frac{(-m)(-m+1)\ldots(-m+n-1)}{1.2\ldots n}+\ldots$$

It is therefore a hypergeometric series, in which $\alpha=-m$, $\beta=\gamma$, $\delta=1$, $x=1$. It follows from last article that the series in question is convergent or divergent according as $-m<>0$, that is, according as m is positive or negative.

This series is the expansion of $(1-x)^m$, when $x=1$.

Example 4. Consider the series

$$1+\frac{m}{1}+\frac{m(m-1)}{1.2}+\ldots+\frac{m(m-1)\ldots(m-n+1)}{1.2\ldots n}+\ldots \quad (1).$$

In this series the terms are ultimately alternatively positive and negative in sign. Hence the rules we have been using are not directly applicable.

1st. Let m be positive; and let $m-r$ be the first negative quantity among m, $m-1$, $m-2$, . . . &c., then, neglecting all the terms of the series before the $(r+1)$th, we have to consider

$$\frac{m(m-1)\ldots(m-r+1)}{1.2\ldots r}\left\{1+\frac{m-r}{r+1}+\frac{(m-r)(m-r-1)}{(r+1)(r+2)}+\ldots\right\} \quad (2).$$

If we change the signs of the alternate terms of the series within brackets, it becomes

$$1+\frac{r-m}{r+1}+\frac{(r-m)(r-m+1)}{(r+1)(r+2)}+\ldots \quad (3).$$

Now (3) is a hypergeometric series, in which $\alpha=r-m$, $\beta=\gamma$, $\delta=r+1$, $x=1$. Hence $\alpha+\beta-\gamma-\delta+1=r-m-(r+1)+1=-m<0$. Therefore (3) is convergent. Hence (2), and therefore (1), is absolutely convergent.

2nd. Let m be negative, $= -\mu$ say. The series (1) then becomes

$$1-\frac{\mu}{1}+\frac{\mu(\mu+1)}{1.2}-\ldots+(-1)^n\frac{\mu(\mu+1)\ldots(\mu+n-1)}{1.2\ldots n}+\ldots \quad (4).$$

Since μ is positive, the hypergeometric series

$$1+\frac{\mu}{1}+\frac{\mu(\mu+1)}{1.2}+\ldots+\frac{\mu(\mu+1)\ldots(\mu+n-1)}{1.2\ldots n}+\ldots \quad (5),$$

is divergent.

Hence (4) cannot be absolutely convergent in the present case.

Since $\rho_n=-(\mu+n)/(n+1)$, the terms will constantly increase in numerical value if $\mu>1$. Hence the series cannot be even semi-convergent unless $\mu<1$.

If μ be less than 1, $\rho_n<1$, and the series will be semi-convergent provided $Lu_n=0$.

Now $$\log u_n=\Sigma\log\frac{\mu+n}{1+n}=\Sigma\log\left\{1+\frac{\mu-1}{n+1}\right\}.$$

Since $L\log\{1+(\mu-1)/(n+1)\}/\{(\mu-1)/(n+1)\}=1$ (see chap. xxv., § 13), the series $\Sigma\log\{1+(\mu-1)/(n+1)\}$ and $\Sigma(\mu-1)/(n+1)$ both diverge to an infinity of the same sign. But the latter series diverges to $-\infty$ or $+\infty$, according as $\mu<$ or >1. Hence $Lu_n=0$ or ∞, according as $\mu<$ or >1.

Hence the series (1) is divergent if $\mu>1$, semi-convergent if $\mu<1$.

It obviously oscillates if $\mu=1$. Hence, to sum up, the series (1)

is absolutely convergent, if	$0\leqq m<+\infty$;
semi-convergent, if	$-1<m<0$;
oscillating, if	$-1=m$;
divergent, if	$-\infty<m<-1$.*

* *Historical Note.*—If we except a number of scattered theorems, given chiefly by Waring in his *Meditationes Analyticæ*, it may be said that Cauchy was the founder of the modern theory of convergent series; and most of the general principles of the subject were first given in his *Analyse Algébrique*. In his *Exercices de Mathématiques*, t. ii. (1827), he gave the following integral criterion from which most of the higher criteria have sprung:—If, for large values of n, $f(n)$ be positive and decrease as n increases, then $\Sigma f(n)$ is convergent if $L_{n=\infty}\int_n^{m+n}dx f(x)=0$ (m arbitrary), otherwise divergent. The second step of the r-criteria was first given by Raabe, *Crelle's Jour.*, Bd. xiii. (1835). De Morgan, in his *Differential Calculus*, p. 323 *et seq.* (1839), first gave the Logarithmic Scale of Functional Dimension, established the Logarithmic Scale of Convergency of Cor. 4, and stated criteria equivalent to, but not identical in form with, those of Cor. 3 and Cor. 5. Continental writers, nevertheless, almost invariably attribute the whole theory to Bertrand. Bertrand, *Liouv. Jour.* (1842), quotes De Morgan, stating that he had obtained independently part of De Morgan's results. His Memoir is very important, because it contains a discussion of various forms of the criteria and a demonstration of their equivalence; we have therefore attached his name, along with De Morgan's, to the two logarithmic criteria. Bonnet, *Liouv. Jour.* (1843), gave elementary demonstrations of Bertrand's formulæ; and Malmsten, *Grunert's Archiv* (1846), gave an elegant elementary demonstration, depending essentially

SERIES WHOSE TERMS HAVE PERIODICALLY RECURRING NEGATIVE SIGNS, OR CONTAIN A PERIODIC FACTOR SUCH AS SIN $n\theta$.

§ 7.] Series which contain an infinite number of negative terms may or may not be absolutely convergent. The former class falls under the cases already discussed. We propose now to give a few theorems regarding the latter class of series, whose convergency depends on the distribution of negative signs throughout the series.

The only cases of much practical importance are those—1st, where the infinity of negative signs has a periodic arrangement; 2nd, where the occurrence of negative signs is caused by the presence in the nth term of a factor, such as $\sin n\theta$, which is a periodic function of n.

In the former case (which might be regarded as a particular instance of the latter) we can always associate into a single term every succession of positive terms and every succession of negative terms. Since the recurrence of the positive and negative terms is periodic, we thus reduce all such series to the simpler case, where the terms are alternately positive and negative.

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on the inequality of chap. xxv., § 13, Cor. 6, that $\Sigma 1/P_r(m+n)\{l'(m+n)\}^a$ (where $l'm$ is positive) is convergent or divergent, according as $a <$ or < 0; and thence deduces Cor. 3. Paucker, *Crelle's Jour.*, Bd. xlii. (1851), deduces both Cor. 3 and Cor. 5 from Cauchy's Condensation Test, much as we have done, except that the actual form in which we have stated the rule of Cor. 5 is taken from Catalan, *Traité Él. d. Séries* (1860). Du Bois-Reymond, *Crelle's Jour.*, Bd. lxxvi. (1873), gives an elegant general theory embracing all the above criteria, and also those of Kummer, *Crelle's Jour.*, xiii. (1835). Abel had shown that, however slightly divergent Σu_n may be, it is always possible to find $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ such that $L\gamma_n = 0$ and yet $\Sigma\gamma_n u_n$ shall be divergent. Du Bois-Reymond shows that, however slowly Σu_n converge, we can always find $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ such that $L\gamma_n = \infty$ and $\Sigma\gamma_n u_n$ nevertheless shall be convergent. He shows that functions can be conceived whose ultimate increase to infinity is slower than that of any step in the logarithmic scale; and concludes definitely that there is a domain of convergency on whose borders the logarithmic criteria entirely fail—a point left doubtful by his predecessors. Finally, Kohn, *Grunert's Archiv* (1882), continuing Du Bois-Reymond's researches, gives a new criterion of a mixed character. The whole matter, although not of great importance as regards the ordinary applications of mathematics, illustrates an exceedingly interesting phase in the development of mathematical thought.

We may carry the process of grouping a step farther, and associate each negative with a preceding or following positive term, and the result will in general be a series whose terms are *ultimately* either all positive or all negative.

The process last indicated often enables us to settle the convergence of the series, but it must be remembered that the series derived by grouping is really a different series from the original one, because the sum of n terms of the original series does not always correspond to the sum of m terms of the derived series. The difference between the two sums will, however, never exceed the sum of a finite number of terms of the original series; and this difference must vanish for $n = \infty$, if the terms of the original series ultimately become infinitely small.

Example. Consider the series

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{3n-2}-\frac{1}{3n-1}-\frac{1}{3n}+\ldots \qquad (1).$$

Compare this with the series

$$\frac{1}{1}-\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{4}-\left(\frac{1}{5}+\frac{1}{6}\right)+\ldots+\frac{1}{3n-2}-\left(\frac{1}{3n-1}+\frac{1}{3n}\right)+\ldots \qquad (2),$$

that is, the series whose $(2n-1)$th term is $1/(3n-2)$, and whose $(2n)$th term is $-(1/(3n-1)+1/3n)$.

If S_n S_n' denote the sums of n terms of (1) and (2) respectively, then $S_{3n-2}=S_{2n'-1}$, $S_{3n-1}=S_{2n'-1}-1/(3n-1)$, $S_{3n}=S_{2n'}$. Since $\mathrm{L}1/(3n-1)=0$, we have in all cases $\mathrm{L}S_n=\mathrm{L}S_n'$. Hence (1) is convergent or divergent according as (2) is convergent or divergent. That (1) is really divergent may be shown by comparing it with the series

$$\Sigma\{1/(3n-2)-1/(3n-1)-1/3n\} \qquad (3).$$

If S_n'' denote the sums of n terms of this last series, we can show as before that $\mathrm{L}S_n''=\mathrm{L}S_n$. But the nth term of (3) can be written in the form $(-9+12/n-2/n^2)/(3-2/n)(3-1/n)3n$; and therefore bears to the nth term of $\Sigma 1/n$ a ratio which is never infinite. But $\Sigma 1/n$ is divergent.

By § 4, II., (3) is therefore also divergent. Hence (1) is divergent.

It should be noticed that in the case of an oscillating series, where $\mathrm{L}u_n \neq 0$, *the grouping of terms may convert a non-convergent into a convergent series; so that we cannot in this case infer the convergency of the original from the convergency of the derived series.**

* This remark is all the more important because the converse process of splitting up the nth term of a series into a group of terms with alternating signs, and using the rules of § 8, often gives a simple means of deciding as to its convergency. The series $1/1.2+1/3.4+1/5.6+1/7.8+\ldots$ may be tested in this way.

Example.

$$\left(1+\frac{1}{2}\right)^2 - \left(1+\frac{1}{3}\right)^2 + \ldots + \left(1+\frac{1}{2n}\right)^2 - \left(1+\frac{1}{2n+1}\right)^2 + \ldots$$

is obviously a non-convergent oscillating series. But

$$\left\{\left(1+\frac{1}{2}\right)^2 - \left(1+\frac{1}{3}\right)^2\right\} + \left\{\left(1+\frac{1}{4}\right)^2 - \left(1+\frac{1}{5}\right)^2\right\} + \ldots + \left\{\left(1+\frac{1}{2n}\right)^2 - \left(1+\frac{1}{2n+1}\right)^2\right\} + \ldots,$$

whose nth term is $(8n^2+8n+1)/(4n^2+2n)^2$, that is $(8+8/n+1/n^2)/16(1+/2n)^2n^2$, is convergent, being comparable in the scale of convergency with $\Sigma 1/n^2$.

§ 8.] The following rule is frequently of use in the discussion of semi-converging series:—

If $u_1 > u_2 > u_3 > \ldots > u_n > \ldots$ *and all be positive, then*

$$u_1 - u_2 + u_3 - \ldots (-)^{n-1}u_n + (-)^n u_{n+1} + \ldots \qquad (1)$$

converges or oscillates according as $\mathrm{L}_{n=\infty} u_n = 0$ *or* $\neq 0$.

Using the notation of § 3, we have

$$\begin{aligned} {}_m\mathrm{R}_n &= \pm (u_{n+1} - u_{n+2} + \ldots \pm u_{n+m}), \\ &= \pm \{u_{n+1} - (u_{n+2} - u_{n+3}) - \ldots\}, \\ &= \pm \{(u_{n+1} - u_{n+2}) + (u_{n+3} - u_{n+4}) + \ldots\}. \end{aligned}$$

Hence we have

$$u_{n+1} > {}_m\mathrm{R}_n > u_{n+1} - u_{n+2} \qquad (2),$$

numerical values being alone in question. If, therefore, $\mathrm{L}u_n = 0$, we have $\mathrm{L}u_{n+1} = \mathrm{L}u_{n+2} = 0$; and it follows that $\mathrm{L}_{n=\infty} {}_m\mathrm{R}_n = 0$ for all values of m. Also

$$u_1 > {}_n\mathrm{R}_0 = \mathrm{S}_n > u_1 - u_2,$$

so that S_n is finite for all values of n. The series (1) is therefore convergent if $\mathrm{L}u_n = 0$.

If $\mathrm{L}u_n = \alpha \neq 0$, then $\mathrm{L}_{n=\infty} {}_m\mathrm{R}_n = \alpha$ or $= 0$ according as m is odd or even. Hence the series is not convergent. We have, in fact, $\mathrm{L}(\mathrm{S}_{2n+1} - \mathrm{S}_{2n}) = \mathrm{L}u_{2n+1} = \alpha$, which shows that the sum of the series oscillates between S and S + α, where $\mathrm{S} = \mathrm{LS}_{2n}$.

Cor. *The series*

$$(u_1 - u_2) + (u_3 - u_4) + \ldots + (u_{2n-1} - u_{2n}) + \ldots$$

where $u_1, u_2, \ldots$ *are as before, is convergent.*

Example 1. The series $\Sigma(-1)^{n-1}/n$ is convergent, notwithstanding the fact, already proved, that $\Sigma 1/n$ is divergent.

Example 2. $\Sigma(-1)^{n-1}(n+1)/n$ is an oscillating series; but $\Sigma(-1)^{n-1}\{(n+1)/n-(n+2)/(n+1)\}$ is convergent.

§ 9.] The most important case of periodic series is $\Sigma a_n \cos(n\theta+\phi)$, where a_n is a function of n, and ϕ is independent of n, commonly spoken of as a *Trigonometrical* or *Fourier's Series*. The question of the convergence of this kind of series is one of great importance owing to their constant application in mathematical physics.

We observe in the first place that

I. *If* Σa_n *be an absolutely converging series then* $\Sigma a_n \cos(n\theta+\phi)$ *is convergent.*

This follows from § 4, I.

II. *If* $\theta = 0$ *or* $2k\pi$ (*k being an integer*), $\Sigma a_n \cos(n\theta+\phi)$ *is convergent or divergent according as* Σa_n *is convergent or divergent.*

This is obvious, since the series reduces to $\Sigma a_n \cos\phi$.

III. *If* $\theta \neq 0$ *or* $2k\pi$, *then* $\Sigma a_n \cos(n\theta+\phi)$ *is convergent if for all values of n greater than a certain finite value,* a_n *constantly decreases as n increases, in such a way that* $\mathrm{L}_{n=\infty} a_n = 0$.

This is a particular case of the following general theorem, which is founded on an inequality given by Abel:—

IV. *If* Σu_n *be convergent or oscillatory, and* $a_1, a_2, \ldots, a_n, \ldots$ *be a series of positive quantities, constantly decreasing as n increases, so that* $\mathrm{L}_{n=\infty} a_n = 0$, *then* $\Sigma a_n u_n$ *is convergent.*

Abel's Inequality is as follows:—If, for all values of n,

$$\mathrm{A} > u_1 + u_2 + \ldots + u_n > \mathrm{B},$$

where $u_1, u_2, \ldots, u_n$ are any real quantities whatever, and if $a_1, a_2, \ldots, a_n$ be a series of positive quantities constantly decreasing as n increases, then

$$a_1\mathrm{A} > a_1u_1 + a_2u_2 + \ldots + a_nu_n > a_1\mathrm{B}.$$

This may be proved as follows:—Let $\mathrm{S}_n = u_1 + u_2 + \ldots + u_n$, $\mathrm{S}_n' = a_1u_1 + a_2u_2 + \ldots + a_nu_n$. Then $u_1 = \mathrm{S}_1$, $u_2 = \mathrm{S}_2 - \mathrm{S}_1$, &c.; and

$$S_n' = a_1S_1 + a_2(S_2 - S_1) + \ldots + a_n(S_n - S_{n-1}),$$
$$= S_1(a_1 - a_2) + S_2(a_2 - a_3) + \ldots + S_{n-1}(a_{n-1} - a_n) + S_n a_n.$$

Hence, since $S_1, S_2, \ldots, S_n$ are each $< A$ and $> B$, and $(a_1 - a_2)$, $(a_2 - a_3), \ldots, (a_{n-1} - a_n), a_n$ are all positive,

$$\{(a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{n-1} - a_n) + a_n\} A$$
$$> S_n' > \{(a_1 - a_2) + (a_2 - a_3) + \ldots + (a_{n-1} - a_n) + a_n\} B;$$

that is,

$$a_1 A > S_n' > a_1 B \qquad (1).$$

Theorem IV. follows at once, for, since Σu_n is not divergent, S_n is not infinite for any value of n. Hence, by (1), S_n' is not infinite. Also, by Abel's Inequality,

$$a_{n+1}C > {}_mR_n' = a_{n+1}u_{n+1} + a_{n+2}u_{n+2} + \ldots + a_{n+m}u_{n+m}$$
$$= S_{n'+m} - S_n' > a_{n+1}D \qquad (2),$$

where C and D are the greatest and least of the values of ${}_mR_n (= u_{n+1} + u_{n+2} + \ldots + u_{n+m} = S_{n+m} - S_n)$ for all different positive values of m. Now, since Σu_n is convergent or oscillatory, $S_{n+m} - S_n$ is either zero or finite, and $\mathrm{L}_{n=\infty} a_{n+1} = 0$, by hypothesis. Therefore, it follows from (2), that $\mathrm{L}_{n=\infty} {}_mR_n' = 0$ for all values of m. Hence $\Sigma a_n u_n$ is convergent.

We shall prove in a later chapter that, when

$$u_n = \cos(n\theta + \phi),$$
$$S_n = \sin \tfrac{1}{2}n\theta \cos\{\tfrac{1}{2}(n+1)\theta + \phi\}/\sin \tfrac{1}{2}\theta.$$

If, therefore, we exclude the cases where $\theta = 0$ or $2k\pi$, we see that S_n cannot be infinite. Theorem III. is thus seen to be a particular case of Theorem IV.

Cor. *If a_n be as above, $\Sigma(-1)^{n-1}a_n \cos(n\theta + \phi)$, $\Sigma a_n \sin(n\theta + \phi)$, and $\Sigma(-1)^{n-1}a_n \sin(n\theta + \phi)$ are all convergent.*

CONVERGENCE OF A SERIES OF COMPLEX TERMS.

§ 10.] If the nth term of a series be of the form $x_n + y_n i$, where i is the imaginary unit, and x_n and y_n are functions of n, we may write the sum of n terms in the form $S_n + T_n i$, where

$$S_n = x_1 + x_2 + \ldots + x_n,$$
$$T_n = y_1 + y_2 + \ldots + y_n.$$

By the sum of the infinite series $\Sigma(x_n + y_n i)$ is meant the limit when $n = \infty$ of $S_n + T_n i$; that is, $(LS_n) + (LT_n)i$.

The necessary and sufficient condition for the convergency of $\Sigma(x_n + y_n i)$ *is therefore that* Σx_n *and* Σy_n *be both convergent.*

For, if the series Σx_n and Σy_n converge to the values S and T respectively, $\Sigma(x_n + y_n i)$ will converge to the value $S + Ti$; and, if either of the series Σx_n, Σy_n diverge or oscillate, then $(LS_n) + (LT_n)i$ will not have a finite definite value.

§ 11.] Let z_n denote $x_n + y_n i$; and let ρ_n be the modulus and θ_n the amplitude of z_n;* so that $z_n = \rho_n(\cos\theta_n + i\sin\theta_n)$, $x_n = \rho_n \cos\theta_n$, $y_n = \rho_n \sin\theta_n$. We have the following theorem, which is sufficient for most elementary purposes :—

The complex series Σz_n *is convergent if the real series* $\Sigma \bmod z_n$ *is convergent.*

For, since $\Sigma\rho_n$ is convergent, and ρ_n by its definition is always positive, it follows from § 4, I., that $\Sigma\rho_n \cos\theta_n$ and $\Sigma\rho_n \sin\theta_n$ are both convergent; that is, Σx_n and Σy_n are both convergent. Hence, by § 10, Σz_n is convergent.

It should be noticed that the condition thus established, *although sufficient, is not necessary.* For example, the series $(1-i)/1 - (1-i)/2 + (1-i)/3 - \ldots$ is convergent since $1/1 - 1/2 + 1/3 - \ldots$ and $-1/1 + 1/2 - 1/3 + \ldots$ are both convergent; but the series of moduli, namely, $\sqrt{2}/1 + \sqrt{2}/2 + \sqrt{2}/3 + \ldots$, is divergent.

When Σz_n *is such that* $\Sigma \bmod z_n$ *is convergent,* Σz_n *is said to be absolutely convergent.* Since the modulus of a real quantity u_n is simply u_n with its sign made positive, if need be, we see that the present definition of absolute convergency includes that formerly given, and that the theorem just proved includes § 4, IV., as a particular case.

Cor. *If* λ_n *be real, and* z_n *a complex number whose modulus is not infinite for any value of* n, *however great, then* $\Sigma(\lambda_n z_n)$ *will be absolutely convergent if* $\Sigma\lambda_n$ *is absolutely convergent.*

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* See chap. xii., § 13.

For $\text{mod}(\lambda_n z_n) = \text{mod}\,\lambda_n \,\text{mod}\, z_n$; and, since $\Sigma\lambda_n$ is absolutely convergent, $\Sigma \,\text{mod}\,\lambda_n$ is convergent. Hence, since $\text{mod}\, z_n$ is always finite, $\Sigma(\text{mod}\,\lambda_n \,\text{mod}\, z_n)$ is convergent by § 4, II.; that is, $\Sigma \,\text{mod}(\lambda_n z_n)$ is convergent. Hence $\Sigma(\lambda_n z_n)$ is absolutely convergent.

Example 1. The series $\Sigma z^n/n!$ is absolutely convergent for all finite values of z.

Example 2. The series $\Sigma z^n/n$ is absolutely convergent provided $\text{mod}\, z < 1$.

Example 3. The series $\Sigma(\cos\theta + i\sin\theta)^n/n$, is convergent if $\theta \neq 0$ or $2k\pi$. For the series $\Sigma\cos n\theta/n$ and $\Sigma\sin n\theta/n$ are convergent by § 9, III.

Example 4. The series $(\cos\theta + i\sin\theta)^n/n^2$ is absolutely convergent. For the series of moduli is $\Sigma 1/n^2$, which is convergent.

APPLICATION OF THE FUNDAMENTAL LAWS OF ALGEBRA TO INFINITE SERIES.

§ 12.] *Law of Association.*—We have already had occasion to observe that the law of association cannot be applied without limitation to an infinite series. It can, however, be applied without limitation provided the series is convergent. For let S_m' denote the sum of m terms of the new series obtained by associating the terms of the original series into groups in any way whatever. Then, if S_n denote the sum of n terms of the original series, we can always assume m so great that S_m includes at least all the terms in S_n. Hence $S_m' - S_n = {}_pR_n$, where p is a certain positive integer. Now, since the original series is convergent, by taking n sufficiently large we can make ${}_pR_n$ as small as we please. It follows therefore that $\underset{m=\infty}{\text{L}}\, S_m' = \underset{n=\infty}{\text{L}}\, S_n$. *Hence the association of terms produces no effect on the sum of the infinite convergent series.*

§ 13.] *Law of Commutation.*—The law of commutation is even more restricted in its application than the law of association. In fact, *the law of commutation can be applied only to absolutely convergent series.*

We shall consider here merely the case where each term of the series is displaced a finite number of steps.* Let Σu_n be

* See below, § 33, Cor. 2.

the original series, $\Sigma u_m'$ the new series obtained by commutation of the terms of Σu_n. Since each term is only displaced by a finite number of steps, we can, whatever n may be, by taking m sufficiently great always secure that S_m' contains all the terms of S_n at least. Under these circumstances $S_m' - S_n$ contains fewer terms than ${}_pR_n$, where p is finite, since m is finite. Now, since Σu_n is absolutely convergent, even if we take the most unfavourable case and suppose all the terms of the same sign, we shall have $\underset{n=\infty}{L}\ {}_pR_n = 0$; and, *a fortiori*, $\underset{m=\infty}{L}\ S_m' - \underset{n=\infty}{L}\ S_n = 0$. Hence $\underset{m=\infty}{L}\ S_m' = \underset{n=\infty}{L}\ S_n$; which establishes our theorem.

The above reasoning would not apply to a semi-convergent series, because the vanishing of $L\ {}_pR_n$ does not depend solely on the individual magnitude of the terms, but partially on the alternation of positive and negative signs.

Riemann has shown that the series $\Sigma(-1)^{n-1}u_n$, where $Lu_n = 0$, and Σu_{2n+1} and Σu_{2n} are both divergent, can, by proper commutation of its terms, be made to converge to any sum we please; and Dirichlet has shown that commutation may render a semi-convergent series divergent.

Example 1. The series

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{(2n-1)}} - \frac{1}{\sqrt{(2n)}} + \ldots \qquad (1)$$

is convergent by § 8; but the series

$$\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{4}}\right) + \ldots + \left(\frac{1}{\sqrt{(4m+1)}} + \frac{1}{\sqrt{(4m+3)}} - \frac{1}{\sqrt{(2m+2)}}\right) + \ldots \qquad (2),$$

which is evidently derivable from (1) by commutation (and an association which is permissible since the terms ultimately vanish), is divergent. For, if $u_m = 1/\sqrt{(4m+1)} + 1/\sqrt{(4m+3)} - 1/\sqrt{(2m+2)}$, and $v_m = 1/\sqrt{m}$, then $Lu_m/v_m = L\{1/\sqrt{(4+1/m)} + 1/\sqrt{(4+3/m)} - 1/\sqrt{(2+2/m)}\} = 1/2 + 1/2 - 1/\sqrt{2} = 1 - \frac{1}{2}\sqrt{2}$. Hence u_m/v_m is always finite; and Σv_m is divergent, by § 6, Cor. 4. Hence Σu_m is divergent. (Dirichlet.)

Example 2. The series

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots + \frac{1}{(2n-1)} - \frac{1}{(2n)} + \ldots \qquad (1),$$

$$\left(\frac{1}{1} + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \ldots + \left(\frac{1}{4m+1} + \frac{1}{4m+3}\right) - \frac{1}{2m+2} + \ldots \qquad (2),$$

are both convergent; but they converge to different sums. For, by taking

successively three and four terms of each series, we see that the sum of (1) lies between ·583 and ·833 ; whereas the sum of (2) lies between ·926 and 1·176.

Addition of two infinite series. If Σu_n *and* Σv_n *be both convergent, and converge to the values* S *and* T *respectively, then* $\Sigma(u_n + v_n)$ *is convergent and converges to the value* S + T.

We may, to secure complete generality, suppose u_n and v_n to be complex quantities. Let S_n, T_n, U_n represent the sums of n terms of Σu_n, Σv_n, $\Sigma(u_n + v_n)$ respectively; then we have, however great n may be, $U_n = S_n + T_n$. Hence, when $n = \infty$, $LU_n = LS_n + LT_n$, which proves the proposition.

§ 14.] *Law of Distribution.*—The application of the law of distribution will be indicated by the following theorems :—

If a be any finite quantity, and Σu_n *converge to the value S, then* $\Sigma a u_n$ *converges to aS.*

The proof of this is so simple that it may be left to the reader.

If Σu_n *and* Σv_n *converge to the values* S *and* T *respectively, and at least one of the two series be absolutely convergent, then the series*

$$u_1v_1 + (u_1v_2 + u_2v_1) + \ldots + (u_1v_n + u_2v_{n-1} + \ldots + u_nv_1) + \ldots \quad (1)$$

converges to the value ST.*

Let S_n, T_n, U_n denote the sums of n terms of Σu_n, Σv_n, $\Sigma(u_1v_n + u_2v_{n-1} + \ldots + u_nv_1)$ respectively; and let us suppose that Σu_n is absolutely convergent. We have

$$S_nT_n = U_n + L_n$$

where

$$\begin{aligned} L_n &= u_2v_n + u_3v_{n-1} + \ldots + u_nv_2 \\ &\quad + u_3v_n + \ldots + u_nv_3 \\ &\quad \ldots \\ &\quad + u_nv_n \\ &= u_2v_n + u_3(v_n + v_{n-1}) + \ldots + u_n(v_n + \ldots + v_2) \quad (2). \end{aligned}$$

* The original demonstration of this theorem given by Cauchy in his *Analyse Algébrique* required that both the series Σu_n, Σv_n be absolutely convergent. Abel's demonstration is subject to the same restriction. The more general form was given by Mertens, *Crelle's Jour.*, lxxix. (1875). Abel had, however, proved a more general theorem (see § 20, Cor.), which partly includes the result in question.

If therefore n be even, $= 2m$ say,

$$L_n = [u_2v_{2m} + u_3(v_{2m} + v_{2m-1}) + \ldots + u_m(v_{2m} + \ldots + v_{m+2})] + [u_{m+1}(v_{2m} + \ldots + v_{m+1}) + \ldots + u_{2m}(v_{2m} + \ldots + v_2)] \quad (3).$$

If n be odd, $= 2m + 1$ say,

$$L_n = [u_2v_{2m+1} + u_3(v_{2m+1} + v_{2m}) + \ldots + u_m(v_{2m} + \ldots + v_{m+3})] + [u_{m+1}(v_{2m+1} + \ldots + v_{m+2}) + \ldots + u_{2m+1}(v_{2m+1} + \ldots + v_2)] \quad (4).$$

Now, since Σv_n is convergent, it is possible, by making m sufficiently great, to make each of the quantities $\mathrm{mod}\, v_{2m}$, $\mathrm{mod}\,(v_{2m-1} + v_{2m})$, . . ., $\mathrm{mod}\,(v_{m+2} + \ldots + v_{2m})$, $\mathrm{mod}\, v_{2m+1}$, $\mathrm{mod}\,(v_{2m} + v_{2m+1})$, . . ., $\mathrm{mod}\,(v_{m+2} + \ldots + v_{2m+1})$ as small as we please. Also, since $\mathrm{mod}\, T_1$, $\mathrm{mod}\, T_2$, $\mathrm{mod}\, T_3$, . . . $\mathrm{mod}\, T_n$, . . . are all finite, and $\mathrm{mod}\,(T_r - T_s) < \mathrm{mod}\, T_r + \mathrm{mod}\, T_s$, therefore

$$\mathrm{mod}\,(v_{m+1} + \ldots + v_{2m}), \ \ldots, \ \mathrm{mod}\,(v_2 + \ldots + v_{2m}),$$
$$\mathrm{mod}\,(v_{m+2} + \ldots + v_{2m+1}), \ \ldots, \ \mathrm{mod}\,(v_2 + \ldots + v_{2m+1}),$$

are all finite. Hence, if ϵ_m be a quantity which can be made as small as we please by sufficiently increasing m, and β a certain finite quantity, we have, from (3) and (4), by chap. xii., § 11,

$$\mathrm{mod}\, L_n < \epsilon_m(\mathrm{mod}\, u_2 + \mathrm{mod}\, u_3 + \ldots + \mathrm{mod}\, u_m) + \beta(\mathrm{mod}\, u_{m+1} + \mathrm{mod}\, u_{m+2} + \ldots + \mathrm{mod}\, u_n).$$

If, therefore, we make n infinite, and observe that, since Σu_n is absolutely convergent, $\mathrm{mod}\, u_2 + \mathrm{mod}\, u_3 + \ldots + \mathrm{mod}\, u_n$ is finite, and $L(\mathrm{mod}\, u_{m+1} + \mathrm{mod}\, u_{m+2} + \ldots + \mathrm{mod}\, u_n) = 0$, we have (seeing that $L\epsilon_m = 0$) $L \,\mathrm{mod}\, L_n = 0$. Hence $LS_nT_n = LU_n$, that is, $LU_n = ST$.

Cauchy has shown that, *if both the series involved be semi-convergent, the multiplication rule does not necessarily apply.*

Suppose, for example, $u_n = v_n = (-1)^{n-1}/\sqrt{n}$. Then both Σu_n and Σv_n are semi-convergent series. The general term of (1) is

$$w_n = \pm\left(\frac{1}{\sqrt{\{n\}}} + \frac{1}{\sqrt{\{(n-1)2\}}} + \ldots + \frac{1}{\sqrt{\{2(n-1)\}}} + \frac{1}{\sqrt{\{n\}}}\right) \quad (5).$$

Now, since $r(n-r+1) = \frac{1}{4}(n+1)^2 - \{\frac{1}{2}(n+1) - r\}^2$, therefore, for all values of r, $r(n-r+1) < \frac{1}{4}(n+1)^2$, except in the case where $r = \frac{1}{2}(n+1)$, and then there is equality. It follows that $\mathrm{mod}\, w_n > n/\frac{1}{2}(n+1) > 2/(1+1/n)$. The terms of Σw_n are therefore ultimately numerically greater than a quantity which is infinitely nearly equal to 2. Hence Σw_n cannot be a convergent series.

SPECIAL DISCUSSION OF THE POWER SERIES $\Sigma a_n x^n$.

§ 15.] As the series $\Sigma a_n x^n$ is of great importance in algebraical analysis we shall give a special discussion of its properties as regards both convergence and continuity. We may speak of it for shortness as the Power Series, and we shall consider both a_n and x to be complex numbers; say $a_n = r_n(\cos \alpha_n + i \sin \alpha_n)$, $x = \rho(\cos \theta + i \sin \theta)$, where r_n and α_n are functions of the integral variable n, but ρ and θ are independent of n.

§ 16.] *$\Sigma a_n x^n$ is convergent if mod $x <$ L $\{$mod a_n/mod $a_{n+1}\}$.*

For the series of moduli is $\Sigma r_n \rho^n$, and this is convergent if L $\{\rho^{n+1} r_{n+1}/\rho^n r_n\} < 1$; that is, if ρL $\{r_{n+1}/r_n\} < 1$; that is, if $\rho <$ L $\{r_n/r_{n+1}\}$.

Three different cases arise according as L $\{r_n/r_{n+1}\}$ is zero, a finite positive quantity R, or ∞. In the first case, $\Sigma a_n x^n$ is not convergent for any value of x other than 0.

In the second case $\Sigma a_n x^n$ is convergent when the point representing x in Argand's Diagram lies *within* a circle whose centre is the origin and whose radius is R. This circle is called the *Circle of Convergence* for the power series in question; and R is called the *Radius of Convergence.* It should be observed that nothing is established for the case where the representative point lies *on* the circle of convergence.

$\Sigma x^n/n$ is an example of this class of series; here R = 1.

In the third case, $\Sigma a_n x^n$ is convergent for all values of x.

The exponential series $\Sigma x^n/n!$ is an example of this class of series.

§ 17.] *If the series $\Sigma a_n x^n$ be absolutely convergent when mod x = R′, it will be absolutely convergent when mod x = R″ < R′.*

For, since $\Sigma a_n x^n$ is absolutely convergent, $\Sigma r_n R'^n$ is convergent. Now, since R″ < R′, $r_n R''^n < r_n R'^n$. Hence, by § 4, I, $\Sigma r_n R''^n$ is convergent; that is, $\Sigma a_n x^n$ is absolutely convergent when mod x = R″.

§ 18.] *Discontinuity and Infinitely slow Convergency.* If the nth term of an infinite series be $f(n,x)$, where $f(n,x)$ is a single valued continuous function of x for all integral values of n, then

the infinite series $\Sigma f(n,x)$ will, if convergent, be a single valued finite function of x, say $\phi(x)$. At first sight, it might be supposed that $\phi(x)$ must necessarily be continuous, seeing that each term of $f(n,x)$ is so. Cauchy took this view; but, as Abel first pointed out, $\phi(x)$ is not necessarily continuous. No doubt $\Sigma f(n,x+h)$ and $\Sigma f(n,x)$, being each convergent, have each definite finite values, and therefore $\Sigma\{f(n,x+h)-f(n,x)\}$ is convergent, and has a definite finite value; *but this value is not necessarily zero for all values of* x. Suppose, for example,* that $f(n,x) = x/(nx+1)\,(nx-x+1)$. Since $f(n,x) = nx/(nx+1) - (n-1)x/\{n-1x+1\}$, we have, in this case, $S_n = nx/(nx+1)$. Hence, provided $x \neq 0$, $LS_n = 1$. If, however, $x = 0$ then $S_n = 0$, however great n may be. The function $\phi(x)$ is, therefore, in this case, discontinuous when $x = 0$.

The discontinuity of the above series is accompanied by another peculiarity which is often, although not always, associated with discontinuity. The Residue of the series, when $x \neq 0$, is given by

$$R_n = 1 - S_n = 1/(nx+1).$$

Now, when x has any given value, we can by making n large enough make $1/(nx+1)$ smaller than any given positive quantity a. But, on the other hand, the smaller x is, the larger must we take n in order that $1/(nx+1)$ may fall under a; and, in general, when x is variable, there is no finite upper limit for n, *independent of* x, say v, such that if $n > v$ then $R_n < a$. When the residue has this peculiarity the series is said to be *non-uniformly convergent;* and, if for a particular value of x, such as $x = 0$, in the present example, the number of terms required to secure a given degree of approximation to the limit is infinite, the series is said to *Converge Infinitely Slowly.*

We are thus led to the following important definition. *If, for values of* x *within a given region in Argand's Diagram, we can for every value of* a, *however small mod* a, *assign for* n *an upper limit* v, INDEPENDENT OF x, *such that when* $n > v$ *mod* $R_n <$ *mod* a,

* Du Bois-Reymond.

then the series $\Sigma f(n,x)$ *is said to be* UNIFORMLY CONVERGENT *within the region in question.**

It can be shown that, so long as $\Sigma f(n,x)$ converges uniformly, $\phi(x)$ cannot be discontinuous; but Du Bois-Reymond has shown by means of the example $\Sigma \{x/n(nx+1)(nx-x+1) - x^2/(nx^2+1)(nx^2-x+1)\}$ that infinitely slow convergence may not involve discontinuity. In point of fact, the sum of this last series is always zero, even when $x=0$; and yet, when $x=0$, the convergence is infinitely slow.

The object of the present paragraph has been, not to introduce the student to a discussion of exceptional cases in the functionality of infinite series, but to lead him to see the necessity of the demonstrations now to be given of the continuity of the Power Series in certain cases.

§ 19.] As regards the power series $\Sigma a_n x^n$ there are two cases of great practical importance—1st, when a_n is independent of x and we regard $\Sigma a_n x^n$ as a function of x, say $\phi(x)$; 2nd, when a_n is a function of n and y, say $f(n,y)$, and x is regarded as constant, so that $\Sigma f(n,y)x^n$ is a function of y, say $\psi(y)$.

The points involved were first raised and discussed by Abel; but the following theorem, together with its elegant demonstration,† will give us at once all that is here required.

Let μ_n *be independent of* z, *and* $w_n(z)$ *be a single valued function of* n *and* z, *finite for all values of* n, *however great, and finite and continuous as regards* z *from* $z=a$ *to* $z=b$, *then, if* $\Sigma\mu_n$ *be absolutely convergent,* $\Sigma\mu_n w_n(z)$ ‡ *is a continuous function of* z *from* $z=a$ *to* $z=b$.

Let $S_n(z) = \mu_1 w_1(z) + \mu_2 w_2(z) + \ldots + \mu_n w_n(z)$, and assume μ_n to be positive for all values of n, which will not limit the

* The distinction here involved was first pointed out by Seidel, *Abhl. d. Bayerischen Akad. d. Wiss.*, Bd. v. (1850). It has assumed great importance in the Theory of Functions developed by Weierstrass and his followers.

† Both due to Du Bois-Reymond. See *Math. Ann.*, iv. (1871). We have presented the original notation and phraseology as closely as possible.

‡ Under the circumstances supposed, $\Sigma\mu_n w_n(z)$ is, of course, convergent by § 4.

generality of the demonstration, since $\Sigma\mu_n$ is absolutely convergent.

Let $\Delta w_p = w_p(z+h) - w_p(z)$, so that $\lim_{h=0} \Delta w_p = 0$ for all values of p. Then we have

$$\begin{aligned} S_n(z+h) - S_n(z) = {} & \mu_1 \Delta w_1 + \mu_2 \Delta w_2 + \ldots + \mu_m \Delta w_m \\ & + \mu_{m+1} w_{m+1}(z+h) + \mu_{m+2} w_{m+2}(z+h) + \ldots \\ & \quad + \mu_n w_n(z+h) \\ & - \mu_{m+1} w_{m+1}(z) - \mu_{m+2} w_{m+2}(z) - \ldots \\ & \quad - \mu_n w_n(z). \end{aligned}$$

Let ΔW_m be a mean among $\Delta w_1, \Delta w_2, \ldots, \Delta w_m$ (that is, be greater than the least and less than the greatest);

W'_{mn} a mean among $w_{m+1}(z+h), w_{m+2}(z+h), \ldots, w_n(z+h)$;

W_{mn} a mean among $w_{m+1}(z), w_{m+2}(z), \ldots, w_n(z)$.

Then

$$S_n(z+h) - S_n(z) = \Delta W_m S'_m + (W'_{mn} - W_{mn})_{n-m} R'_m$$

where S'_m and ${}_{n-m}R'_m$ have the usual meanings as regards $\Sigma\mu_m$.

If, now, we make n infinite, $W'_{m\infty}$ and $W_{m\infty}$ become, by virtue of our hypotheses, finite determinate quantities for every value of m and z; and we have

$$S_\infty(z+h) - S_\infty(z) = \Delta W_m S'_m + (W'_{m\infty} - W_{m\infty}) R'_m.$$

Our object is to prove that $\lim_{h=0} \{S_\infty(z+h) - S_\infty(z)\}$ (say, $S_\infty(z \pm 0) - S_\infty(z)) = 0$. Now when $h = 0$, $\Delta w_1 = 0$, $\Delta w_2 = 0, \ldots, \Delta w_m = 0$, since all the functions $w_1(z), w_2(z), \ldots, w_m(z)$ are continuous. Therefore, since S'_m is finite for all values of m, owing to the convergency of $\Sigma\mu_m$, we have

$$S_\infty(z \pm 0) - S_\infty(z) = (\lim_{h=0} W'_{m\infty} - W_{m\infty}) R'_m \qquad (1).$$

We cannot be sure that $\lim W'_{m\infty} = W_{m\infty}$; but since both are finite their difference is finite. Hence, since R'_m is the residue of the convergent series $\Sigma\mu_m$, by making m sufficiently great we can make R'_m and therefore the right-hand side of (1) as small as we please. It follows that the left-hand side of (1) must be numerically less than any assignable quantity; that is, must be zero.

We have supposed all the quantities involved to be real; but the extension to the case where μ_n and $w_n(x)$ are complex is obvious after what has been said in §§ 10, 11.

As an example, take the first of the series discussed in § 18. The nth term may be written $\{1/n^2\}\{x/(x+1/n)(x-x/n+1/n)\}$. Hence, if we take $\mu_n = 1/n^2$, $w_n(x) = x/(x+1/n)(x-x/n+1/n)$, we see that all the conditions of Du Bois-Reymond's Theorem are fulfilled, except when $x=0$; for $w_\infty(x) = 1/x$, which becomes infinite when $x=0$. We conclude therefore that this infinite series is a continuous function of x for all positive values of x except $x=0$, in which case the theorem does not apply.

Cor. 1. *If the power series* $\Sigma a_n x^n$ *be absolutely convergent when* $\mod x = \mathrm{R}$, *then, for all values of* x *such that* $\mod x < \mathrm{R}$, $\Sigma a_n x^n$ *is a continuous function of* x.

We have $a_n x^n = a_n \mathrm{R}^n (x/\mathrm{R})^n$. Now $\Sigma a_n \mathrm{R}^n$ is an absolutely convergent series by hypothesis. Hence, if we take $\mu_n = a_n \mathrm{R}^n$, $w_n(x) = (x/\mathrm{R})^n$, all the conditions of Du Bois-Reymond's Theorem will be satisfied, and the corollary follows.

Cor. 2. *If the power series* $\Sigma f(n,y) x^n$ *be convergent when* $\mod x = \mathrm{R}\ (<1)$, *and* $f(n,y)$ *be a function of* y *which is finite and single-valued for all values of* n, *and finite, single-valued, and continuous as regards* y *from* $y=a$ *to* $y=b$, *then, from* $y=a$ *to* $y=b$, $\psi(y) = \Sigma f(n,y) x^n$ *is a continuous function of* y *so long as* $\mod x \not> \mathrm{R}$.

This follows at once from Du Bois-Reymond's Theorem, if we take $\mu_n = x^n$, $z = y$, and $w_n(z) = f(n,y)$.

§ 20.] We have seen that, so long as x lies within the circle of convergence, the power series $\Sigma a_n x^n$ is a continuous function of x. Nothing, however, has been established regarding values of x that lie on the circumference of the circle of convergence. Hence the importance of the following theorem of Abel's, which we prove for real series, but which can at once (see § 10) be extended to imaginary series.

If the series Σa_n *be convergent, and if* $\Sigma a_n x^n$ *be convergent for all values of* x *less than* 1, *then* $\mathrm{L}_{x=1-0} \Sigma a_n x^n = \Sigma a_n$.

This is tantamount to asserting the continuity of $\Sigma a_n x^n$ up to the circumference of its circle of convergency, so far as real values are concerned.* If $f(x)$ denote $\Sigma a_n x^n$ we have to show that $\mathrm{L}_{x=1-0} f(x)$, say $f(1-0)$, $= \Sigma a_n$.

* Proofs have been given by Abel, Dirichlet, Du Bois-Reymond, and others. The above is a modification of Dirichlet's demonstration (see *Liouville's Jour.*)

Since Σa_n is convergent, if $s_0 = a_0$, $s_1 = a_0 + a_1$, . . ., $s_n = a_0 + a_1 + \ldots + a_n$, . . . then s_0, s_1, . . ., s_n . . . are all finite, and have for their limit S, the sum of the infinite series Σa_n. Also we have $a_0 = s_0$, $a_1 = s_1 - s_0$, $a_2 = s_2 - s_1$, . . ., $a_n = s_n - s_{n-1}$, . . .

Hence

$$f(x) = s_0 + (s_1 - s_0)x + (s_2 - s_1)x^2 + \ldots + (s_n - s_{n-1})x^n + \ldots$$
$$= s_0(1 - x) + s_1x(1 - x) + \ldots + s_nx^n(1 - x) + \ldots$$

This transformation will be legitimate so long as x is less than 1, by however little.

Let $x = 1 - \xi$, then we have, however small ξ may be,

$$f(1 - \xi) = s_0\xi + s_1(1 - \xi)\xi + \ldots + s_{n-1}(1 - \xi)^{n-1}\xi + s_n(1 - \xi)^n\xi + s_{n+1}(1 - \xi)^{n+1}\xi + \ldots,$$

wherein n may be taken as large as we please.

Let now σ'_n be a mean among s_0, $(1 - \xi)s_1$, . . ., $(1 - \xi)^{n-1}s_{n-1}$, and σ_n a mean among s_n, s_{n+1}, s_{n+2} . . .

Then $L\sigma'_n$ is finite, and $L\sigma_n = s$.

We have

$$f(1 - \xi) = n\xi\sigma'_n + \{1 + (1 - \xi) + (1 - \xi)^2 + \ldots\}\xi(1 - \xi)^n\sigma_n,$$
$$= n\xi\sigma'_n + (1 - \xi)^n\sigma_n.$$

Since n may be made as large as we please, we may cause ξ to approach the limit zero by putting $\xi = 1/n^2$, and then making $n = \infty$. Hence

$$f(1 - 0) = L\sigma'_n/n + L(1 - 1/n^2)^n\sigma_n.$$

Now, $L\sigma'_n/n = 0$, since $L\sigma'_n$ is finite.

Also $L(1 - 1/n^2)^n = L\{(1 - 1/n^2)^{-n^2}\}^{-1/n} = Le^{-1/n} = e^0 = 1$; and $L\sigma_n = s$. Hence $L(1 - 1/n^2)^n\sigma_n = s$; and we have finally

$$f(1 - 0) = s,$$
$$= \Sigma a_n.$$

It should be observed that Σa_n need not be absolutely convergent, but if it be semi-convergent the order of its terms must not be altered.

By considering the series Σu_nx^n, Σv_nx^n, and the series $\Sigma(u_nv_1 + u_{n-1}v_2 + \ldots + u_1v_n)x^{n+1}$, which is their product when both of them are absolutely convergent, and applying the theorem just established, we easily arrive at the following theorem, also due to Abel.

Cor. *If each of the series* Σu_n *and* Σv_n *converge to the limits* u *and* v *respectively, then, if the series* $\Sigma(u_n v_1 + u_{n-1}v_2 + \ldots + u_1 v_n)$ *be convergent, it will converge to* uv*; and this will hold even if all the three series be only semi-convergent.*

§ 21.] *Principle of Indeterminate Coefficients.*

If the real series $\Sigma a_n x^n$ *be convergent for all values of* x *such that mod* $x \not> R$, *and if for all the values in question* $a_0 + \Sigma a_n x^n = 0$, then $a_0 = 0$, $a_1 = 0$, $a_2 = 0, \ldots, a_n = 0, \ldots$

Since $\Sigma a_n x^n$ is convergent, it follows that $\underset{x=0}{\mathrm{L}}\, \Sigma a_n x^n = 0$. Since $a_0 + \Sigma a_n x^n = 0$ when $x = 0$, we must have $a_0 = 0$. Therefore, by our original hypothesis, we have $\Sigma a_n x^n = 0$ for all values of x such that mod $x \not> R$. Now, by § 14, $\Sigma a_n x^n = x \Sigma a_n x^{n-1}$, where $\Sigma a_n x^{n-1}$ is a convergent series for any value of x which renders $\Sigma a_n x^n$ convergent. Since, then, we have $x \Sigma a_n x^{n-1} = 0$ for values of x other than 0, it follows that $\Sigma a_n x^{n-1} = 0$. But, since $\Sigma a_n x^{n-1}$ is convergent, $\underset{x=0}{\mathrm{L}}\, \Sigma a_n x^{n-1} = a_1$. Thus we must have $a_1 = 0$. Proceeding in this way, we can show that all the coefficients must vanish.

Cor. *If, for all values of* x *such that mod* $x \not> R$, $a_0 + \Sigma a_n x^n = b_0 + \Sigma b_n x^n$, *both series being convergent, then* $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, $\ldots, a_n = b_n, \ldots$

For we must have $(a_0 - b_0) + \Sigma(a_n - b_n)x^n = 0$ where, by § 13, $\Sigma(a_n - b_n)x^n$ is a convergent series. Hence, by the main theorem, $a_0 - b_0 = 0$, $a_1 - b_1 = 0$, &c.

INFINITE PRODUCTS.

§ 22.] The product of an infinite number of factors formed in given order according to a definite law is called an Infinite Product. Since, as we shall presently see, it is only when the factors ultimately become unity that the most important case arises, we shall write the nth factor in the form $1 + u_n$.

By the value of the infinite product is meant the limit of

$$(1 + u_1)(1 + u_2) \ldots (1 + u_n),$$

(which may be denoted by $\overset{n}{\Pi}(1 + u_n)$, or simply by P_n), when n is increased without limit.

It is obvious that if Lu_n were numerically greater than unity, then LP_n would be either zero or infinite. As neither of these cases is of any importance we shall, in what follows, suppose $\text{mod}\, u_n$ to be always less than unity. Any *finite* number of factors at the commencement of the product for which this is not true, may be left out of account in discussing the convergency. We also suppose any factor that becomes zero to be set aside; the question as to convergency then relates merely to the product of all the remaining factors.

Four essentially distinct cases arise—

1st. LP_n may be 0.

2nd. LP_n may be a finite definite quantity, which we may denote by $\Pi(1 + u_n)$, or simply by P.

3rd. LP_n may be infinite.

4th. LP_n may have no definite value; but assume one or other of a series of values according to the integral character of n.

In cases 1 and 2 the infinite product might be said to be *convergent*; it is, however, usual to confine the term *convergent* to the 2nd case, and to this convenient usage we shall adhere; in case 3 *divergent*; in case 4 *oscillatory*.

§ 23.] If, instead of considering P_n, we consider its logarithm, we reduce the whole theory of infinite products (so far as real positive factors are concerned *) to the theory of infinite series; for we have

$$\log P_n = \log(1 + u_1) + \log(1 + u_2) + \ldots + \log(1 + u_n)$$
$$= \overset{n}{\Sigma} \log(1 + u_n);$$

and we see at once that

1st. If $\Sigma \log(1 + u_n)$ is divergent, and $L\overset{n}{\Sigma} \log(1 + u_n) = -\infty$, then $\Pi(1 + u_n) = 0$; and conversely.

2nd. If $\Sigma \log(1 + u_n)$ be convergent, then $\Pi(1 + u_n)$ is convergent.

3rd. If $\Sigma \log(1 + u_n)$ is divergent, and $L\overset{n}{\Sigma} \log(1 + u_n) = +\infty$, then $\Pi(1 + u_n)$ is divergent.

* The logarithm of a complex number has not yet been *defined*, much less discussed.

4th. If $\Sigma \log(1 + u_n)$ oscillates, then $\Pi(1 + u_n)$ oscillates.

§ 24.] If we confine ourselves to the case where u_n has ultimately always the same sign, it is easy to deduce a simple criterion for the convergency of $\Pi(1 + u_n)$.

If $Lu_n < 0$, then $\Sigma \log(1 + u_n) = -\infty$, and $\Pi(1 + u_n) = 0$.

If $Lu_n > 0$, $\Sigma \log(1 + u_n) = +\infty$, and $\Pi(1 + u_n)$ is divergent.

It is therefore a necessary condition for the convergency of $\Pi(1 + u_n)$ *that* $Lu_n = 0$.

Since $Lu_n = 0$, $L(1 + u_n)^{1/u_n} = e$; hence $L \log(1 + u_n)/u_n = 1$. It therefore follows from § 4 that $\Sigma \log(1 + u_n)$ is convergent or divergent according as Σu_n is convergent or divergent. Moreover, if u_n be ultimately negative, the last and infinite parts of Σu_n and $\Sigma \log(1 + u_n)$ will be negative; and if u_n be ultimately positive the last and infinite part of Σu_n and $\Sigma \log(1 + u_n)$ will be positive. Hence the following conclusions—

If the terms of Σu_n *become ultimately infinitely small, and have ultimately the same sign, then*

1st. $\Pi(1 + u_n)$ *is convergent, if* Σu_n *be convergent; and conversely.*

2nd. $\Pi(1 + u_n) = 0$, *if* Σu_n *diverge to* $-\infty$; *and conversely.*

3rd. $\Pi(1 + u_n)$ *diverges to* $+\infty$, *if* Σu_n *diverge to* $+\infty$; *and conversely.*

Since in the case contemplated, where u_n is ultimately of invariable sign, the convergency of $\Pi(1 + u_n)$ does not depend on any arrangement of signs but merely on the ultimate magnitude of the factors, the infinite product, if convergent, is said to be *absolutely convergent.* It is obvious that *any infinite product in which the sign of* u_n *is not ultimately invariable, but which is convergent when the signs of* u_n *are made all alike, will be, a fortiori, convergent in its original form, and is therefore said to be absolutely convergent; and we have in general, for infinite products of real factors, the theorem that* $\Pi(1 + u_n)$ *is absolutely convergent when* Σu_n *is absolutely convergent; and conversely.*

Cor. *If either of the two infinite products* $\Pi(1 + u_n)$, $\Pi(1 - u_n)$ *be absolutely convergent, the other is absolutely convergent.*

For, if Σu_n is absolutely convergent, so is $\Sigma(-u_n)$; and conversely.

Example 1. $(1+1/1^2)(1+1/2^2)\ldots(1+1/n^2)\ldots$ is absolutely convergent since $\Sigma 1/n^2$ is absolutely convergent.

Example 2. $(1-1/2)(1-1/3)\ldots(1-1/n)\ldots$ has zero for its value since $\Sigma(-1/n)$ diverges to $-\infty$.

Example 3. $(1+1/\sqrt{2})(1+1/\sqrt{3})\ldots(1+1/\sqrt{n})\ldots$ diverges to $+\infty$ since $\Sigma(1/\sqrt{n})$ diverges to $+\infty$.

Example 4. $(1+1/\sqrt{1})(1-1/\sqrt{2})(1+1/\sqrt{3})(1-1/\sqrt{4})\ldots$ Since the sign of u_n is not ultimately invariable, and since the series $\Sigma(-1)^{n-1}/\sqrt{n}$ is not absolutely convergent, the rules of the present paragraph do not apply. We must therefore examine the series $\Sigma\log(1+(-1)^{n-1}/\sqrt{n})$. The terms of this series become ultimately infinitely small; therefore we may (see § 12) associate every odd term with the following even term. We thus replace the series by the equivalent series

$$\Sigma\log\{1+1/\sqrt{(2n-1)}-1/\sqrt{(2n)}-1/\sqrt{(4n^2-2n)}\}.$$

It is easy to show that

$$1/\sqrt{(2n-1)}-1/\sqrt{(2n)}-1/\sqrt{(4n^2-2n)}<0,$$

for all values of $n>1$. Hence the terms of the series in question ultimately become negative. Moreover, $1/\sqrt{(2n-1)}-1/\sqrt{(2n)}-1/\sqrt{(4n^2-2n)}$ is ultimately comparable with $-1/2n$. Hence $\Sigma\log(1+(-1)^{n-1}/\sqrt{n})$ diverges to $-\infty$. The value of $(1+1/\sqrt{1})(1-1/\sqrt{2})(1+1/\sqrt{3})(1-1/\sqrt{4})\ldots$ is therefore 0. This is an example of a semi-convergent product.

Example 5. $e^{1+1}e^{-1-\frac{1}{2}}e^{1+\frac{1}{3}}e^{-1-\frac{1}{4}}\ldots$ The series $\Sigma\log(1+u_n)$ in this case becomes

$$(1+1)-(1+\tfrac{1}{2})+(1+\tfrac{1}{3})-(1+\tfrac{1}{4})+\ldots$$

which oscillates. The infinite product therefore oscillates also.

Example 6. $\Pi(1-x^{n-1}/n)$ is absolutely convergent if $x<1$, and has 0 for its value when $x=1$.

§ 25.] We have deduced the theory of the convergence of infinite products of real factors from the theory of infinite series by means of logarithms; and this is probably the best course for the learner to follow, because the points in the new theory are suggested by the points in the old. All that is necessary is to be on the outlook for discrepancies that arise here and there, mainly owing to the imperfectness of the analogy between the properties of 0 (that is, $+a-a$) and 1 (that is, $\times a\div a$).

It is quite easy, however, by means of a few simple inequality theorems,* to deduce all the above results directly from the definition of the value of $\Pi(1+u_n)$.

* See Weierstrass, *Abhandlungen aus d. Functionenlehre*, p. 203; or *Crelle's Jour.*, Bd. 51.

If P_n have the meaning of § 22, then we see, by exactly the same reasoning as we used in dealing with infinite series, that the necessary and sufficient conditions for the convergency of $\Pi(1+u_n)$ are that P_n be not infinite for any value of n, however large, and that $\underset{n=\infty}{L}(P_{n+m}-P_n)=0$.

If we exclude the exceptional case where $\underset{n=\infty}{L}P_n=0$, then, since P_n is always finite, the condition $\underset{n=\infty}{L}(P_{n+m}-P_n)=0$ is equivalent to $\underset{n=\infty}{L}(P_{n+m}/P_n-1)=0$, that is, $LP_{n+m}/P_n=1$.

If, therefore, we denote $(1+u_{n+1})(1+u_{n+2})\ldots(1+u_{n+m})$ by ${}_mQ_n$, we may state the criteria as follows—

The necessary and sufficient conditions for the convergency of $\Pi(1+u_n)$ *are that* P_n *be not infinite for any value of* n, *however large, and that* $\underset{n=\infty}{L}{}_mQ_n=1$.

If u_n *be complex, then the two conditions obviously* (see chap. xii.) *are that mod* P_n *be not infinite for any value of* n, *however large, and that* $L\,mod\,({}_mQ_n-1)=0$.

We shall not stop to re-prove the results of § 24 by direct deduction from these criteria, but proceed at once to complete the theory by deducing conditions for the absolute convergence of an infinite product of complex factors.

§ 26.] $\Pi(1+u_n)$ *is convergent if* $\Pi(1+mod\,u_n)$ *is convergent.*

Let $\rho_n=\mathrm{mod}\,u_n$, so that ρ_n is positive for all values of n, then, since $\Pi(1+\rho_n)$ is convergent,

$$L\{(1+\rho_{n+1})(1+\rho_{n+2})\ldots(1+\rho_{n+m})-1\}=0 \qquad (1).$$

Now

$$\begin{aligned}{}_mQ_n-1&=(1+u_{n+1})(1+u_{n+2})\ldots(1+u_{n+m})-1,\\&=\Sigma u_{n+1}+\Sigma u_{n+1}u_{n+2}+\ldots+u_{n+1}u_{n+2}\ldots u_{n+m}.\end{aligned}$$

Hence, by chap. xii., §§ 9, 11, we have

$$\begin{aligned}0\not>\mathrm{mod}\,({}_mQ_n-1)&\not>\Sigma\rho_{n+1}+\Sigma\rho_{n+1}\rho_{n+2}+\ldots+\rho_{n+1}\rho_{n+2}\ldots\rho_{n+m},\\&\not>(1+\rho_{n+1})(1+\rho_{n+2})\ldots(1+\rho_{n+m})-1.\end{aligned}$$

Hence, by (1), $L\,\mathrm{mod}\,({}_mQ_n-1)=0$.

Also

$$\begin{aligned}\operatorname{mod} P_n &= \operatorname{mod}(1+u_1)(1+u_2)\ldots(1+u_n),\\ &= \operatorname{mod}(1+u_1)\operatorname{mod}(1+u_2)\ldots\operatorname{mod}(1+u_n),\\ &\neq (1+\rho_1)(1+\rho_2)\ldots(1+\rho_n).\end{aligned}$$

Hence $\operatorname{mod} P_n$ is finite, since $\Pi(1+\rho_n)$ is convergent.

Remark.—The converse of this theorem is not true; as may be seen at once by considering the product $(1+1)(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{4})\ldots$, which converges to a finite limit $\neq 0$; although $(1+1)(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{4})\ldots$ is not convergent.

When $\Pi(1+u_n)$ *is such that* $\Pi(1+\operatorname{mod} u_n)$ *is convergent,* $\Pi(1+u_n)$ *is said to be absolutely convergent. If* $\Pi(1+u_n)$ *be convergent, but* $\Pi(1+\operatorname{mod} u_n)$ *non-convergent,* $\Pi(1+u_n)$ *is said to be semi-convergent.* The present use of these terms includes as a particular case the use formerly made in § 24.

§ 27.] *If* $\Sigma \operatorname{mod} u_n$ *be convergent, then* $\Pi(1+u_n)$ *is absolutely convergent; and conversely.*

For, if $\Sigma \operatorname{mod} u_n$ be convergent, it is absolutely convergent, seeing that $\operatorname{mod} u_n$ is by its nature positive. Hence, by § 24, $\Pi(1+\operatorname{mod} u_n)$ is convergent. Therefore, by § 26, $\Pi(1+u_n)$ is absolutely convergent.

Again, if $\Pi(1+u_n)$ be absolutely convergent, $\Pi(1+\operatorname{mod} u_n)$ is convergent; that is, since $\operatorname{mod} u_n$ is positive, $\Pi(1+\operatorname{mod} u_n)$ is absolutely convergent. Therefore, by § 24, $\Sigma \operatorname{mod} u_n$ is absolutely convergent.

Cor. *If* Σu_n *be absolutely convergent,* $\Pi(1+u_n x)$ *is absolutely convergent, where* x *is either independent of* n *or is such a function of* n *that* $\mathrm{L} \operatorname{mod} x \neq \infty$ *when* $n = \infty$.

Example 1. $\Pi(1-x^n/n)$ is absolutely convergent for all complex values such that $\operatorname{mod} x < 1$, but is not absolutely convergent when $\operatorname{mod} x = 1$.

Example 2. $\Pi(1-x/n^2)$, where x is independent of n, is absolutely convergent.

§ 28.] After what has been done for infinite series, it is not necessary to discuss in detail the application of the laws of algebra to infinite products. We can at once deduce the following results—

I. *The law of association may be safely applied to the factors of* $\Pi(1+u_n)$ *provided* $\mathrm{L} u_n = 0$; *but not otherwise.*

II. *The law of commutation may be safely applied to* $\Pi(1+u_n)$, *provided it be absolutely convergent, but not in general otherwise.*

III. *If both* $\Pi(1+u_n)$ *and* $\Pi(1+u'_n)$ *be absolutely convergent, then* $\Pi\{(1+u_n)(1+u'_n)\}$ *is absolutely convergent and has for its limit* $\{\Pi(1+u_n)\}\times\{\Pi(1+u'_n)\}$; *also* $\Pi\{(1+u_n)/(1+u'_n)\}$ *is absolutely convergent, and has for its limit* $\{\Pi(1+u_n)\}/\{\Pi(1+u'_n)\}$ *provided none of the factors of* $\Pi(1+u'_n)$ *vanish.*

Since $\Sigma\log\{1+\mu_n w_n(z)\}=\Sigma\mu_n w_n(z)\log\{1+\mu_n w_n(z)\}^{1/\mu_n w_n(z)}$. If μ_n and $w_n(z)$ satisfy all the conditions of Du Bois-Reymond's theorem, given in § 19, we have $L\mu_n w_n(z)=0$, $L\log\{1+\mu_n w_n(z)\}^{1/\mu_n w_n(z)}=1$, and $w_n(z)\log\{1+\mu_n w_n(z)\}^{1/\mu_n w_n(z)}$ satisfies all the conditions imposed upon $w_n(z)$ alone. Hence $\Sigma\log\{1+\mu_n w_n(z)\}$ is a continuous function of z from $z=a$ to $z=b$. Hence

IV. *If* μ_n *and* $w_n(z)$ *satisfy the conditions of* § 19, $\Pi(1+\mu_n w_n(z))$ *is a continuous function of* z *from* $z=a$ *to* $z=b$.

Cor. 1. *If* $\Sigma a_n x^n$ *be convergent when mod* $x=\mathrm{R}$, *then* $\Pi(1+a_n x^n)$ *converges to* $\phi(x)$, *where* $\phi(x)$ *is a finite continuous function of* x *for all values of* x *such that mod* $x<\mathrm{R}$.

Cor. 2. *If* $f(n,y)$ *be finite and single-valued as regards* n, *and finite, single-valued, and continuous as regards* y *from* $y=a$ *to* $y=b$, *and if* $\Sigma f(n,y)x^n$ *be absolutely convergent when mod* $x=\mathrm{R}$ (<1), *then, so long as mod* $x\not>\mathrm{R}$, $\Pi(1+f(n,y)x^n)$ *converges to* $\psi(y)$, *where* $\psi(y)$ *is a finite continuous function of* y *from* $y=a$ *to* $y=b$.

Cor. 3. *If* Σa_n *be absolutely convergent, then* $\Pi(1+a_n x)$ *converges to* $\psi(x)$, *where* $\psi(x)$ *is a finite and continuous function of* x *for all finite values of* x, *however large.*

We can also establish for infinite products the following theorem, which is analogous to the principle of indeterminate coefficients.

V. *If, for a continuum of values of* x *including* 0, $\Pi(1+a_n x^n)$ *and* $\Pi(1+b_n x^n)$ *be both absolutely convergent, and* $\Pi(1+a_n x^n)=\Pi(1+b_n x^n)$, *then* $a_1=b_1$, $a_2=b_2$, . . ., $a_n=b_n$, . . .

For we have

$$\Sigma\log(1+a_n x^n)=\Sigma\log(1+b_n x^n),$$

both the series being convergent.

Hence for any value of x, however small, we have, after dividing by x, $\Sigma a_n x^{n-1} \log(1 + a_n x^n)^{1/a_n x^n} = \Sigma b_n x^{n-1} \log(1 + b_n x^n)^{1/b_n x^n}$.

Since $\mathop{\mathrm{L}}_{x=0} \log(1 + a_n x^n)^{1/a_n x^n} = 1$, we have, for very small values of x,

$$a_1 \mathrm{P}_1 + a_2 \mathrm{P}_2 x + a_3 \mathrm{P}_3 x^2 + \ldots = b_1 \mathrm{Q}_1 + b_2 \mathrm{Q}_2 x + b_3 \mathrm{Q}_3 x^2 + \ldots \quad (1),$$

where $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{Q}_1, \mathrm{Q}_2$ differ very little from unity, and all have unity for their limit when $x = 0$.

Hence, since $\Sigma a_n x^{n-1}$ and $\Sigma b_n x^{n-1}$ are, by virtue of our hypotheses, absolutely convergent, we have

$$\mathop{\mathrm{L}}_{x=0} (a_2 \mathrm{P}_2 x + a_3 \mathrm{P}_3 x^2 + \ldots) = 0$$

$$\mathop{\mathrm{L}}_{x=0} (b_2 \mathrm{Q}_2 x + b_3 \mathrm{Q}_3 x^2 + \ldots) = 0$$

Hence, if in (1) we put $x = 0$, we must have

$$a_1 \mathop{\mathrm{L}}_{x=0} \mathrm{P}_1 = b_1 \mathop{\mathrm{L}}_{x=0} \mathrm{Q}_1.$$

But $\mathrm{LP}_1 = \mathrm{LQ}_1 = 1$; therefore $a_1 = b_1$. Removing now the common factor $1 + a_1 x$ from both products, and proceeding as before, we can show that $a_2 = b_2$; and so on.

§ 29.] The following theorem gives an extension of the Factorisation Theorem of chap. v., § 15, to Infinite Products.

If $\psi(x) = \Pi(1 + a_n x)$ *be convergent for all values of* x, *in the sense that* $\mathrm{L} \bmod \mathrm{P}_n \neq \infty$, *and* $\mathrm{L} \bmod ({}_m\mathrm{Q}_n - 1) = 0$, *when* $n = \infty$, *no matter what value* m *may have, then* $\psi(x)$ *will vanish if* x *have one of the values* $-1/a_1, -1/a_2, \ldots, -1/a_r, \ldots$, *and, if* $\psi(x) = 0$, *then* x *must have one of the values* $-1/a_1, -1/a_2, \ldots, -1/a_r, \ldots$

In the first place, we remark that, by our conditions, the vanishing of $\mathrm{L}_m\mathrm{Q}_n$ when $n = \infty$ is precluded. For, if ${}_m\mathrm{Q}_n = \rho(\cos\phi + i \sin\phi)$, $\bmod ({}_m\mathrm{Q}_n - 1) = \{(\rho\cos\phi - 1)^2 + (\rho \sin\phi)^2\}^{1/2}$. Hence we must have $\mathrm{L}\{(\rho\cos\phi - 1)^2 + (\rho\sin\phi)^2\} = 0$, which leads to $\mathrm{L}(\rho\cos\phi - 1) = 0$, $\mathrm{L}\rho\sin\phi = 0$; that is, to $\mathrm{L}\rho\cos\phi = 1$, $\mathrm{L}\rho\sin\phi = 0$. Hence, ${}_m\mathrm{Q}_n = 1 + h + ki$, where h and k have each 0 for limit when $n = \infty$. The exceptional case, mentioned in § 23, where $\Sigma \log(1 + a_n x)$ diverges to $-\infty$, and $\Pi(1 + a_n x)$ converges to 0 for all values of x, is thus excluded.

Now, whatever n may be, we have

$$\psi(x) = P_n \, {}_\infty Q_n \qquad (1).$$

Suppose that we cause x to approach the value $-1/a_r$. We can always in the equation (1) take n greater than r; so that $1 + a_r x$ will occur among the factors of the integral function P_n. Hence, when $x = -1/a_r$, we have $P_n = 0$, and therefore, since ${}_\infty Q_n \neq \infty$, $\psi(-1/a_r) = 0$.

Again, suppose that $\psi(x) = 0$. Then, by (1), $P_n \, {}_\infty Q_n = 0$. But, since n may be as large as we please, and $L \, {}_\infty Q_n = 1$ when $n = \infty$, we can take n so large that ${}_\infty Q_n \neq 0$. Hence, if only n be large enough, the integral function P_n will vanish. Hence x must have a value which will make some one of the factors of P_n vanish; that is to say, x must have some one of the values $-1/a_1, -1/a_2, \ldots, -1/a_r, \ldots$

It should be noticed that nothing in the above reasoning prevents any two or more of the quantities $a_1, a_2, \ldots, a_r, \ldots$ from being equal to one another; and the equal members of the series may, or may not, be contiguous. If there be μ_n contiguous factors identical with $1 + a_n x$, the product $\psi(x)$ will take the form $\Pi(1 + a_n x)^{\mu_n}$; and it can always be brought into this form if it be absolutely convergent, for in that case the commutation of its factors does not affect its value.

Cor. 1. *If x lie within a continuum (x) which includes all the values*

$$-1/a_1, \quad -1/a_2, \quad \ldots, \quad -1/a_n, \ldots \qquad (A),$$

and

$$-1/b_1, \quad -1/b_2, \quad \ldots, \quad -1/b_n, \ldots \qquad (B),$$

if $\Pi(1 + a_n x)^{\mu_n}$ and $\Pi(1 + b_n x)^{\nu_n}$ be absolutely convergent for all values of x in (x), if $f(x)$ and $g(x)$ be definite functions of x which become neither zero nor infinite for any of the values (A) *or* (B), *and if, for all values of x in (x),*

$$f(x)\Pi(1 + a_n x)^{\mu_n} = g(x)\Pi(1 + b_n x)^{\nu_n} \qquad (1),$$

then must each factor in the one product occur in the other raised to the same power; and, for all the values of x in (x),

$$f(x) = g(x) \qquad (2).$$

For, since, by (1), each of the products must vanish for each

of the values (A) or (B), it follows that each of the quantities (A) must be equal to one of the quantities (B); and *vice versa*. The two series (A) and (B) are therefore identical.

Since the two infinite products are absolutely convergent, we may now arrange them in such an order that $a_1 = b_1$, $a_2 = b_2$, . . ., &c., so that we now have

$$f(x)(1 + a_1x)^{\mu_1}(1 + a_2x)^{\mu_2} \ldots = g(x)(1 + a_1x)^{\nu_1}(1 + a_2x)^{\nu_2} \ldots \quad (3).$$

Suppose that $\mu_1 \neq \nu_1$, but that μ_1, say, is the greater; then we have, from (3),

$$f(x)(1 + a_1x)^{\mu_1 - \nu_1}(1 + a_2x)^{\mu_2} \ldots = g(x)(1 + a_2x)^{\nu_2} \ldots \quad (4).$$

Now this is impossible, because the left-hand side tends to 0 as limit when $x = -1/a_1$, whereas the right-hand side does not vanish when $x = -1/a_1$. We must therefore have $\mu_1 = \nu_1$; and, in like manner, $\mu_2 = \nu_2$; and so on.

We may therefore clear the first n factors out of each of the products in (1), and thus deduce the equation

$$f(x)\,{}_\infty Q_n = g(x)\,{}_\infty Q'_n \quad (5),$$

where ${}_\infty Q_n$ and ${}_\infty Q'_n$ have the usual meaning. The equation (5) will hold, however large n may be. Hence, since $L\,{}_\infty Q_n = L\,{}_\infty Q'_n = 1$, we must have

$$f(x) = g(x).$$

Cor. 2. *From this it follows that a given function of x which vanishes for any of the values* (A) *and for no others within the continuum* (x), *can be expressed within* (x) *as a convergent infinite product of the form* $f(x)\Pi(1 + a_nx)^{\mu_n}$ (*where $f(x)$ is finite and not zero for all finite values of x within* (x)), *if at all, in one way only.*

If the infinite product be only semi-convergent, the above demonstration fails.

It may be remarked that it is not in general possible to express a function, having given zero points, in the form described in the corollary. On this subject the student should consult Weierstrass, *Abhandlungen aus der Functionenlehre*, p. 14 *et seq.*

ESTIMATION OF THE RESIDUE OF A CONVERGING SERIES OR INFINITE PRODUCT.

§ 30.] For many theoretical, and for some practical purposes, it is often required *to assign an upper limit to the residue of an infinite series.* This is easily done in what are by far the two most important cases, namely :—(1) Where the ratio of convergence ($\rho_n = u_{n+1}/u_n$) ultimately becomes less than unity, and the terms are all ultimately of the same sign ; (2) Where the terms ultimately continually diminish in value, and alternate in sign.

Case (1). It is essential to distinguish two varieties of series under this head, namely :—(*a*) That in which ρ_n descends to its limit ρ ; (*b*) That in which ρ_n ascends to its limit ρ.

In case (*a*), let n be taken so large that, on and after n, ρ_n is always numerically less than 1, and never increases in numerical value. Then

$$\begin{aligned} R_n &= u_{n+1} + u_{n+2} + u_{n+3} + \ldots, \\ &= u_{n+1}\left\{1 + \frac{u_{n+2}}{u_{n+1}} + \frac{u_{n+3}}{u_{n+2}} \cdot \frac{u_{n+2}}{u_{n+1}} + \ldots\right\}, \\ &= u_{n+1}\{1 + \rho_{n+1} + \rho_{n+1}\rho_{n+2} + \rho_{n+1}\rho_{n+2}\rho_{n+3} + \ldots\}. \end{aligned}$$

Therefore, if dashes be used to denote the numerical values, or moduli, of the respective quantities, we have

$$\begin{aligned} R'_n &\geqslant u'_{n+1}\{1 + \rho'_{n+1} + \rho'^{2}_{n+1} + \ldots\}, \\ &\geqslant u'_{n+1}/(1 - \rho'_{n+1}), \\ &\geqslant u'_{n+1}/(1 - u'_{n+2}/u'_{n+1}) \end{aligned} \qquad (1).$$

And also, for a lower limit,

$$R'_n \leqslant u'_{n+1}/(1 - \rho) \qquad (2).$$

In case (*b*), let n be so large that, after n, ρ_n is numerically less than 1, and never decreases in numerical value. Then

$$\begin{aligned} R_n &= u_{n+1}\{1 + \rho_{n+1} + \rho_{n+2}\rho_{n+1} + \ldots\}. \\ R'_n &\geqslant u'_{n+1}\{1 + \rho + \rho^2 + \ldots\}, \\ &\geqslant u'_{n+1}/(1 - \rho) \end{aligned} \qquad (3);$$

And we have also

$$\begin{aligned} R'_n &\leqslant u'_{n+1}/(1 - \rho'_{n+1}), \\ &\leqslant u'_{n+1}/(1 - u'_{n+2}/u'_{n+1}) \end{aligned} \qquad (4).$$

Case (2). When the terms of the series ultimately decrease and alternate in sign, the estimation of the residue is still

simpler. Let n be so large that, on and after n, the terms never increase in numerical value, and always alternate in sign. Then we have

$$R'_n = u'_{n+1} - u'_{n+2} + u'_{n+3} - \ldots$$
$$> u'_{n+1} \quad (5);$$
$$< u'_{n+1} - u'_{n+2} \quad (6).$$

§ 31.] *Residue of an Infinite Product.* Let us consider the infinite products, $\Pi(1+u_n)$ and $\Pi(1-u_n)$, *in which* u_n *becomes ultimately positive and less than unity.* If the series Σu_n converge in such a way that the limit of the convergency-ratio ρ_n is a positive quantity ρ less than 1, then it is easy to obtain an estimate of the residue. Let Q_n, Q'_n denote the product of all the factors after the nth in $\Pi(1+u_n)$ and $\Pi(1-u_n)$ respectively, so that $Q_n > 1$, and $Q'_n < 1$. We suppose n so great that, on and after n, u_n is positive, ρ_n less than 1, and either (*a*) ρ_n never increases, or else (*b*) ρ_n never decreases. In case (*a*), Σu_n falls under case (1) (*a*) of last paragraph; in case (*b*), Σu_n falls under case (1) (*b*) of last paragraph. We shall, as usual, denote the residue of Σu_n by R_n; and we shall suppose that n is so large that mod $R_n < 1$.

Now (by chap. xxiv., § 7, Example 2),

$$Q_n = (1+u_{n+1})(1+u_{n+2}) \ldots,$$
$$> 1 + u_{n+1} + u_{n+2} + \ldots,$$
$$> 1 + R_n \quad (1).$$
$$Q'_n = (1-u_{n+1})(1-u_{n+2}) \ldots,$$
$$> 1 - R_n \quad (2).$$

Also,

$$1/Q_n = \{1 - u_{n+1}/(1+u_{n+1})\}\{1 - u_{n+2}/(1+u_{n+2})\} \ldots,$$
$$> 1 - u_{n+1}/(1+u_{n+1}) - u_{n+2}/(1+u_{n+2}) - \ldots,$$
$$> 1 - u_{n+1} - u_{n+2} - \ldots,$$
$$> 1 - R_n.$$

Whence $$Q_n - 1 < R_n/(1-R_n) \quad (3).$$

In like manner,

$$1/Q'_n = \{1 + u_{n+1}/(1-u_{n+1})\}\{1 + u_{n+2}/(1-u_{n+2})\} \ldots,$$
$$> 1 + u_{n+1}/(1-u_{n+1}) + u_{n+2}/(1-u_{n+2}) + \ldots,$$
$$> 1 + u_{n+1} + u_{n+2} + \ldots,$$
$$> 1 + R_n.$$

Whence $$1 - Q'_n > R_n/(1+R_n) \quad (4).$$

From (1), (2), (3), and (4) we have

$$R_n < Q_n - 1 < R_n/(1 - R_n) \qquad (5);$$
$$R_n/(1 + R_n) < 1 - Q'_n < R_n \qquad (6).$$

Since upper and lower limits for R_n can be calculated by means of the inequalities of last paragraph, (5) and (6) enable us to estimate the residues of the infinite products $\Pi(1 + u_n)$ and $\Pi(1 - u_n)$.

Example. Find an upper limit to the residue of $\Pi(1 - x^n/n)$, $x<1$.

Here $u_n = x^n/n$, $\rho_n = x/(1 + 1/n)$, $\rho = x$. The series has an ascending convergency-ratio; and we have $R_n < u_{n+1}/(1 - \rho) < x^{n+1}/(n + 1)(1 - x)$. Therefore, $1 - Q'_n < x^{n+1}/(n + 1)(1 - x)$. Hence, if P'_n be the nth approximation to $\Pi(1 - x^n/n)$, P'_n differs from the value of the whole product by less than $100\, x^{n+1}/(n + 1)(1 - x)$ % of P'_n itself.

CONVERGENCY OF DOUBLE SERIES.

§ 32.] It will be necessary in some of the following chapters to refer to certain properties of series which have a doubly infinite number of terms. We proceed therefore to give a brief sketch of the elementary properties of this class of series. The theory originated with Cauchy, and the greater part of what follows is taken with slight modifications from note viii. of the *Analyse Algébrique*, and § 8 of the *Résumés Analytiques*.

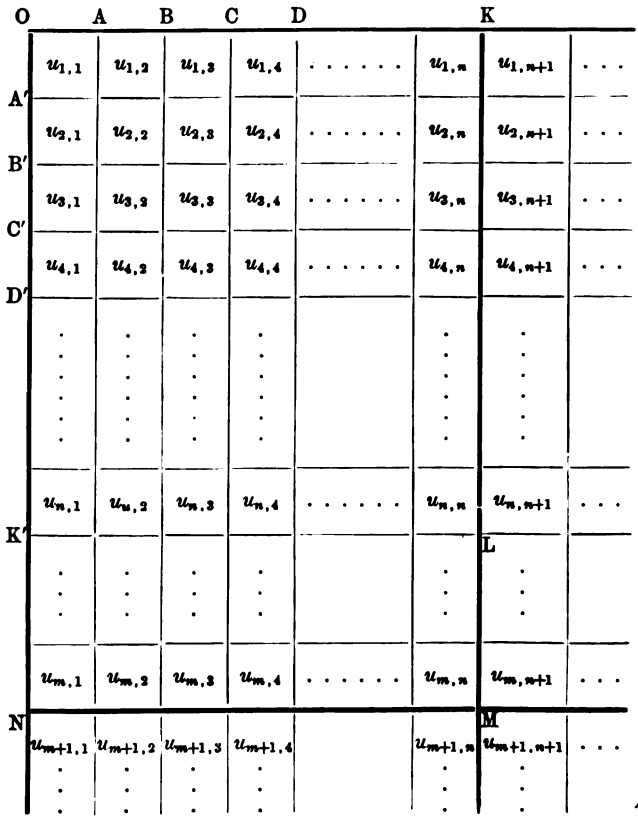
Let us consider the doubly infinite series of terms represented in (1). We may take as the general, or specimen term, $u_{m,n}$, where the first index indicates the row, and the second the column, to which the term belongs. The assemblage of such terms we may denote by $\Sigma u_{m,n}$; and we shall speak of this assemblage as a *Double Series.**

A great variety of definitions might obviously be given of the sum to a finite number of terms of such a series; and, corresponding to every such definition, there would arise a definite question regarding the sum to infinity, that is, regarding the convergency of the series.

There are, however, only four ways of taking the sum of the double series which are of any importance for our purposes.

* Sometimes the term "Series of Double Entry" is used.

First Way.—We may define the finite sum to be the sum of all the mn terms within the rectangular array OKMN. This we denote by $S_{m,n}$. Then we may take the limit of this by first making m and finally n infinite, or by first making n infinite and finally m infinite. If the result of both these limit operations is the same definite quantity S, then we say that $\Sigma u_{m,n}$ *converges to* S *in the first way.*



It may, however, happen—1st, that both these operations lead to an infinite value; 2nd, that neither leads to a definite value; 3rd, that one leads to a definite finite value, and the

other not; 4th, that one leads to one definite finite value, and the other to another definite finite value.* In all these cases we say that the series is *non-convergent for the first way of summing.*

Second Way.—Sum to n terms each of the series formed by taking the terms in the first m horizontal rows of (1); and call the sums $T_{1,n}$, $T_{2,n}$, . . ., $T_{m,n}$. Define

$$S'_{m,n} = T_{1,n} + T_{2,n} + \ldots + T_{m,n} \qquad (2)$$

as the finite sum.

Then, supposing each of the horizontal series to converge to T_1, T_2, . . ., T_m respectively, and ΣT_m to be a convergent series, define

$$S' = T_1 + T_2 + \ldots + T_m + \ldots \text{ ad } \infty \qquad (3)$$

as the *sum to infinity in the second way.*

Third Way.—Sum to m terms each of the series in the first n columns; and let these sums be $U_{1,m}$, $U_{2,m}$, . . ., $U_{n,m}$. Define

$$S''_{m,n} = U_{1,m} + U_{2,m} + \ldots + U_{n,m} \qquad (4)$$

as the finite sum.

Then, supposing these vertical series to converge to U_1, U_2, . . ., U_n respectively, and ΣU_n to be a convergent series, define

$$S'' = U_1 + U_2 + \ldots + U_n + \ldots \text{ ad } \infty \qquad (5)$$

as the *sum to infinity in the third way.*

So long as m and n are finite, it is obvious that we have

$$S'_{m,n} = S''_{m,n} = S_{m,n};$$

so that, for finite summation, the second and third ways of summing are each equivalent to the first.

The case is not quite so simple when we sum to infinity. It is clear, however, that

$$S' = \operatorname*{L}_{m=\infty} \{ \operatorname*{L}_{n=\infty} S_{m,n} \} \qquad (6);$$

and

$$S'' = \operatorname*{L}_{n=\infty} \{ \operatorname*{L}_{m=\infty} S_{m,n} \} \qquad (7);$$

* Examples of some of these cases are given in § 35 below.

so that S′ and S″ will be equal to each other and to S when the two ways of taking the limit of $S_{m,n}$ both lead to the same definite finite result.*

Fourth Way.—Sum the terms which lie in the successive diagonal lines of the array, namely, AA′, BB′, CC′, . . ., KK′; and let these sums be $D_2, D_3, \ldots, D_{n+1}$ respectively; that is, $D_2 = u_{1,1}$, $D_3 = u_{1,2} + u_{2,1}$, . . ., $D_{n+1} = u_{1,n} + u_{2,n-1} + \ldots + u_{n,1}$.

Define

$$S'''_n = D_2 + D_3 + \ldots + D_n \qquad (8)$$

as the finite sum; and, supposing ΣD_n to be convergent, define

$$S''' = D_2 + D_3 + \ldots + D_n + \ldots \text{ ad } \infty \qquad (9)$$

as the *sum to infinity in the fourth way.*

The finite sum according to this last definition includes all the terms in the triangle OKK′; it can therefore never (except for $m = n = 1$) coincide with the finite sum according to the former definitions. Whether the sum to infinity (S‴) according to the fourth definition will coincide with S, S′, or S″, depends on the nature of the series. It may, in fact, happen that the limits S, S′, S″ exist and are all equal, and that the limit S‴ is infinite.†

§ 33.] *Double series in which the terms are all ultimately of the same sign.* By far the most important kind of double series is that in which, for all values of m and n greater than certain fixed limits, $u_{m,n}$ has always the same sign, say always the positive sign. Since, by adding or subtracting a *finite* quantity to the sum (however defined), we can always make any finite number of terms have the same sign as the ultimate terms of the series, we may, so far as questions regarding convergency are concerned, suppose all the terms of $\Sigma u_{m,n}$ to have the same (say positive) sign from the beginning. Suppose now (1) to represent the array of terms under this last supposition; and let us farther suppose that $\Sigma u_{m,n}$ is convergent in the first way.

Then, since $L(S_{m+p,n+q} - S_{m,n}) = S - S = 0$, when $m = \infty$, $n = \infty$ whatever p and q may be, it follows that the sum of all

* For an illustration of the case when this is not so, see below, § 35.

† See below, § 35.

the terms in the gnomon between NMK and two parallels to NM and MK below and to the right of these lines respectively, must become as small as we please when we remove NM sufficiently far down and MK sufficiently far to the right.

From this it follows, *a fortiori*, seeing that all the terms of the array are positive, that, if only m and n be sufficiently great, the sum of any group of terms taken in any way from the residual terms lying outside OKMN will be as small as we please.

Hence, in particular,

1st. The total or partial residue of each of the horizontal series vanishes when $n = \infty$.

2nd. The same is true for each of the vertical series.

3rd. The same is true for the series ΣD_n.

The last inference holds, since S'''_n obviously lies between $S_{q,n-q}$ and $S_{n-1,n-1}$.

Hence

Theorem I. *If all the terms of* $\Sigma u_{m,n}$ *be positive, and if the series be convergent in the first sense, then each of the horizontal series, each of the vertical series, and the diagonal series will be convergent, and the double series will be convergent in the remaining three ways, always to the same limit.*

If we commutate the terms of a double series so that the term $u_{m,n}$ becomes the term $u_{m',n'}$, where $m' = f(m,n)$, $n' = g(m,n)$, $f(m,n)$ *and* $g(m,n)$ *being functions of* m *and* n, *each of which has a distinct value for every distinct pair of values of* m *and* n (say non-repeating functions), *and each of which is finite for all finite values of* m *and* n (Restriction A*), then we shall obviously leave the convergency of the series unaffected. Hence

Cor. 1. *If* $\Sigma u_{m,n}$ *be a series of positive terms convergent in the first way, then any commutation of its terms (under Restriction A) will leave its convergency unaffected; that is to say, it will converge in all the four ways to the same limit* S *as before.*

---

* No such restriction is usually mentioned by writers on this subject; but some such restriction is obviously implied when it is said that the terms of an absolutely convergent series are commutative; otherwise the characteristic property of a convergent series, namely, that it has a vanishing residue, would not be conserved.

Cor. 2. *If the terms (all positive) of a convergent single series* Σu_n *be arranged into a double series* $\Sigma u_{m',n'}$, *where* m' *and* n' *are functions of* n *subject to Restriction A, then* $\Sigma u_{m',n'}$ *will converge in all four ways to the same limit as* Σu_n.

It should be noticed that this last corollary gives a further extension of the laws of commutation and association to a series of positive terms; and therefore, as we shall see presently, to any absolutely convergent series.

Let us next assume that the series $\Sigma u_{m,n}$ is convergent in the second way. Then, since ΣT_m is convergent, we can, by sufficiently increasing m, make the residue of this series, that is, the sum of as many as we choose of the terms below the infinite horizontal line NM, less than $\frac{1}{2}\epsilon$, where ϵ is as small as we please. Also, since each of the horizontal series is, by our hypothesis, convergent, we can, by sufficiently increasing n, make the residue of each of them, less than $\epsilon/2m$; and therefore the sum of their residues, that is, as many as we please of the terms above NM produced and right of MK, less than $\frac{1}{2}\epsilon$. Hence, by sufficiently increasing both m and n, we can make the sum of the terms outside OKMN, less than ϵ, that is, as small as we please. From this it follows that $\Sigma u_{m,n}$ is convergent in the first way, and, therefore, by Theorem I., in all the four ways.

In exactly the same way, we can show that, if $\Sigma u_{m,n}$ is convergent in the third way, it is convergent in all four ways.

Finally, let us assume that $\Sigma u_{m,n}$ is convergent in the fourth way. It follows that the residue of the diagonal series ΣD_p can, by making p large enough, be made as small as we please. Now, if only m and n be each large enough, the residue of $S_{m,n}$, that is, the sum of as many as we please of the terms outside OKMN, will contain only terms outside OKK', all of which are terms in the residue of S'''_p. Hence, since all the terms in the array (1) are positive, we can make the sum of as many as we please of the terms outside OKMN as small as we please, by

sufficiently increasing both m and n. Therefore $\Sigma u_{m,n}$ is convergent in the first way, and consequently in all four ways.

Combining these results with Theorem I., we now arrive at the following :—

Theorem II. *If a double series of positive terms converge in any one of the four ways to the limit* S, *it also converges in all the other three ways to the same limit* S; *and the subsidiary single series, horizontal, vertical, and diagonal, are all convergent.*

Cor. *Any single series* $\Sigma u_{n'}$ *consisting of terms selected from* $\Sigma u_{m,n}$ (*under Restriction A*) *will be a convergent series, if* $\Sigma u_{m,n}$ *be convergent.*

Restriction A will here take the form that n' must be a function of m and n whose values do not repeat, and which is finite for finite values of m and n.

Example. The double series $\Sigma x^m y^n$ is convergent for all values of x and y, such that $0 < x < +1$, $0 < y < +1$.

For the $(m+1)$th horizontal series is $x^m \Sigma y^n$, which converges to $x^m/(1-y)$ since $0 < y < +1$. Also $\Sigma x^m/(1-y)$ converges to $1/(1-x)(1-y)$ since $0 < x < +1$.

§ 34.] *Absolutely Convergent Double Series.*—When a double series is such that it remains convergent when all its terms are taken positively, it is said to be *Absolutely Convergent.*

Any convergent series whose terms are all ultimately of the same sign is of course an absolutely convergent series according to this definition.

It is also obvious that all the propositions which we have proved regarding the convergency of double series consisting solely of positive terms are, *a fortiori*, true of absolutely convergent double series, for restoring the negative signs will, if it affect the residues at all, merely render them less than before.

In particular, from Theorem II. we deduce the following, which we may call *Cauchy's test for the absolute convergency of a double series.*

Theorem III. *If* $u'_{m,n}$ *be the numerical or positive value of* $u_{m,n}$, *and if all the horizontal series of* $\Sigma u'_{m,n}$ *be convergent, and the sum of their sums to infinity also convergent, then*

1st. *The Horizontal Series of* $\Sigma u_{m,n}$ *are all absolutely convergent,*

and the sum of their sums to infinity converges to a definite finite limit S.

2nd. $\Sigma u_{m,n}$ *converges to* S *in the first way.*

3rd. *All the Vertical Series are absolutely convergent, and the sum of their sums to infinity converges to* S.

4th. *The Diagonal Series is absolutely convergent, and converges to* S.

5th. *Any series formed by taking terms from* $\Sigma u_{m,n}$ *(under Restriction A) is absolutely convergent.*

The like conclusions also follow, if all the vertical series, or if the diagonal series of $\Sigma u'_{m,n}$ *be convergent.*

Cor. *If* Σu_n *and* Σv_n *be each absolutely convergent, and converge to* u *and* v *respectively, then* $\Sigma(u_n v_1 + u_{n-1}v_2 + \ldots + u_1 v_n)$ *is absolutely convergent, and converges to* uv.

For the series in question is the diagonal series of the double series $\Sigma u_m v_n$, which, as may be easily shown, satisfies Cauchy's conditions.

This is, in a more special form, the theorem already proved in § 14.

Example 1. Find the condition that the double series $1 + \Sigma(-)^m {}_nC_m x^{n-m} y^m$ $(n > m)$ be absolutely convergent; and find its sum.

The series may be arranged thus:—

$$\begin{aligned}
&1 + x + x^2 + \ldots + x^n + \ldots \\
&- y - 2yx - 3yx^2 - \ldots - (n+1)yx^n - \ldots \\
&+ y^2 + 3y^2x + 6y^2x^2 + \ldots + \tfrac{1}{2}(n+1)(n+2)y^2x^n + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&(-)^m y^m + (-)^m {}_{m+1}C_1 y^m x + (-)^m {}_{m+2}C_2 y^m x^2 + \ldots + (-)^m {}_{m+n}C_n y^m x^n + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

If x' and y' be the moduli, or positive values, of x and y, then the series $\Sigma u'_{m,n}$ corresponding to the above will be

$$\begin{aligned}
&1 + x' + x'^2 + \ldots + x'^n + \ldots \\
&+ y' + 2y'x' + 3y'x'^2 + \ldots + (n+1)y'x'^n + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

In order that the horizontal series in this last may be convergent, it is necessary and sufficient that $x' < 1$.

Also $T'_m = y'^m/(1-x')^{m+1}$; hence the necessary and sufficient condition that $\Sigma T'_m$ be convergent is that $y' < 1 - x'$, which implies, of course, that $y' < 1$.

The given series will therefore satisfy Cauchy's conditions of absolute convergency if $\operatorname{mod} x < 1$, $\operatorname{mod} x + \operatorname{mod} y < 1$, and consequently also $\operatorname{mod} y < 1$.

These being fulfilled, we have $T_m = (-)^m y^m/(1-x)^{m+1}$;

$$S' = \frac{1}{1-x}\left\{1 - \frac{y}{1-x} + \ldots (-)^m \left(\frac{y}{1-x}\right)^m \ldots\right\},$$
$$= \frac{1}{1-x+y};$$

and the sum of the series in whatever order we take its terms is $1/(1-x+y)$.

Example 2. If $u_r = x^{2^r} + x^{2^{r+1}} + x^{2^{r+2}} + \ldots$, where $x<1$, show that

$$\frac{u_0}{2^0} + \frac{u_1}{2^1} + \frac{u_2}{2^2} + \ldots = 2u_0 - \frac{x^{2^0}}{2^0} - \frac{x^{2^1}}{2^1} - \frac{x^{2^2}}{2^2} - \ldots$$

Let S denote the series on the left. Then S may be written as a double series thus,

$$\frac{1}{2^0}(x^{2^0} + x^{2^1} + x^{2^2} + \ldots + x^{2^n} + \ldots)$$
$$+ \frac{1}{2^1}(0 + x^{2^1} + x^{2^2} + \ldots + x^{2^n} + \ldots)$$
$$+ \frac{1}{2^2}(0 + 0 + x^{2^2} + \ldots + x^{2^n} + \ldots)$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Now each of the vertical series is absolutely convergent, and we have $U_n = x^{2^n}(1 - 1/2^{n+1})/(1-\frac{1}{2}) = x^{2^n}(2 - 1/2^n)$. ΣU_n is of the same order of convergence as Σx^{2^n}, hence it is absolutely convergent. Also all the terms of the double series are positive. The double series therefore satisfies Cauchy's conditions; and its sum is the same as that of ΣU_n, or of ΣT_n. Now

$$\Sigma T_n = u_0/2^0 + u_1/2^1 + u_2/2^2 + \ldots;$$

and

$$\Sigma U_n = \Sigma x^{2^n}(2 - 1/2^n),$$
$$= 2\Sigma x^{2^n} - \Sigma x^{2^n}/2^n,$$
$$= 2u_0 - x^{2^0}/2^0 - x^{2^1}/2^1 - \ldots$$

Hence the theorem.

§ 35.] *Examples of the exceptional cases that arise when a double series is not absolutely convergent.* It may help to accentuate the points of the foregoing theory if we give an example or two of the anomalies that arise when the conditions of absolute convergency are not fulfilled.

Example 1. It is easy to construct double series whose horizontal and vertical series are absolutely convergent, and which nevertheless have not a definite sum of the first kind; but, on the other hand, have one definite sum of the second kind and another of the third kind.

If the finite sum of the first kind, $S_{m,n}$, of a double series be $A + f(m,n)$, where A is independent of m and n, then it is easy to see that

$$u_{m,n} = f(m,n) - f(m-1,n) - f(m,n-1) + f(m-1,n-1).$$

Hence we have only to give $f(m,n)$ such a form that

$$\lim_{m=\infty}\{\lim_{n=\infty} f(m,n)\} \neq \lim_{n=\infty}\{\lim_{m=\infty} f(m,n)\},$$

and we shall have a series whose sums of the second and third kind are not alike, and which consequently has no definite sum of the first kind.

Suppose, for example, that $f(m,n)=(m+1)/(m+n+2)$, then

$$u_{m,n}=(m+1)/(m+n+2)-m/(m+n+1)-(m+1)/(m+n+1)+m/(m+n),$$
$$=(m-n)/(m+n)(m+n+1)(m+n+2).$$

It is at once obvious that the sums of the second, third, and fourth kind for this series are all different. For in the first place we observe that $u_{m,n}=-u_{n,m}$. Hence there is a "skew" arrangement of the terms in the array (1), such that the terms equidistant from the dexter diagonal of the array and on the same perpendicular to this diagonal are equal and of opposite sign, those on the diagonal itself being zero. Each term of the diagonal series ΣD_n is therefore zero; and the sum of the fourth kind is 0.

Also, owing to the arrangement of signs, we have $T_{m,n}=-U_{m,n}$; and, since each of the horizontal and each of the vertical series in this case is convergent, $T_m=-U_m$, and therefore $S'=-S''$.

Now

$$T_{m,n}=\sum_{n=1}^{n}[(m+1)\{1/(m+n+2)-1/(m+n+1)\}-m\{1/(m+n+1)-1/(m+n)\}],$$
$$=(m+1)\{1/(m+n+2)-1/(m+2)\}-m\{1/(m+n+1)-1/(m+1)\}.$$

Hence

$$T_m=-(m+1)/(m+2)+m/(m+1)=-1/(m+1)(m+2).$$

The series ΣT_m is therefore absolutely convergent; and its sum to infinity is obviously $-1+1/2=-1/2$. Hence the double series has for its sum $-1/2$, $+1/2$, or 0, according as we sum it in the second, third, or fourth way.

At first sight, the reader might suppose (seeing that the horizontal series are all absolutely convergent, and that the sum of their actual sums is also absolutely convergent) that this case is a violation of Cauchy's criterion. But it is not so. For, if we take all the terms in the mth horizontal series positively, and notice that the terms begin to be negative after $m=n$, then we see that T'_m the sum of the positive values of the terms in the mth series is given by

$$T'_m=\sum_{n=1}^{m}u_{m,n}-\sum_{n=m+1}^{\infty}u_{m,n},$$
$$=(m+1)\{1/(2m+2)-1/(m+2)\}-m\{1/(2m+1)-1/(m+1)\}$$
$$-(m+1)\{0-1/(2m+2)\}+m\{0-1/(2m+1)\},$$
$$=1-2m/(2m+1)-(m+1)/(m+2)+m/(m+1),$$
$$=(m^2+m+1)/(m+1)(m+2)(2m+1).$$

Now the convergence of $\Sigma T'_m$ is of the same order as that of $\Sigma 1/m$, that is to say, $\Sigma T'_m$ is divergent. Hence Cauchy's conditions are not fully satisfied; and the anomaly pointed out above ceases to be surprising. The present case is an excellent example of the care required in dealing with double series which are wont to be used somewhat recklessly by beginners in mathematics.*

* Before Cauchy the reckless use of double series and consequent perplexity was not confined to beginners. See a curious paper by Babbage, *Phil. Trans. R.S.L.* (1819).

Example 2. The double series $\Sigma(-)^{m+n}1/m^n$, whose horizontal and vertical series are each semi-convergent, converges to the sum $(\log 2)^2$ in the second, third, or fourth way (see chap. XXVIII., § 9, and Exercises XIII. 14). But alteration in the order of the terms in the array would alter the sum (see chap. XXVIII., § 4, Example 3).

Example 3. If the two series Σa_n and Σb_n converge to a and b respectively, and at least one of them be absolutely convergent, then it follows from § 14 that the double series $\Sigma a_n b_n$ converges to the same sum, namely ab, in all the four ways, although it is not absolutely convergent, and its sum is not independent of the order of its terms.

The same also follows by § 20, Cor., provided Σa_n, Σb_n, $\Sigma(a_n b_1 + a_{n-1}b_2 + \ldots + a_1 b_n)$ be all convergent, even if no one of the three be absolutely convergent.*

If, however, both Σa_n and Σb_n be semi-convergent, then the diagonal series may be divergent, although the series converges to the same limit in the second and third way. This happens with the series $\Sigma(-)^{m+n}1/(mn)^\alpha$ where α is a quantity lying between 0 and $\frac{1}{2}$. This series obviously converges to the finite limit $(1 - 1/2^\alpha + 1/3^\alpha - \ldots)^2$ in the second and third ways. For the diagonal series we have

$$D_n = \sum_{r=1}^{n} 1/r^\alpha (n-r)^\alpha.$$

Now, since $0 < \alpha < 1$, we have, by chap. XXIV., § 9, $r^\alpha + (n-r)^\alpha < 2^{1-\alpha}\{r + (n-r)\}^\alpha < 2^{1-\alpha}n^\alpha$.
Therefore

$$Dn = \frac{1}{2^{1-\alpha}n^\alpha} \Sigma \frac{2^{1-\alpha}n^\alpha}{r^\alpha(n-r)^\alpha} \prec \frac{1}{2^{1-\alpha}n^\alpha} \Sigma \frac{r^\alpha + (n-r)^\alpha}{r^\alpha(n-r)^\alpha},$$

$$\prec \frac{2}{2^{1-\alpha}n^\alpha} \sum_{r=1}^{n} \frac{1}{r^\alpha} \prec \frac{2}{2^{1-\alpha}n^\alpha} \cdot \frac{n}{n^\alpha},$$

$$\prec 2^\alpha n^{1-2\alpha}.$$

Hence, if $\alpha = \frac{1}{2}$, $LD_n \prec 2^\alpha$; and, if $\alpha < \frac{1}{2}$, $LD_n = \infty$, when $n = \infty$. Therefore ΣD_n diverges if $0 < \alpha \not> \frac{1}{2}$.

IMAGINARY DOUBLE SERIES.

§ 36.] After what has been laid down in § 10, it will be obvious that, in the first instance, the convergency of a double series of imaginary terms involves simply the convergency of two double series, each consisting of real terms only.

It is at once obvious that each of the two double series, $\Sigma\alpha_{m,n}$, $\Sigma\beta_{m,n}$, will be absolutely convergent if the double series

* See Stolz, *Allgemeine Arithmetik*, Th. I., p. 248.

$\Sigma \sqrt{(\alpha^2_{m,n} + \beta^2_{m,n})}$ is convergent. Hence, if $u'_{m,n}$ denote the modulus of $u_{m,n} = \alpha_{m,n} + i\beta_{m,n}$, we see that $\Sigma u_{m,n}$ will converge to the same limit in all four ways if $\Sigma u'_{m,n}$ be convergent. In this case we say that the imaginary series is absolutely convergent.

Since all the terms $u'_{m,n}$ are positive, we deduce from Theorem II. the following:—

Theorem IV. *If all the horizontal series in the double series formed by the moduli of the terms of $\Sigma u_{m,n}$ be convergent, and the sum of their sums to infinity be also convergent, then the series $\Sigma u_{m,n}$ is absolutely convergent, and all its subsidiary series are also absolutely convergent.*

Here subsidiary series may mean any series formed by selecting terms from $\Sigma u_{m,n}$ under Restriction A. Theorem IV., of course, includes Theorem III. as a particular case.

§ 37.] The following simple general theorem regarding the convergency of the double series $\Sigma a_{m,n} x^m y^n$ will be of use in a later chapter.

If the moduli of the coefficients of the series $\Sigma a_{m,n} x^m y^n$ have a finite upper limit λ, then $\Sigma a_{m,n} x^m y^n$ is absolutely convergent for all values of x and y such that mod $x < 1$, mod $y < 1$.

For, if dashes be used to indicate moduli, we have, by hypothesis, $a'_{m,n} \not> \lambda$. Hence the series $\Sigma a'_{m,n} x'^m y'^n$ is, *a fortiori*, convergent if the series $\Sigma \lambda x'^m y'^n$ is convergent; that is, if $\Sigma x'^m y'^n$ is convergent. Now, as we have already seen (§ 33), this last series is convergent provided $x' < 1$ and $y' < 1$. Hence the theorem in question.

Exercises VIII.

Examine the convergency of the series whose nth terms are the following:—

(1.) $(1+n)/(1+n^2)$.
(2.) $n^p/(n^p+a)$.
(3.) $e^{-n^2 x}$.
(4.) $1/(n^2 \pm 1)$.
(5.) $1/\sqrt{(n^2-n)}\{\sqrt{n} - \sqrt{(n-1)}\}$.
(6.) $\alpha^n/(\alpha^n + x^n)$.
(7.) $(n!)^2 x^n/(2n)!$.
(8.) $n^4/n!$.
(9.) $\{(y+x^n)/(z-x^n)\}^{1/n}$.
(10.) $n \log\{(2n+1)/(2n-1)\} - 1$.
(11.) $1 \cdot 3 \cdot 5 \ldots (2n-1)/2 \cdot 4 \cdot 6 \ldots 2n$.
(12.) $\{1/1^\alpha + 1/2^\alpha + \ldots + 1/n^\alpha\}/n^\alpha$.

(13.) $1/(an+b)$. (14.) $n/(an^2+b)$.

(15.) $m(m-1)\ldots(m-n+1)/n^n$. (16.) $\{(n+1)/(n+2)\}^n/n$.

(17.) Show that $\frac{m}{n}+\frac{m(m+1)}{n(n+1)}+\frac{m(m+1)(m+2)}{n(n+1)(n+2)}+\ldots$ is convergent or divergent according as $n-m>$ or $\not>1$.

(18.) Show that $a^{1/m}+a^{1/m+1/(m+1)}+a^{1/m+1/(m+1)+1/(m+2)}+\ldots$ is convergent or divergent according as $a<$ or $\not<1/e$. (Bourguet, *Nouv. Ann.*, ser. ii., t. 18.)

(19.) Examine the convergency of $\Sigma 1/n^{(n+1)/n}$.

(20.) Show that $\Sigma n^a/(n+1)^{a+\alpha}$ is convergent or divergent according as $a>$ or $\not>1$. (Bertrand.)

(21.) Show that $\Sigma 1/n\log n\{\log\log n\}^a$ is convergent or divergent according as $a>$ or <1.

(22.) Show that $\Sigma 1/(n+1+\cos n\pi)^2$ is convergent. (Catalan, *Traité Él. d. Séries*, p. 28.)

Examine the convergency of the following infinite products:—

(23.) $\Pi\{1+f(n)r^n\}$, where $f(n)$ is an integral function of n.

(24.) $\Pi\{(x^{2n}+x)/(x^{2n}+1)\}$. (25.) $\Pi\{n^{z+1}/(n-1)^z(n+z)\}$.

(26.) If $\Sigma f(n)$ be convergent, show that, when $n=\infty$,

$$\mathrm{L}\{\overset{n}{\underset{1}{\Pi}}(x+f(n))\}^{1/n}=x.$$

(27.) If p denote one of the series of primes 2, 3, 5, 7, 11, . . ., then $\Sigma f(p)$ is convergent if $\Sigma f(p)/\log p$ is convergent. (Bonnet, *Liouville's Jour.*, viii. (1843), and Tchebichef, *ib.*, xvii. (1852).)

(28.) If $x<1$, show that the remainder after n terms of the series

$$1^r x+2^r x^2+3^r x^3+\ldots$$

is $<(n+1)^r x^{n+1}/\{1-(1+1/n)^r x\}$.

(29.) If $u_0, u_1, \ldots, u_n$ be all positive, and $\Sigma u_n x^n$ be convergent for all values of $x^2<a^2$, then

$$\Sigma x^n\left\{u_n-(n+1)au_{n+1}+\frac{(n+1)(n+2)}{1\cdot 2}a^2u_{n+2}-\&c.\right\}$$

will be convergent between the same limits of x.

(30.) Point out the fallacy of the following reasoning:—

Let $\Sigma=1+\frac{1}{2}+\frac{1}{3}+\ldots$ ad ∞,

then $\log_e 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

$=(1+\frac{1}{2}+\frac{1}{3}+\ldots)-2(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots)$

$=\Sigma-2\Sigma/2=0$.

(31.) If ρ and ρ' be the ratios of convergence of $\Sigma 1/\mathrm{P}_{r-1}(n)\{l^{r-1}n\}^{1+\alpha}$ and $\Sigma 1/\mathrm{P}_r(n)\{l^r n\}^{1+\alpha'}$ (see § 6), then $\mathrm{L}(\rho_n'-\rho_n)\mathrm{P}_{r-1}(n)=a$, when $n=\infty$. What conclusion follows regarding the convergence of the two series?

(32.) If Σu_n is divergent, then $\Sigma u_n/\mathrm{S}_{n-1}^{\alpha}$ is divergent if $a\not>1$ (where $\mathrm{S}_n=u_1+u_2+\ldots+u_n$), and $\Sigma u_n/\mathrm{S}_n^{a+1}$ is convergent if $a>0$. Hence show

that there can be no function $\phi(n)$ such that every series Σu_n is convergent or divergent, according as $\mathrm{L}_{n=\infty}\, \phi(n)u_n =$ or $\neq 0$. (Abel, *Œuvres*, ii., p. 197.)

(33.) If Σu_n be any convergent series whose terms are ultimately positive, we can always find another convergent series, Σv_n, whose terms are ultimately positive, and such that $\mathrm{L}\, v_n/u_n = \infty$.

If Σu_n be any divergent series whose terms are ultimately positive, we can always find another divergent series whose terms are ultimately positive, and such that $\mathrm{L}\, u_n/v_n = \infty$.

(These theorems are due to Du Bois-Reymond and Abel respectively; for concise demonstrations, see Thomae, *Elementare Theorie der Analytischen Functionen*. Halle, 1880.)

(34.) If $u_{n+1}/u_n = (n^\alpha + \mathrm{A}n^{\alpha-1} + \ldots)/(n^\alpha + \mathrm{A}'n^{\alpha-1} + \ldots)$, then Σu_n will be convergent or divergent according as $\mathrm{A} - \mathrm{A}' >$ or $\not> 1$. (Gauss, *Werke*, Bd. iii., p. 139.)

(35.) If $u_{n+1}/u_n = \alpha - \beta/n + \gamma/n^2 + \delta/n^3 + \ldots$, then Σu_n is convergent or divergent according as $\alpha <$ or > 1. If $\alpha = 1$, Σu_n is convergent only if $\beta > 1$. (Schlömilch, *Zeitschr. f. Math.*, x., p. 74.)

(36.) $\Sigma 1/u_n$ is convergent if $u_{n+2} - 2u_{n+1} + u_n$ is constant or ultimately increases with n. (Laurent, *Nouv. Ann.*, ser. ii., t. 8.)

(37.) If the terms of Σu_n are ultimately positive, then—

(I.) If $\psi(n)$ can be found such that $\psi(n)$ is positive, $\mathrm{L}\psi(n)u_n = 0$, and $\mathrm{L}\{\psi(n)u_n/u_{n+1} - \psi(n+1)\} > 0$, Σu_n is convergent.

(II.) If $\psi(n)$ be such that $\mathrm{L}\psi(n)u_n = 0$, $\mathrm{L}\{\psi(n)u_n/u_{n+1} - \psi(n+1)\} = 0$, and $\mathrm{L}\psi(n)u_n/\{\psi(n)u_n/u_{n+1} - \psi(n+1)\} \neq 0$, Σu_n is divergent.

(III.) If u_n/u_{n+1} can be expanded in descending powers of n, Σu_n is convergent or divergent according as $\mathrm{L}\{nu_n/u_{n+1} - (n+1)\} >$ or $\not> 0$.

(IV.) If u_n/u_{n+1} can be expanded in descending powers of n, Σu_n is convergent or divergent according as $\mathrm{L}nu_n =$ or $\neq 0$. (Kummer's Criteria, *Crelle's Jour.*, xiii. (1835) and xvi.)

(38.) If the terms of Σu_n be ultimately positive, and if, on and after a certain value of n, $a_n u_n/u_{n+1} - a_{n+1} > \mu$, where a_n is a function of n which is always positive for values of n in question, and μ is a positive constant, then Σu_n is convergent.

From this rule can be deduced the rules of Cauchy, De Morgan, and Bertrand. (Jensen, *Comptes Rendus*, c. vi., p. 729. 1888.)

Discuss the convergence of the following double series:—

(39.) $\Sigma(-)^{n-1}r^m/n$. (40.) $\Sigma(-1)^{n-1}r^m/n$!.

(41.) $\Sigma\{(n-1)^m/n^{m+1} - n^m/(n+1)^{m+1}\}$.

(42.) $\Sigma x^m y^n/(m+n)$. (43.) $\Sigma 1/(m+n)^2$.

(44.) $\Sigma 1/(m+n)$. (45.) $\Sigma 1/(m^2 - n^2)$.

(46.) Under what restrictions can $1/(1+x+y)$ be expanded in a double series of the form $1 + \Sigma \mathrm{A}_{m,n} x^m y^n$?

(47.) If $\Sigma u_{m,n}$ converge to S in the first way, and if its diagonal series be convergent, show that the diagonal series converges to S also.

Deduce Abel's Theorem regarding the product of two semi-convergent series. (See Stolz, *Math. Ann.*, xxiv.)

(48.) If u_n/u_{n-1} can be expanded in a series of the form $1+a_1/n+a_2/n^2+\ldots$, show that

1°. If $a_1=0, a_2=0, \ldots, a_{\mu-1}=0, a_\mu \neq 0$, then $u_n=\omega+r_n/n$, where ω is a definite constant $\neq 0$ and $\neq \infty$, and Lr_n is finite when $n=\infty$.

2°. If $a_1 \neq 0$, and the real part of a_1 be positive, then $Lu_n=\infty$ when $n=\infty$.

3°. If $a_1 \neq 0$, and the real part of $a_1=0$, then Lu_n is not infinite, but is not definite.

4°. If $a_1 \neq 0$, and the real part of a_1 be negative, then $Lu_n=0$.

Apply these results to the discussion of the convergency of $\Sigma u_n x^n$, and, in particular, to the Hypergeometric Series, and to the following series:—

$$\Sigma_{\mu+\nu}C_n(x+yi)^n, \quad \Sigma x^n/n^{\mu+\nu i}, \quad \Sigma_m C_n/(m+n)^p, \quad \Sigma(-)^n{}_m C_n/(m+n)^p.$$

(See Weierstrass, *Uber die Theorie die Analytischen Facultat.—Crelle's Jour.*, li.)

(49.) Discuss the convergence of $\Sigma_m C_n(\alpha-n\beta)^{n-1}x+n\beta^n$.

(50.) If u_n and v_n be positive for all values of n, never increase when n increases, and be such that $Lu_n=0$, $Lv_n=0$, when $n=\infty$, find the necessary and sufficient condition that $\Sigma(u_nv_1+u_{n-1}v_2+\ldots+u_1v_n)=\Sigma u_n \times \Sigma v_n$. (See Pringsheim, *Math. Ann.*, Bd. xxi.)

CHAPTER XXVII.

Binomial and Multinomial Series for any Index.

BINOMIAL SERIES.

§ 1.] We have already shown that, when m is a positive integer,

$$(1+x)^m = 1 + {}_mC_1x + {}_mC_2x^2 + \ldots + {}_mC_nx^n + \ldots + {}_mC_mx^m \quad (1),$$

where $${}_mC_n = m(m-1)\ \ldots\ (m-n+1)/n! \quad (2).$$

When m is not a positive integer, ${}_mC_n$, although it has still a definite analytical meaning, can no longer be taken to denote the number of n-combinations of m things; hence our former demonstration is no longer applicable. Moreover, the right-hand side of (1) then becomes an infinite series, and has, according to the principles of last chapter, no definite meaning unless the series be convergent. In cases where the series is divergent there cannot be any question, in the ordinary sense at least, regarding the equivalence of the two sides of (1).

As has already been shown, the series

$$1 + {}_mC_1x + {}_mC_2x^2 + \ldots + {}_mC_nx^n + \ldots \quad (3)$$

is convergent when x has any real value between -1 and $+1$; also when $x = +1$, provided $m > -1$; and when $x = -1$, provided $m > 0$. We propose now to inquire, whether in these cases the series (3) still represents $(1+x)^m$ in any legitimate sense.

In what follows, we suppose the numerical value of m to be a commensurable number;* also, for the present, we consider

* If m be incommensurable we must suppose it replaced by a commensurable approximation of sufficient accuracy.

only real values of x, and understand $(1+x)^m$ to be real and positive.

§ 2.] If we assume that $(1+x)^m$ can be expanded in a convergent series of ascending powers of x, then it is easily shown that the coefficient of x^n must be $m(m-1)\ldots(m-n+1)/n!$.

For, let

$$(1+x)^m = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \quad (1)$$

where

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \quad (2)$$

is convergent so long as $\operatorname{mod} x < \mathrm{R}$ (it will ultimately appear that $\mathrm{R} = 1$). Then, if h be so small that $\operatorname{mod}(x+h) < \mathrm{R}$, we have

$$(1+x+h)^m = a_0 + a_1(x+h) + a_2(x+h)^2 + \ldots + a_n(x+h)^n + \ldots \quad (3),$$

the series in (3) being convergent by hypothesis.

Hence by the principles of last chapter, we have

$$\frac{(1+x+h)^m - (1+x)^m}{(1+x+h)-(1+x)} = a_1\frac{(x+h)-x}{(x+h)-x} + a_2\frac{(x+h)^2-x^2}{(x+h)-x} + \ldots + a_n\frac{(x+h)^n-x^n}{(x+h)-x} + \ldots \quad (4),$$

the series in (4) being still convergent. Hence, if we make $h=0$, and observe that

$$\mathrm{L}\frac{(1+x+h)^m-(1+x)^m}{(1+x+h)-(1+x)} = m(1+x)^{m-1},$$

$$\mathrm{L}\frac{(x+h)^n-x^n}{(x+h)-x} = nx^{n-1},$$

by chap. XXV., § 12, we have

$$m(1+x)^{m-1} = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots \quad (5),$$

where the series on the right must still be convergent since $\mathrm{L}(n+1)a_{n+1}/na_n = \mathrm{L}a_{n+1}/a_n$ when $n=\infty$. Hence, multiplying by $1+x$, we deduce

$$m(1+x)^m = a_1 + (a_1+2a_2)x + \ldots + \{na_n + (n+1)a_{n+1}\}x^n + \ldots,$$

that is,

$$ma_0 + ma_1x + \ldots + ma_nx^n + \ldots = a_1 + (a_1+2a_2)x + \ldots + \{na_n + (n+1)a_{n+1}\}x^n + \ldots \quad (6).$$

By chap. xxvi., § 21, the coefficients of the powers of x on both sides of (6) must be equal. Hence

$$a_1 = ma_0,\ 2a_2 = (m-1)a_1,\ \ldots,\ (n+1)a_{n+1} = (m-n)a_n,\ \ldots \quad (7).$$

From (7) we deduce at once

$$a_1 = ma_0,\quad a_2 = m(m-1)a_0/2\,!,\ \ldots$$
$$a_n = m(m-1)\ \ldots\ (m-n+1)a_0/n\,!,\ \ldots$$

To determine a_0 we may put $x = 0$. We then get from (1), $a_0 = 1^m = 1$ (if we suppose, as usual, the real positive value of any root involved to be alone in question). We therefore have

$$(1+x)^m = 1 + \Sigma_m C_n x^n \qquad (8).$$

The theorem is therefore established; and we see that the hypothesis under which we started is not contradicted provided $\mod x < 1$, this being a sufficient condition for the convergency of $\Sigma_m C_n x^n$.

§ 3.] Although the assumption that $(1+x)^m$ can be expanded in a series of ascending powers of x leads to no contradiction in the process of determining the coefficients, so long as $\mod x < 1$; this fact can scarcely be regarded as sufficient evidence for the validity of a theorem so fundamentally important. We proceed, therefore, to establish the following theorem in which we start from the series in the first instance.

Whenever the series $1 + \Sigma_m C_n x^n$ *is convergent, its sum is the real positive value of* $(1+x)^m$.

The fundamental idea of the following demonstration is due to Euler;* but it involves important additions, due mainly to Cauchy, which were necessary to make it accurate according to the modern view of the nature of infinite series.

Let us denote the series

$$1 + {}_mC_1 x + {}_mC_2 x^2 + \ldots + {}_mC_n x^n + \ldots \qquad (1)$$

by the symbol $f(m)$.

So long as $-1 < x < +1$, $f(m)$ is an absolutely convergent series, and (by chap. xxvi., § 19) is a continuous function both of m and of x.

* *Nov. Comm. Petrop.*, t. xix. (1775).

Hence, m_1 and m_2 being any real values of m, we have

$$f(m_1)f(m_2) = \{1 + \Sigma_{m_1}C_n x^n\}\{1 + \Sigma_{m_2}C_n x^n\},$$

$$= 1 + \Sigma({}_{m_1}C_n + {}_{m_2}C_1\,{}_{m_1}C_{n-1} + {}_{m_2}C_2\,{}_{m_1}C_{n-2} + \ldots + {}_{m_2}C_n)x^n \quad (2),$$

where the last written series is convergent (by chap. xxvi., § 14), since the two series, $1 + \Sigma_{m_1}C_n x^n$ and $1 + \Sigma_{m_2}C_n x^n$, are absolutely convergent.

Now, by chap. xxiii., § 8, Cor. 5,

$${}_{m_1}C_n + {}_{m_2}C_1\,{}_{m_1}C_{n-1} + {}_{m_2}C_2\,{}_{m_1}C_{n-2} + \ldots + {}_{m_2}C_n = {}_{m_1+m_2}C_n;$$

hence
$$f(m_1)f(m_2) = 1 + \Sigma_{m_1+m_2}C_n x^n,$$
$$= f(m_1 + m_2) \quad (3).$$

In like manner, we can show that

$$f(m_1 + m_2)f(m_3) = f(m_1 + m_2 + m_3).$$

Hence
$$f(m_1)f(m_2)f(m_3) = f(m_1 + m_2 + m_3);$$

and, in general, v being any positive integer,

$$f(m_1)f(m_2)\ \ldots\ f(m_v) = f(m_1 + m_2 + \ldots + m_v) \quad (4).$$

This result may be called the *Addition Theorem for the Binomial Series.*

If in (4) we put $m_1 = m_2 = \ldots = m_v = 1$, then we deduce

$$\{f(1)\}^v = f(v) \quad (5),$$

where v is any positive integer.

If in (4) we put $m_1 = m_2 = \ldots = m_v = p/q$, where p and q are any positive integers, and also put $v = q$, we deduce

$$\{f(p/q)\}^q = f(p) \quad (6).$$

Hence, by (5),
$$\{f(p/q)\}^q = \{f(1)\}^p \quad (7).$$

Again, if in (3) we put $m_1 = m$, $m_2 = -m$, we deduce

$$f(m)f(-m) = f(m - m) = f(0) \quad (8).$$

Hence
$$f(-m) = f(0)/f(m) \quad (9).$$

These properties of the series (1) hold so long as $-1 < x < +1$, and they are sufficient to determine its sum for all real values of m.

For, since ${}_1C_1 = 1$, ${}_1C_2 = 0, \ldots, {}_1C_n = 0, \ldots {}_0C_1 = 0$, ${}_0C_2 = 0, \ldots, {}_0C_n = 0, \ldots$ we have

$$f(1) = 1 + x, \quad f(0) = 1.$$

Suppose, now, m to be a positive integer. Then, by (5),

$$(1+x)^m = f(m) = 1 + {}_mC_1 x + {}_mC_2 x^2 + \ldots + {}_mC_m x^m \quad (10),$$

where the series terminates, since ${}_mC_{m+1} = 0$, ${}_mC_{m+2} = 0, \ldots$, when m is a positive integer. This is another demonstration of that part of the theorem with which we are already familiar.

Next, let m be any positive commensurable quantity, say p/q, where p and q are positive integers. Then, by (7),

$$\{f(p/q)\}^q = (1+x)^p \quad (11).$$

Hence $f(p/q)$ is one of the qth roots of the positive* quantity $(1+x)^p$. But $f(p/q)$ is necessarily real; hence, if $(1+x)^{p/q}$ denote, as usual, the real positive qth root of $(1+x)^p$, we must have

$$f(p/q) = \pm (1+x)^{p/q} \quad (12).$$

The only remaining question is the sign of the right-hand side of (12).

Since $f(p/q)$ is a continuous function both of p/q and of x, its equivalent $\pm (1+x)^{p/q}$ must be a continuous function both of p/q and of x. Now $(1+x)^{p/q}$ does not vanish (or become infinite) for any values of p/q or of x admissible under our present hypothesis; and being the equivalent of a continuous function it cannot change sign without passing through 0. Hence only one of the two possible signs is admissible; and we can settle which by considering any particular case. Now, when $x = 0$, $f(p/q) = +1$. Hence the positive sign must be taken; and we establish finally that

$$f(p/q) = +(1+x)^{p/q},$$

that is,

$$(1+x)^m = 1 + {}_mC_1 x + {}_mC_2 x^2 + \ldots + {}_mC_n x^n + \ldots \quad (13),$$

when m is any positive commensurable quantity.

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* Positive, since $-1 < x < 1$, by hypothesis.

Finally, let m be any negative commensurable quantity, say $m = -m'$, where m' is a real positive commensurable quantity.

By (9) we have

$$f(-m') = f(0)/f(m') = 1/f(m').$$

Hence, by (13),

$$f(-m') = 1/(1+x)^{m'},$$
$$= (1+x)^{-m'},$$

that is,

$$(1+x)^m = 1 + {}_mC_1x + {}_mC_2x^2 + \ldots + {}_mC_nx^n + \ldots \quad (14),$$

where m is any commensurable negative quantity.

The results of (10), (13), and (14) establish the Binomial Theorem for all values of x such that $-1 < x < +1$. It remains to consider the extreme cases.

When $x = +1$, the series (1) reduces to

$$1 + {}_mC_1 + {}_mC_2 + \ldots + {}_mC_n + \ldots$$

This series is semi-convergent if $-1 < m < 0$, absolutely convergent if $m > 0$. Hence, by Abel's Theorem, chap. XXVI., § 20,

$$(1+\overline{1-0})^m = \operatorname*{L}_{x=1-0} \{1 + {}_mC_1x + {}_mC_2x^2 + \ldots + {}_mC_nx^n + \ldots\},$$

that is,

$$2^m = 1 + {}_mC_1 + {}_mC_2 + \ldots + {}_mC_n + \ldots \quad (15),$$

provided $m > -1$, with the condition that, when $-1 < m < 0$, the order of the terms in the series of (15) must not be altered.

If $0 < x < 1$, we have, by the general case already established,

$$(1-x)^m = 1 - {}_mC_1x + {}_mC_2x^2 - \ldots (-)^n {}_mC_nx^n + \ldots$$

Hence, since the series

$$1 - {}_mC_1 + {}_mC_2 - \ldots (-)^n {}_mC_n + \ldots$$

is convergent if $m > 0$, we have, by Abel's Theorem,

$$(1-\overline{1-0})^m = \operatorname*{L}_{x=1-0} (1 - {}_mC_1x + {}_mC_2x^2 - \ldots (-)^n {}_mC_nx^n + \ldots),$$

that is,

$$0 = 1 - {}_mC_1 + {}_mC_2 - \ldots (-)^n {}_mC_n + \ldots \quad (16),$$

provided m be positive.

The results of (15) and (16) complete the demonstration of

the Binomial Theorem in all cases where its validity is in question.

Cor. If $x \neq y$, it follows from the above result that we can always expand $(x+y)^m$ in an absolutely convergent series. We have in fact, if $x > y$, that is, $y/x < 1$,

$$(x+y)^m = x^m(1+y/x)^m,$$
$$= x^m\{1 + {}_mC_1(y/x) + {}_mC_2(y/x)^2 + \ldots + {}_mC_n(y/x)^n + \ldots\},$$
$$= x^m + {}_mC_1x^{m-1}y + {}_mC_2x^{m-2}y^2 + \ldots + {}_mC_nx^{m-n}y^n + \ldots \quad (17);$$

and if $x < y$, that is, $x/y < 1$,

$$(x+y)^m = y^m(1+x/y)^m,$$
$$= y^m\{1 + {}_mC_1(x/y) + {}_mC_2(x/y)^2 + \ldots + {}_mC_n(x/y)^n + \ldots\},$$
$$= y^m + {}_mC_1y^{m-1}x + {}_mC_2y^{m-2}x^2 + \ldots + {}_mC_ny^{m-n}x^n + \ldots \quad (18).$$

If m be a positive integer, both the formulæ (17) and (18) will be admissible because both series terminate. But, if m be not a positive integer, only one of the two series will be convergent.

§ 4.] The general formulæ of last paragraph contain a vast number of particular cases. To help the student to detect these particular cases under the various disguises which they assume, we proceed to draw his attention to several of the more commonly occurring. The difficulties of identification are in reality in most cases much smaller than they at first sight appear. We assume in all cases that the values of the variables are such that the series are convergent.

Example 1.

$$(1+x)^{-1} = 1 - x + x^2 - \ldots + (-)^n x^n + \ldots;$$
$$(1-x)^{-1} = 1 + x + x^2 + \ldots + x^n + \ldots$$

For $(1+x)^{-1} = 1 + \Sigma_{-1}C_n x^n$;

and $${}_{-1}C_n = -1(-1-1)(-1-2)\ldots(-1-n+1)/n!,$$
$$= (-)^n 1.2.3\ldots n/n!,$$
$$= (-)^n 1.$$

$$(1-x)^{-1} = 1 + \Sigma_{-1}C_n(-x)^n;$$

and $${}_{-1}C_n(-x)^n = (-)^n(-)^n x^n = (-)^{2n}x^n$$
$$= x^n.$$

Example 2.

$$(1+x)^{-2} = 1 - 2x + 3x^2 - \ldots + (-)^n(n+1)x^n + \ldots;$$
$$(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots + (n+1)x^n + \ldots$$

For $${}_{-2}C_n = -2(-2-1)\ldots(-2-n+1)/n!,$$
$$= (-)^n(n+1).$$

Example 3.

$$(1+x)^{-3}=1-3x+6x^2-\ldots+(-)^n\tfrac{1}{2}(n+1)(n+2)x^n+\ldots;$$
$$(1-x)^{-3}=1+3x+6x^2+\ldots+\tfrac{1}{2}(n+1)(n+2)x^n+\ldots$$

Example 4.

$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\ldots+(-)^{n-1}\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots 2n}x^n+\ldots;$$
$$(1-x)^{\frac{1}{2}}=1-\tfrac{1}{2}x-\tfrac{1}{8}x^2-\tfrac{1}{16}x^3-\ldots-\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots 2n}x^n+\ldots$$

Example 5.

$$(1+x)^{-\frac{1}{2}}=1-\tfrac{1}{2}x+\tfrac{3}{8}x^2-\tfrac{5}{16}x^3+\ldots+(-)^n\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}x^n+\ldots;$$
$$(1-x)^{-\frac{1}{2}}=1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\tfrac{5}{16}x^3+\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}x^n+\ldots$$

Example 6.

$$(1+x)^{m/2}=1+\frac{m}{1}\frac{x}{2}+\frac{m(m-2)}{2!}\left(\frac{x}{2}\right)^2+\ldots+\frac{m(m-2)(m-4)\ldots(m-2n+2)}{n!}\left(\frac{x}{2}\right)^n+\ldots;$$
$$=1+\frac{m}{2}x+\frac{m(m-2)}{2.4}x^2+\ldots+\frac{m(m-2)(m-4)\ldots(m-2n+2)}{2.4.6\ldots 2n}x^n+\ldots$$
$$(1+x)^{-m/2}=1+\Sigma(-)^n\frac{m(m+2)(m+4)\ldots(m+2n-2)}{2.4.6\ldots 2n}x^n.$$

Example 7.

$$(1+x)^{p/q}=1+\Sigma\frac{p(p-q)(p-2q)\ldots(p-nq+q)}{q.2q.3q\ldots nq}x^n;$$
$$(1-x)^{-p/q}=1+\Sigma\frac{p(p+q)(p+2q)\ldots(p+nq-q)}{q.2q.3q\ldots nq}x^n.$$

Example 8.

$$(1-x)^{-m}=1+\Sigma\frac{m(m+1)\ldots(m+n-1)}{n!}x^n.$$

It will be observed that the coefficient of x^n in this last expansion, when m is integral, is (see chap. xxiv., § 10) the number $({}_mH_n)$ of n-combinations of m things when repetition is allowed. It is therefore usual to denote this coefficient by the symbol ${}_mH_n$, m being now unrestricted in value. We shall return to this function later on.

Example 9.

$$\tfrac{1}{2}\{(1+x)^m+(1-x)^m\}=1+{}_mC_2x^2+{}_mC_4x^4+\ldots+{}_mC_{2n}x^{2n}+\ldots;$$
$$\tfrac{1}{2}\{(1+x)^m-(1-x)^m\}={}_mC_1x+{}_mC_3x^3+\ldots+{}_mC_{2n-1}x^{2n-1}+\ldots$$

Ultimate Sign of the Terms.—Infinite Binomial Series belong to one or other of two classes as regards the ultimate sign of the terms—1st, those in which the signs of the terms are ultimately alternately positive and negative; 2nd, those in which all the terms are ultimately of the same sign.

If x and m denote positive quantities (m of course not a positive integer),

1st. The expansions of $(1+x)^m$ and $(1+x)^{-m}$ both belong to the first class. In $(1+x)^m$ the first negative term will be that containing x^{n+1}, where n is the least integer which exceeds m. In $(1+x)^{-m}$ the first negative term is of course the second.

2nd. The expansions of $(1-x)^m$, $(1-x)^{-m}$, both belong to the second class. In $(1-x)^m$ the terms will have the same sign on and after the term in x^n, n being the least integer which exceeds m, and this sign will be + or − according as n is even or odd. In $(1-x)^{-m}$ all the terms are positive after the first.

§ 5.] A great variety of series suitable for various purposes can be readily deduced from the Binomial Series; and, conversely, many series can be summed by identifying them with particular cases of the Binomial Series itself, or with some series deducible from it.

The following cases deserve special attention, because they include so many of the series usually treated in elementary text-books as particular cases, and because the methods by which the summation is effected are typical.

Consider the series $\Sigma\phi_r(n)_m\mathrm{C}_n x^n$, where $\phi_r(n)$ is any integral function of n of the rth degree. Such a series stands in the same relation to the simple Binomial Series as does the Integro-Geometric to the simple Geometric Series. We may therefore speak of it as an *Integro-Binomial Series.*

We may always, by the process of chap. v., § 22, establish an identity of the following kind,

$$\phi_r(n) \equiv \mathrm{A}_0 + \mathrm{A}_1 n + \mathrm{A}_2 n(n-1) + \ldots + \mathrm{A}_r n(n-1)\ldots(n-r+1) \quad (1),$$

where $\mathrm{A}_0, \mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_r$ are constants, that is, are independent of n.

We can therefore write the general term of the Integro-Binomial Series in the following form :—

$$\begin{aligned}\phi_r(n)_m\mathrm{C}_n x^n &\equiv \mathrm{A}_0{}_m\mathrm{C}_n x^n + \mathrm{A}_1 n\,{}_m\mathrm{C}_n x^n + \ldots + \mathrm{A}_r n(n-1)\ldots \\ &\qquad\qquad (n-r+1)_m\mathrm{C}_n x^n, \\ &\equiv \mathrm{A}_0{}_m\mathrm{C}_n x^n + m\mathrm{A}_1 x_{m-1}\mathrm{C}_{n-1}x^{n-1} \\ &\quad + m(m-1)\mathrm{A}_2 x^2{}_{m-2}\mathrm{C}_{n-2}x^{n-2} + \ldots + m(m-1)\ldots \\ &\qquad\qquad (m-r+1)\mathrm{A}_r x^r{}_{m-r}\mathrm{C}_{n-r}x^{n-r} \quad (2).\end{aligned}$$

Hence, if the summation proceed from 0 to ∞, we evidently have

$$\sum_0^\infty \phi_r(n)_m C_n x^n = A_0 \sum_0^\infty {}_m C_n x^n + mA_1 x \sum_1^\infty {}_{m-1}C_{n-1} x^{n-1} + \ldots$$
$$+ m(m-1)\ldots(m-r+1)A_r x^r \sum_r^\infty {}_{m-r}C_{n-r} x^{n-r} \quad (3),$$
$$= A_0(1+x)^m + mA_1 x(1+x)^{m-1} + \ldots$$
$$+ m(m-1)\ldots(m-r+1)A_r x^r (1+x)^{m-r}.$$

Since all the Binomial Series are evidently complete.* Hence

$$\sum_0^\infty \phi_r(n)_m C_n x^n = \{A_0 + mA_1 x/(1+x) + m(m-1)A_2 x^2/(1+x)^2 + \ldots$$
$$+ m(m-1)\ldots(m-r+1)A_r x^r/(1+x)^r\}(1+x)^m \quad (4);$$

and the summation to infinity of the Integro-Binomial Series is effected.†

The formula will still apply when m is a positive integer, although in that case the series on the left of (4) has not an infinite number of terms. The only peculiarity is that a number of the terms within the crooked bracket on the right-hand side of (4) may become zero.

Cor. *We can in general sum the series* $\sum_0^\infty \phi_r(n)_m C_n x^n/(n+a)(n+b)\ldots(n+k)$, *where* $a, b, \ldots, k$ *are unequal positive integers, in ascending order of magnitude.*

For by introducing the factors $n+1, n+2, \ldots, n+a-1, n+a+1, n+a+2, \ldots, n+b-1$, &c., we can reduce the general term to the form

$$\psi(n)_{m+k} C_{n+k} x^{n+k}/(m+1)(m+2)\ldots(m+k)x^k \quad (5);$$

where $\psi(n)$ is an integral function of n, namely, $\phi_r(n)$ multiplied by all the factors introduced.

---

* If the lower limit of summation be not 0, then the Binomial Series on the right-hand side of (3) will not all be complete, and the sum will not be quite so simple as in (4).

† It may be remarked that the series is evidently convergent when $x<1$. The examination of the convergence when $x=1$ will form a good exercise on chap. XXVI.

Hence

$$\overset{\infty}{\Sigma}\phi_r(n)_m C_n x^n/(n+a)(n+b)\ldots(n+k)$$
$$= \{\overset{\infty}{\Sigma}\psi(n)_{m+k} C_{n+k} x^{n+k}\}/(m+1)(m+2)\ldots(m+k)x^k \quad (6).$$

The summation of the series inside the crooked bracket may be effected; for it is an Integro-Binomial Series. Hence the summation originally proposed is always possible.

We have not indicated the lower limit of the summation, and it is immaterial what it is. Even if the lower limit of summation be 0, the Binomial Series into which the right-hand side of (6) is decomposed will not all be complete (see Example 6, below).

It should also be noticed that this method will not apply if m be such that any of the factors $m+1$, $m+2$, . . ., $m+k$ vanish. In such cases the right-hand side of (6) would become indeterminate, and the evaluation of its limit would be troublesome.

The above method can be varied in several ways, which need not be specified in detail. It is sufficient to add that by virtue of Abel's Theorem (chap. xxvi., § 20) all the above summations hold when $x = \pm 1$, provided the series involved remain convergent.

Example 1. To expand $(x+y)^m$ in a highly convergent series when x and y are nearly equal. From the obvious identities

$$\{(x+y)/2x\}^m = \{2x/(x+y)\}^{-m} = \{1+(x-y)/(x+y)\}^{-m},$$
$$\{(x+y)/2y\}^m = \{2y/(x+y)\}^{-m} = \{1-(x-y)/(x+y)\}^{-m},$$
$$(x+y)^m\{1/(2x)^m \pm 1/(2y)^m\} = \{1+(x-y)/(x+y)\}^{-m} \pm \{1-(x-y)/(x+y)\}^{-m},$$

we deduce at once

$$(x+y)^m = 2^m x^m\left\{1+\Sigma_m(-)^n H_n\left(\frac{x-y}{x+y}\right)^n\right\},$$
$$= 2^m y^m\left\{1+\Sigma_m H_n\left(\frac{x-y}{x+y}\right)^n\right\},$$

where $_mH_n = m(m+1)\ldots(m+n-1)/n!$,

$$= \frac{2^{m+1}x^m y^m}{x^m+y^m}\left\{1+\frac{m(m+1)}{2!}\left(\frac{x-y}{x+y}\right)^2+\frac{m(m+1)(m+2)(m+3)}{4!}\left(\frac{x-y}{x+y}\right)^4 + \ldots\right\},$$

$$= \frac{2^{m+1}x^m y^m}{x^m-y^m}\left\{\frac{m}{1!}\left(\frac{x-y}{x+y}\right)+\frac{m(m+1)(m+2)}{3!}\left(\frac{x-y}{x+y}\right)^3+\ldots\right\}.$$

All these series are highly convergent, since $(x-y)/(x+y)$ is small.

Example 2. To sum the series

$$\frac{2}{9}+\frac{2}{2!}\left(\frac{2}{9}\right)^2+\frac{2.5}{3!}\left(\frac{2}{9}\right)^3+\frac{2.5.8}{4!}\left(\frac{2}{9}\right)^4\ldots$$

If we denote this series by $u_1+u_2+u_3+\ldots$, we see that

$$u_n=\frac{2.5\ldots\{2+(n-2)3\}}{n!}\frac{2^n}{3^{2n}},$$

$$=\frac{\frac{1}{3}.\frac{2}{3}.\frac{5}{3}\ldots(-\frac{1}{3}+n-1)}{n!}\left(\frac{2}{3}\right)^n,$$

$$=-\frac{(-\frac{1}{3})(-\frac{1}{3}+1)(-\frac{1}{3}+2)\ldots(-\frac{1}{3}+n-1)}{n!}\left(\frac{2}{3}\right)^n,$$

$$=-(-)^n\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)\ldots(\frac{1}{3}-n+1)}{n!}\left(\frac{2}{3}\right)^n.$$

Hence

$$1-(u_1+u_2+u_3+\ldots)=(1-\tfrac{2}{3})^{1/3},$$
$$=1/\sqrt[3]{3}.$$

Therefore, $$u_1+u_2+u_3+\ldots=1-1/\sqrt[3]{3}.$$

Example 3. To sum the series

$$m+\frac{m(m-1)}{1}+\frac{m(m-1)(m-2)}{1.2}+\ldots,$$

whenever it is convergent.

Here we have

$$u_{n+1}=\frac{m(m-1)(m-2)\ldots(m-n)}{n!},$$

$$=\frac{m(m-1)(m-1-1)\ldots(m-1-n+1}{n!},$$

$$=m_{m-1}C_n.$$

Hence

$$u_1+u_2+u_3+\ldots=m\{1+{}_{m-1}C_1+{}_{m-1}C_2+\ldots\}$$
$$=m\{1+1\}^{m-1}=m2^{m-1},$$

provided $m-1>-1$, that is $m>0$.

It should be observed that we have at once from § 2 (5) the equation

$$m(1+x)^{m-1}=1_mC_1+2_mC_2x+\ldots+n_mC_nx^{n-1}+\ldots \quad (1),$$

from which the above result follows by putting $x=1$.

By repeating the process of § 2, we should deduce the equation

$$m(m-1)\ldots(m-k+1)(1+x)^{m-k}=1.2\ldots k_mC_k+2.3\ldots(k+1)_mC_{k+1}x+\ldots$$
$$+(n-k+1)(n-k+2)\ldots n_mC_nx^{n-k}+\ldots \quad (2),$$

whence it follows that

$$m(m-1)\ldots(m-k+1)2^{m-k}=1.2\ldots k_mC_k+2.3\ldots(k+1)_mC_{k+1}+\ldots \quad (3),$$

provided $m>k$. These results might also be easily established by the method first used.

Example 4. To sum the series

$$\frac{1}{1.2\ldots k}+\frac{{}_mC_1x}{2.3\ldots(k+1)}+\frac{{}_mC_2x^2}{3.4\ldots(k+2)}+\ldots$$

Here we have

$$u_{n+1}=\frac{{}_mC_nx^n}{(n+1)(n+2)\ldots(n+k)},$$

$$=\frac{{}_{m+k}C_{n+k}x^{n+k}}{(m+1)(m+2)\ldots(m+k)x^k}.$$

Hence

$$\frac{(1+x)^{m+k}}{(m+1)(m+2)\ldots(m+k)x^k}=\frac{1}{(m+1)(m+2)\ldots(m+k)x^k}\{1+{}_{m+k}C_1x+{}_{m+k}C_2x^2+\ldots+{}_{m+k}C_{k-1}x^{k-1}\}+\{u_1+u_2+u_3+\ldots\}.$$

Therefore

$$u_1+u_2+u_3+\ldots=\frac{(1+x)^{m+k}-1-{}_{m+k}C_1x-{}_{m+k}C_2x^2-\ldots-{}_{m+k}C_{k-1}x^{k-1}}{(m+1)(m+2)\ldots(m+k)x^k} \quad (4).$$

If $m>-k-1$, this gives as a particular case

$$\Sigma_m C_n/(n+1)(n+2)\ldots(n+k)=\{2^{m+k}-1-\sum_{s=1}^{s=k-1}{}_{m+k}C_s\}/(m+1)(m+2)\ldots(m+k) \quad (5).$$

The formulæ (1), (2), (3), (4), and (5) contain of course a considerable variety of particular cases.

Example 5. Evaluate $\sum_0^\infty n^3\,{}_mC_nx^n$.

Let $n^3\equiv A_0+A_1n+A_2n(n-1)+A_3n(n-1)(n-2)$, then we have the following calculation to determine A_0, A_1, A_2, A_3 (see chap. v., § 22).

	1 +0 +0 | +0		$A_0=0,$
1	0 +1 +1		
	1 +1 | +1		$A_1=1,$
2	0 +2		
	1 | +3		$A_2=3,\ A_3=1.$

Hence

$$\sum_0^\infty n^3\,{}_mC_nx^n=0\cdot\sum_0^\infty {}_mC_nx^n+1mx\sum_1^\infty {}_{m-1}C_{n-1}x^{n-1}+3m(m-1)x^2\sum_2^\infty {}_{m-2}C_{n-2}x^{n-2}+m(m-1)(m-2)x^3\sum_3^\infty {}_{m-3}C_{n-3}x^{n-3},$$

$$=mx(1+x)^{m-1}+3m(m-1)x^2(1+x)^{m-2}+m(m-1)(m-2)x^3(1+x)^{m-3},$$

$$=\{m^3x^3+m(3m-1)x^2+mx\}(1+x)^{m-3}.$$

Example 6. Evaluate $\sum_0^\infty {}_mC_nx^n/(n+2)(n+4)$.

$$\frac{{}_mC_nx^n}{(n+2)(n+4)}=\frac{(n+1)(n+3)\,{}_{m+4}C_{n+4}x^{n+4}}{x^4(m+1)(m+2)(m+3)(m+4)},$$

$$(n+1)(n+3)\equiv n^2+4n+3,$$

$$\equiv A_0+A_1(n+4)+A_2(n+4)(n+3).$$

	1 +4 +3		
−4	0 −4 +0		
	1 +0 | +3		$A_0=3,$
−3	0 −3		
	1 | −3		$A_1=-3,\ A_2=1.$

We therefore have

$$\sum_0^\infty \frac{{}_mC_n x^n}{(n+2)(n+4)} = \frac{1}{x^4(m+1)(m+2)(m+3)(m+4)} \{3\sum_0^\infty {}_{m+4}C_n x^{n+4} - 3(m+4)x \sum_0^\infty {}_{m+3}C_n x^{n+3} + (m+4)(m+3)x^2 \sum_0^\infty {}_{m+2}C_n x^{n+2}\},$$

$$= \frac{1}{x^4(m+1)\ldots(m+4)} [3\{(1+x)^{m+4} - 1 - {}_{m+4}C_1 x - {}_{m+4}C_2 x^2 - {}_{m+4}C_3 x^3\} - 3(m+4)x\{(1+x)^{m+3} - 1 - {}_{m+3}C_1 x - {}_{m+3}C_2 x^2\} + (m+4)(m+3)x^2\{(1+x)^{m+2} - 1 - {}_{m+2}C_1 x\}],$$

$$= \frac{1}{x^4(m+1)(m+2)(m+3)(m+4)} [\{(m+1)(m+3)x^2 - 3(m+2)x + 3\}(1+x)^{m+2} + \{\tfrac{1}{2}(m+3)(m+4)x^2 - 3\}].$$

EXERCISES IX.

Expand each of the following in ascending powers of x to 5 terms; and in each case write down and simplify the coefficient of x^r.

(1.) $(1+x)^{\frac{1}{2}}$. (2.) $(1-x)^{-\frac{1}{2}}$. (3.) $(1-x)^{-\frac{3}{2}}$.

(4.) $(2-\frac{1}{2}x)^{\frac{3}{2}}$. (5.) $(a+3x)^{\frac{1}{3}}$. (6.) $\sqrt[3]{(a^3-x^3)}$.

(7.) $\sqrt[n]{(1-nx)}$. (8.) $1/(1-3x^2)^{\frac{1}{3}}$. (9.) $(x-1/x)^{-n}$.

(10.) Write down the first four terms in the expansion of $\{(a+x)/(a-x)\}^{\frac{1}{2}}$ in ascending powers of x.

Determine the numerically greatest term in

(11.) $(3+x)^{\frac{1}{2}}$, $x<3$. (12.) $(2-3/2)^{11/2}$. (13.) $(1-5/7)^{-12/3}$.

(14.) Find the greatest term in $(1+x)^{-n}$, when $x=\frac{2}{3}$, $n=4$.

(15.) If n be a positive integer, find the greatest term in $(n-1/n)^{2n+1}$.

(16.) The sum of the middle terms of $(1+x)^m$ for all even values of m (including 0) is $(1-4x)^{-\frac{1}{2}}$.

(17.) $x^n = 1 + n\left(1-\frac{1}{x}\right) + \frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2 + \ldots$

(18.) Show that, if m exceed a certain value, then

$$2^m = 1 + \frac{(m+1)m}{2!} + \frac{(m+1)m(m-1)(m-2)}{4!} + \ldots$$

(19.) Sum the series

$$a - (a+b)m + (a+2b)\frac{m(m-1)}{2!} - (a+3b)\frac{m(m-1)m-2)}{3!} + \ldots,$$

for such values of m as render the series convergent.

(20.) $$\sqrt{27} = 2 + \frac{5}{2!3} + \frac{5.7}{3!3^2} + \ldots$$

(21.) $$\frac{23}{24} - \frac{2}{3}\sqrt{2} = \frac{1}{2^2 3!} - \frac{1.3}{2^4 4!} + \frac{1.3.5}{2^6 5!} - \ldots$$

(22.) Sum to infinity

$$\frac{1}{6}+\frac{1.4}{6.12}+\frac{1.4.7}{6.12.18}+\ldots$$

(23.) Sum the series

$$m(m-1)+\frac{m(m-1)(m-2)}{1!}+\ldots+\frac{m(m-1)\ldots(m-r+1)}{(r-2)!}+\ldots,$$

for such values of m as render the series convergent.

(24.) If n be even, show that

$$n(n+2)\ldots(2n-2)/1.3\ldots(n-1)=2^{n-1}.$$

(25.) In the expansion of $(1-x)^{-m}$ no coefficient can be equal to the next following unless all the coefficients are equal.

(26.) Prove by induction that

$$1+m+\frac{m(m+1)}{2!}+\ldots+\frac{m(m+1)\ldots(m+r-1)}{r!}=\frac{(m+r)!}{m!\,r!},$$

where r is a positive integer. Hence show that, if $x<1$,

$$(1-x)^{-m}=\Sigma\frac{(m+r-1)!\,x^r}{(m-1)!\,r!}.$$

(27.) The sum of the first r coefficients in $1/\sqrt[n]{(1-x)}$: the coefficient of the rth term $=1+n(r-1):1$.

(28.) If $F(a)=1+\frac{ax}{1!}+\frac{a(a+k)}{2!}x^2+\frac{a(a+k)(a+2k)}{3!}x^3+\ldots$, the series being absolutely convergent, then

$$F(a)F(b)=F(a+b).$$

What is the condition for the convergency of the series?

(29.) Show that

$$\frac{x^2}{2}-{}_nC_1\frac{x^3}{3}+{}_nC_2\frac{x^4}{4}-\ldots=[1-\{(n+1)x+1\}(1-x)^{n+1}]/(n+1)(n+2).$$

Sum the following series, so far as they are convergent:—

(30.) $\Sigma(n-1)^2m(m-1)\ldots(m-n+1)x^n/n!$, from $n=1$ to $n=\infty$.

(31.) $\Sigma(-)^{n-1}(n+1)(n+2)1.3.5\ldots(2n-5)x^n/n!$, from $n=0$ to $n=\infty$.

(32.) $\Sigma m(m+1)\ldots(m+n-1)x^n/(n+3)n!$, from $n=0$ to $n=\infty$.

(33.) $\Sigma(n-1)^2 1.4.7\ldots(3n-2)/(n+2)(n+3)n!$, from $n=1$ to $n=\infty$.

(34.) Why does the method of summation given in § 5 not apply to $\sum^{\infty}x^n/(n+1)$?

SERIES DEDUCED BY EXPANSION OF RATIONAL FUNCTIONS OF x.

§ 6.] Since every rational function of x can be expressed in the form $I+F$, where I is an integral function of x, and F a proper rational fraction, and since F can, by chap. viii., § 7, be

expressed in the form $\Sigma A(x-a)^{-n}$, where A is constant, it follows that for certain values of x a rational function of x can be expanded in a series of ascending powers of x, and for certain other values of x in a series of descending powers of x.* We shall have occasion to dwell more on the general consequences of this result in a later chapter, where we deal with the theory of Recurring Series. There are, however, certain particular cases which may with advantage be studied here.

§ 7.] *Series for expressing* $\alpha^n+\beta^n$ *and* $(\alpha^{n+1}-\beta^{n+1})/(\alpha-\beta)$ *in terms of* $\alpha\beta$ *and* $\alpha+\beta$, *n being a positive integer.*

If we denote the elementary symmetric functions $\alpha+\beta$ and $\alpha\beta$ by p and q respectively, it follows from chap. xviii., § 2, that we can express the symmetric functions $\alpha^n+\beta^n$, $(\alpha^{n+1}-\beta^{n+1})/(\alpha-\beta)$ as follows:—

$$\alpha^n+\beta^n = a_0p^n + a_1p^{n-2}q + \ldots + a_rp^{n-2r}q^r + \ldots \quad (1),$$

$$(\alpha^{n+1}-\beta^{n+1})/(\alpha-\beta) = b_0p^n + b_1p^{n-2}q + \ldots + b_rp^{n-2r}q^r + \ldots \quad (2),$$

where both series terminate.

By the methods of chap. viii., § 8, or by direct verification we can establish the identity

$$\frac{2-px}{1-px+qx^2} \equiv \frac{2-(\alpha+\beta)x}{(1-\alpha x)(1-\beta x)} \equiv \frac{1}{1-\alpha x}+\frac{1}{1-\beta x} \quad (3).$$

Now if x be (as it obviously always may be) taken so small that $px-qx^2<1$, we have by the Binomial Theorem

$$\frac{2-px}{1-px+qx^2} \equiv (2-px)\{1-(px-qx^2)\}^{-1} = (2-px)\{1+(px-qx^2) + (px-qx^2)^2+\ldots+(px-qx^2)^n+\ldots\} \quad (4).$$

Now (by chap. xxvi., § 34) if x be taken between $-a$ and $+a$, a being such that the numerical value of $\pm pa \pm qa^2 < 1$, that arrangement of signs being taken which makes $\pm pa \pm qa^2$ greatest, then each of the terms on the right-hand side may be expanded in powers of x and the whole rearranged as a convergent series proceeding by ascending powers of x.

* Strictly speaking, this is as yet established only for cases where a is real. The cases where a is imaginary will, however, be covered by the extension of the Binomial Theorem given in chap. xxix.

We thus find that

$$\frac{2-px}{1-px+qx^2} = (2-px)\{1 + \Sigma(p^n - {}_{n-1}C_1 p^{n-2}q + {}_{n-2}C_2 p^{n-4}q^2 - \ldots + (-)^r {}_{n-r}C_r p^{n-2r}q^r + \ldots)x^n\} \quad (5),$$

$$= 2\{1 + \Sigma \text{ \&c.}\} - px\{1 + \Sigma \text{ \&c.}\} \quad (6).$$

The coefficient of x^n on the right-hand side of (6) is

$$2\{p^n - {}_{n-1}C_1 p^{n-2}q + {}_{n-2}C_2 p^{n-4}q^2 + \ldots + (-)^r {}_{n-r}C_r p^{n-2r}q^r + \ldots\} - p\{p^{n-1} - {}_{n-2}C_1 p^{n-3}q + {}_{n-3}C_2 p^{n-5}q^2 + \ldots + (-)^r {}_{n-r-1}C_r p^{n-2r-1}q^r + \ldots\}.$$

Now

$$2\,{}_{n-r}C_r - {}_{n-r-1}C_r = n(n-r-1)(n-r-2)\ldots(n-2r+1)/r!.$$

Hence

$$\frac{2-px}{1-px+qx^2} = 2 + \Sigma\left\{p^n - \frac{n}{1!}p^{n-2}q + \frac{n(n-3)}{2!}p^{n-4}q^2 - \ldots + (-)^r \frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}p^{n-2r}q^r + \ldots\right\}x^n \quad (7).$$

Again

$$\frac{1}{1-\alpha x} + \frac{1}{1-\beta x} = \{1 + \alpha x + \alpha^2 x^2 + \ldots + \alpha^n x^n + \ldots\} + \{1 + \beta x + \beta^2 x^2 + \ldots + \beta^n x^n + \ldots\},$$

$$= 2 + \Sigma(\alpha^n + \beta^n)x^n \quad (8).$$

All the series involved in (8) will be absolutely convergent, provided x be taken so small that mod αx and mod βx are each < 1. Now, by (3), the series in (7) and (8) must be identical. Hence, comparing the coefficients of x^n, we must have (by chap. xxvi., § 21)

$$\alpha^n + \beta^n \equiv p^n - \frac{n}{1!}p^{n-2}q + \frac{n(n-3)}{2!}p^{n-4}q^2 - \ldots + (-1)^r \frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}p^{n-2r}q^r + \ldots \quad (9),$$

As we have indicated (by using $\equiv$), the equation (9) is an algebraical identity, on the understanding that p stands for $\alpha + \beta$

and q for $\alpha\beta$. The last term will or will not contain p according as n is odd or even.

In like manner, from the identity

$$\frac{x}{1 - px + qx^2} \equiv \frac{x}{1 - (\alpha + \beta)x + \alpha\beta x^2} \equiv \left\{ \frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x} \right\} \frac{1}{\alpha - \beta}$$

we deduce

$$(\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) \equiv p^n - \frac{n-1}{1!} p^{n-2}q + \frac{(n-2)(n-3)}{2!} p^{n-4}q^2 - \ldots$$

$$+ (-1)^r \frac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!} p^{n-2r}q^r + \ldots \quad (10),$$

subject to the same remarks as (9).

If we write the series (9) in the reverse order, and observe that, when n is even, $= 2m$ say, only even powers of p occur, and that the term which contains p^{2s} is

$$(-)^{m-s} \frac{2m(m+s-1)(m+s-2) \ldots (2s+1)}{(m-s)!} p^{2s}q^{m-s},$$

that is,

$$(-)^{m-s} \frac{2m(m+s-1)(m+s-2)\ldots(m+1)m(m-1)\ldots(m-s+1)}{(2s)!}$$

$$p^{2s}q^{m-s},$$

that is,

$$(-)^{m-s} 2 \frac{m^2(m^2-1^2) \ldots (m^2 - \overline{s-1}^2)}{(2s)!} p^{2s}q^{m-s},$$

then we have

$$\alpha^{2m} + \beta^{2m} \equiv (-)^m 2 \left\{ q^m - \frac{m^2}{2!} p^2 q^{m-1} + \frac{m^2(m^2-1^2)}{4!} p^4 q^{m-2} - \ldots \right.$$

$$\left. + (-)^s \frac{m^2(m^2-1^2)\ldots(m^2 - \overline{s-1}^2)}{(2s)!} p^{2s}q^{m-s} - \ldots \right\} \quad (9').$$

Similarly, we have

$$\alpha^{2m+1} + \beta^{2m+1} \equiv (-)^m (2m+1) \left\{ pq^m - \frac{(m+1)m}{3!} p^3 q^{m-1} \right.$$

$$+ \frac{(m+2)m(m^2-1^2)}{5!} p^5 q^{m-2} - \ldots$$

$$\left. + (-)^{s-1} \frac{(m+s-1)m(m^2-1^2)\ldots(m^2 - \overline{s-2}^2)}{(2s-1)!} p^{2s-1}q^{m-s+1} \ldots \right\}$$

$$(9'').$$

$$\frac{\alpha^{2m}-\beta^{2m}}{\alpha-\beta}\equiv(-)^{m-1}\left\{\frac{m}{1!}pq^{m-1}-\frac{m(m^2-1^2)}{3!}p^3q^{m-2}+\ldots\right.$$

$$\left.+(-)^{s-1}\frac{m(m^2-1^2)\ldots(m^2-\overline{s-1}^2)}{(2s-1)!}p^{2s-1}q^{m-s}+\ldots\right\} \quad (10').$$

$$\frac{\alpha^{2m+1}-\beta^{2m+1}}{\alpha-\beta}\equiv(-)^m\left\{q^m-\frac{(m+1)m}{2!}p^2q^{m-1}+\frac{(m+2)m(m^2-1^2)}{4!}\right.$$

$$p^4q^{m-2}-\ldots+(-)^s\frac{(m+s)m(m^2-1^2)\ldots(m^2-\overline{s-1}^2)}{(2s)!}$$

$$\left.p^{2s}q^{m-s}+\ldots\right\} \quad (10'').$$

Since α and β are the roots of the quadratic function z^2-pz+q, we may replace α and β in the above identities by $\frac{1}{2}\{p+\sqrt{(p^2-4q)}\}$, and $\frac{1}{2}\{p-\sqrt{(p^2-4q)}\}$ respectively. If this be done, and we at the same time put $p=x$ and $-4q=y^2$, we deduce the following :—

$$\{x+\sqrt{(x^2+y^2)}\}^n+\{x-\sqrt{(x^2+y^2)}\}^n$$

$$\equiv 2^n\left\{x^n+\frac{n}{1!\,2^2}x^{n-2}y^2+\frac{n(n-3)}{2!\,2^4}x^{n-4}y^4+\ldots\right.$$

$$\left.+\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!\,2^{2r}}x^{n-2r}y^{2r}+\ldots\right\},$$

$$\equiv 2\left\{y^n+\frac{n^2}{2!}x^2y^{n-2}+\frac{n^2(n^2-2^2)}{4!}x^4y^{n-4}+\ldots\right.$$

$$\left.+\frac{n^2(n^2-2^2)(n^2-4^2)\ldots(n^2-\overline{2s-2}^2)}{(2s)!}x^{2s}y^{n-2s}+\ldots\right\}, \quad (9''').$$

if n be even ;

$$\equiv 2\left\{nxy^{n-1}+\frac{n(n^2-1^2)}{3!}x^3y^{n-3}+\frac{n(n^2-1^2)(n^2-3^2)}{5!}\right.$$

$$x^5y^{n-5}+\ldots+\frac{n(n^2-1^2)(n^2-3^2)\ldots(n^2-\overline{2s-1}^2)}{(2s+1)!}$$

$$\left.x^{2s+1}y^{n-2s-1}+\ldots\right\}, \text{ if } n \text{ be odd.}$$

$$\left.\begin{aligned}
&\{x+\sqrt{(x^2+y^2)}\}^n - \{x-\sqrt{(x^2+y^2)}\}^n \\
&\equiv 2^n\sqrt{(x^2+y^2)}\left\{x^{n-1} + \frac{(n-2)}{1!\,2^2}x^{n-3}y^2 + \frac{(n-3)(n-4)}{2!\,2^4}\right. \\
&\quad x^{n-5}y^4 + \ldots + \frac{(n-r-1)(n-r-2)\ldots(n-2r)}{r!\,2^{2r}} \\
&\quad \left. x^{n-2r-1}y^{2r} + \ldots\right\}, \\
&\equiv 2\sqrt{(x^2+y^2)}\left\{\frac{n}{1!}xy^{n-2} + \frac{n(n^2-2^2)}{3!}x^3y^{n-4} + \ldots\right. \\
&\quad \left. + \frac{n(n^2-2^2)\ldots(n^2-2s-2^2)}{(2s-1)!}x^{2s-1}y^{n-2s} + \ldots\right\}, \\
&\quad \text{if } n \text{ be even;} \\
&\equiv 2\sqrt{(x^2+y^2)}\left\{y^{n-1} + \frac{(n^2-1^2)}{2!}x^2y^{n-3} + \frac{(n^2-1^2)(n^2-3^2)}{4!}\right. \\
&\quad x^4y^{n-5} + \ldots + \frac{(n^2-1^2)(n^2-3^2)\ldots(n^2-2s-1^2)}{(2s)!} \\
&\quad \left. x^{2s}y^{n-2s-1} + \ldots\right\}, \text{ if } n \text{ be odd.}
\end{aligned}\right\} \quad (10''').$$

These series are important in connection with the theory of the circular and hyperbolic functions.

§ 8.] A slight extension of the method of last paragraph enables us to find *expressions for the sum and for the number of r-ary products of n letters* (repetition of each letter being allowed).

The inverse method of partial fractions gives us the identity

$$1/(1-a_1x)(1-a_2x)\ldots(1-a_nx) \equiv \Sigma A_s(1-a_sx)^{-1} \quad (1),$$

where $A_s = a_s^{n-1}/(a_s-a_1)(a_s-a_2)\ldots(a_s-a_n)$.

Also, since $(1-a_sx)^{-1} = 1 + \Sigma a_s^r x^r$, we have (by chap. xxvi., § 14), provided x be taken small enough to secure the absolute convergency of all the series involved,

$$\begin{aligned}
&1/(1-a_1x)(1-a_2x)\ldots(1-a_nx) \\
&\quad = (1+\Sigma a_1^r x^r)(1+\Sigma a_2^r x^r)\ldots(1+\Sigma a_n^r x^r) \quad (2), \\
&\quad = 1 + \Sigma_n K_r x^r \quad (3),
\end{aligned}$$

where ${}_nK_r$ is obviously the sum of all the r-ary products of $a_1, a_2, \ldots a_n$. Since the coefficients of x^r on the right-hand sides of (1) and (3) must be equal, we have

$${}_nK_r \equiv \Sigma a_s^{n+r-1}/(a_s-a_1)(a_s-a_2)\ldots(a_s-a_n) \quad (4).$$

If, for example, there be three letters, a_1, a_2, a_3, we have

$$ {}_3K_r \equiv \frac{a_1^{r+2}}{(a_1-a_2)(a_1-a_3)} + \frac{a_2^{r+2}}{(a_2-a_1)(a_2-a_3)} + \frac{a_3^{r+2}}{(a_3-a_1)(a_3-a_2)} $$
$$ \equiv -\frac{a_1^{r+2}(a_2-a_3) + a_2^{r+2}(a_3-a_1) + a_3^{r+2}(a_1-a_2)}{(a_2-a_3)(a_3-a_1)(a_1-a_2)} \qquad (5). $$

If we put $a_1 = a_2 = \ldots = a_n = 1$, then each of the terms in ${}_nK_r$ reduces to 1, and ${}_nK_r$ becomes ${}_nH_r$. Hence, from (3),

$$ (1-x)^{-n} = 1 + \Sigma {}_nH_r x^r \qquad (6). $$

Equating coefficients of x^r on both sides of (6), we have

$$ {}_nH_r = n(n+1) \ldots (n+r-1)/r!, $$

a result already found by another method in chap. xxiii., § 10.

§ 9.] Some interesting results can be obtained by expanding $1/(y+x)(y+x+1) \ldots (y+x+n)$ in descending, and in ascending powers of y.

If we write

$$ 1/(y+x)(y+x+1) \ldots (y+x+n) = \sum_{r=0}^{r=n} A_r (y+x+r)^{-1}, $$

then we find, by the method of chap. viii., § 6, that

$$ 1 = A_r(-r)(-r+1) \ldots (-1)1.2 \ldots (n-r). $$

Hence $A_r = (-)^r {}_nC_r/n!$.

Therefore

$$ n!/(y+x)(y+x+1)\ldots(y+x+n) = \Sigma(-)^r {}_nC_r (y+x+r)^{-1} \qquad (1). $$

Hence, if P_1, P_2, P_3, . . . denote respectively the sum of x, $x+1$, . . ., $x+n$, and of their products taken 2, 3, . . . at a time (without repetition), we have

$$ \frac{n!}{y^n}\left\{1 + \frac{P_1}{y} + \frac{P_2}{y^2} + \ldots\right\}^{-1} = \Sigma(-)^r {}_nC_r \left(1 + \frac{x+r}{y}\right)^{-1}, $$

$$ \frac{n!}{y^n}\left\{1 - \left(\frac{P_1}{y} + \frac{P_2}{y^2} + \ldots\right) + \left(\frac{P_1}{y} + \frac{P_2}{y^2} + \ldots\right)^2 - \ldots\right\} $$
$$ = \Sigma(-)^r {}_nC_r \left\{1 + \Sigma(-)^s \left(\frac{x+r}{y}\right)^s\right\} \qquad (2), $$

where we suppose y to have a value so large that all the series involved are convergent.

Since there is no power of $1/y$ less than the nth on the left of (2), the coefficient of any such power on the right must vanish. Therefore

$$(x+n)^s - {}_nC_1(x+n-1)^s + {}_nC_2(x+n-2)^s - \ldots (-)^n x^s = 0 \quad (3),$$

where s is any positive integer $< n$.

Equating coefficients of $1/y^n$, $1/y^{n+1}$, and $1/y^{n+2}$, we find

$$(x+n)^n - {}_nC_1(x+n-1)^n + {}_nC_2(x+n-2)^n - \ldots (-)^n x^n = n! \quad (4);$$

$$(x+n)^{n+1} - {}_nC_1(x+n-1)^{n+1} + {}_nC_2(x+n-2)^{n+1} - \ldots (-)^n x^{n+1} = n!P_1, = (n+1)!(x+\tfrac{1}{2}n) \quad (5);$$

$$(x+n)^{n+2} - {}_nC_1(x+n-1)^{n+2} + {}_nC_2(x+n-2)^{n+2} - \ldots (-)^n x^{n+2} = n!(P_1^2 - P_2), = \tfrac{1}{2}(n+2)!\{x^2 + nx + \tfrac{1}{12}n(3n+1)\} \quad (6);$$

and so on.

Again from (1) we have

$$\frac{n!}{x(x+1)\ldots(x+n)}\{1 + Q_1 y + Q_2 y^2 + \ldots\}^{-1} = \sum_{r=0}^{r=n} (-)^r \frac{{}_nC_r}{x+r}\left\{1 + \frac{y}{x+r}\right\}^{-1} \quad (7),$$

where Q_1, Q_2, Q_3 . . . are respectively the sum of $1/x$, $1/(x+1)$, . . ., $1/(x+n)$, and the sums of their products taken 2, 3, . . . at a time. From (7), by expanding and equating coefficients of y, we get

$$\frac{n!}{x(x+1)\ldots(x+n)}\left\{\frac{1}{x} + \frac{1}{x+1} + \ldots + \frac{1}{(x+n)}\right\} = \frac{1}{x^2} - \frac{{}_nC_1}{(x+1)^2} + \frac{{}_nC_2}{(x+2)^2} - \ldots (-)^n \frac{1}{(x+n)^2} \quad (8).$$

If we put $x = 1$, we get the following curious relation between the sum of the reciprocals of 1, 2, . . ., $n+1$, and the reciprocals of their squares :—

$$\frac{1}{n+1}\left\{\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n+1}\right\}=\frac{1}{1^2}-\frac{{}_nC_1}{2^2}+\frac{{}_nC_2}{3^2}-\ldots$$

$$(-)^n\frac{1}{(n+1)^2} \qquad (9).$$

§ 10.] We have now exemplified most of the elementary processes used in the transformation of Binomial Series. The following additional examples may be useful in helping the student to thread the intricacies of this favourite field of exercise for the tyro in Mathematics.

Example 1. Find the coefficient of x^n in the expansion of $(1-x)^2/(1+x)^{3/2}$ in ascending powers of x.

If $(1+x)^{-3/2}=1+\Sigma a_n x^n$, then $(1-x)^2/(1+x)^{3/2}=(1-2x+x^2)(1+\Sigma a_n x^n)$. Hence the coefficient required is $a_n-2a_{n-1}+a_{n-2}$. If we substitute the actual values of a_n, a_{n-1}, a_{n-2}, we find that

$$a_n-2a_{n-1}+a_{n-2}=(-)^n(16n^2-8n-1)\frac{3.5\ldots(2n-3)}{2.4.6\ldots 2n}.$$

Example 2. If $f(x)=a_0+a_1x+a_2x^2+\ldots$, then the coefficient of x^r in the expansion of $f(x)/(1-x)^m$ in ascending powers of x is $a_0\,{}_mH_r+a_1\,{}_mH_{r-1}+a_2\,{}_mH_{r-2}+\ldots+a_r$. This follows at once from the equation

$$f(x)/(1-x)^m=(a_0+\Sigma a_r x^r)(1+\Sigma\,{}_mH_r x^r).$$

In particular, if we put $f(x)\equiv(1-x)^{-n}$ and $m=1$, we deduce that

$${}_{n+1}H_r={}_nH_r+{}_nH_{r-1}+{}_nH_{r-2}+\ldots+1;$$

and, if we put $f(x)\equiv(1-x)^{-n}$, we deduce that

$${}_{m+n}H_r={}_mH_r+{}_mH_{r-1}\,{}_nH_1+{}_mH_{r-2}\,{}_nH_2+\ldots+{}_nH_r$$

results which have already appeared, in the particular case where m and n are integral (see chap. xxiii., § 10).

Example 3. Show that

$${}_mC_n/2+{}_{m+1}C_n/2^2+{}_{m+2}C_n/2^3+\ldots \text{ ad } \infty=1+{}_mC_1+{}_mC_2+\ldots+{}_mC_n \quad (1).$$

The left-hand side of (1) is obviously the coefficient of x^n in

$$X=(1+x)^m/2+(1+x)^{m+1}/2^2+(1+x)^{m+2}/2^3+\ldots \text{ ad } \infty.$$

Now
$$X=\tfrac{1}{2}(1+x)^m[1+\{(1+x)/2\}+\{(1+x)/2\}^2+\ldots \text{ ad } \infty],$$
$$=(1+x)^m/2\{1-(1+x)/2\}, \text{ if we suppose } x<1.$$
$$=(1+x)^m/(1-x),$$
$$=1+\Sigma(1+{}_mC_1+{}_mC_2+\ldots+{}_mC_n)x^n,$$

by last example. Hence the theorem follows.

Example 4. Sum the series

$$S=1-\frac{n-3}{2!}+\frac{(n-4)(n-5)}{3!}-\frac{(n-5)(n-6)(n-7)}{4!}+\ldots,$$

n being a positive integer.

The equations (9''') of § 7 being algebraical identities, we may substitute therein any values of x and y we choose, so long as no ambiguity arises in the determination of the functions involved. We may, for example, put $x=-1$ and $y=2i$. We thus find

$$\left\{\frac{-1+\sqrt{3}i}{2}\right\}^n+\left\{\frac{-1-\sqrt{3}i}{2}\right\}^n\equiv(-)^n\{1-n\mathrm{S}\}.$$

Hence, if ω and ω^2 denote, as usual, the two imaginary cube roots of $+1$, we have

$$\mathrm{S}=\{1+(-)^{n-1}(\omega^n+\omega^{2n})\}/n.$$

If we evaluate $\omega^n+\omega^{2n}$ for the four cases where n has the forms $6m$, $6m\pm1$, $6m\pm2$, $6m+3$ (remembering that $\omega^{6m}=1$, $\omega^{-1}=\omega^2$, $\omega^{-2}=\omega$), we find that S has the values $-1/n$, 0, $2/n$, and $3/n$ respectively.

Example 5. Sum the series

$$\mathrm{S}=1+\frac{n(n-1)}{2(2r+1)}+\frac{n(n-1)(n-2)(n-3)}{2.4(2r+1)(2r+3)}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2.4.6(2r+1)(2r+3)(2r+5)}+\ldots$$

n being a positive integer.

If we denote the series by $1+u_1+u_2+u_3+\ldots$, then

$$u_s=\frac{n(n-1)\ldots(n-2s+1)}{2.4\ldots2s(2r+1)(2r+3)\ldots(2r+2s-1)},$$

$$=\frac{n!(2r)!(r+1)(r+2)\ldots(r+s)}{(n-2s)!(2r+2s)!\,s!},$$

restricting r for the present to be a positive integer. We may therefore write

$$u_s=\frac{n!(2r)!}{(n+2r)!}{}_{n+2r}\mathrm{C}_{2r+2s}\cdot{}_{r+s}\mathrm{C}_s.$$

Now ${}_{r+s}\mathrm{C}_s$ is the coefficient of x^{2r} in the expansion of $x^{2r+2s}(1+1/x^2)^{r+s}$; that is, in the expansion of $x^{2r+2s}\{\sqrt{(1+1/x^2)}\}^{2r+2s}$. Hence $2u_s$ is one part of the coefficient of x^{2r} in the expansion of

$$\frac{n!(2r)!}{(n+2r)!}[\{1+x\sqrt{(1+1/x^2)}\}^{n+2r}+\{1-x\sqrt{(1+1/x^2)}\}^{n+2r}].$$

Hence 2S is the whole coefficient of x^{2r} in the expansion of

$$\frac{n!(2r)!}{(n+2r)!}[\{1+\sqrt{(1+x^2)}\}^{n+2r}+\{1-\sqrt{(1+x^2)}\}^{n+2r}].$$

Now, by § 7,

$$\{1+\sqrt{(1+x^2)}\}^{n+2r}+\{1-\sqrt{(1+x^2)}\}^{n+2r}$$

$$=2^{n+2r}\left\{1+\Sigma\frac{(n+2r)(n+2r-s-1)(n+2r-s-2)\ldots(n+2r-2s+1)}{(s)!}\frac{x^{2s}}{2^{2s}}\right\},$$

the coefficient of x^{2r} in which is

$$\frac{(n+2r)(n+r-1)(n+r-2)\ldots(n+1)}{r!\,2^{2r}}.$$

Hence

$$S = 2^{n+2r-1}\frac{n!\,(2r)!\,(n+2r)(n+r-1)!}{(n+2r)!\,r!\,n!\,2^{2r}},$$

$$= 2^{n-1}\frac{(n+r-1)(n+r-2)\ldots(r+1)}{(n+2r-1)(n+2r-2)\ldots(2r+1)}.$$

The summation is thus effected for all integral values of r. So far, however, as r is concerned, the formula arrived at might be reduced to an identity between two integral functions of r of finite degree. Since we have shown that this identity holds for an infinite number of particular values of r, it must (chap. v., § 16) hold for all values of r. The summation is therefore general so far as r is concerned.

Exercises X.

Find the coefficient of x^r in the expansion of the following in ascending powers of x.

(1.) $x/(x-a)(x-b)(x-c)$. (2.) $x^{r+3}/(x-a)(x-b)(x-c)$.

(3.) $x^{m+3}/(x-a)(x-b)(x-c)$, where m is a positive integer $< r-3$.

(4.) $(3-x)/(2-x)(1-x)^2$. (5.) $2x^2/(x-1)^2(x^2+1)$.

(6.) $(1-px)^m(1-qx)^{-n}$.

(7.) If $(1-3x)^n/(1-2x)^2$ be expanded in ascending powers of x, the coefficient of x^{n+r-1} is $(-1)^n(r-2n)2^{r-1}$, n and r being positive integers.

(8.) Find the numerically greatest term in the expansion of $(a-x)/(b+x)^2$ in ascending powers of x.

(9.) Show that

$$\frac{(x+\beta)(x+2\beta)\ldots(x+n\beta)}{(x-\beta)(x-2\beta)\ldots(x-n\beta)} = 1 + \sum_{r=1}^{r=n}(-)^{n-r}\frac{n(n+r)(n^2-1^2)(n^2-2^2)\ldots(n^2-\overline{r-1}^2)}{(r!)^2}\,\frac{r\beta}{x-r\beta};$$

and hence show that

$$\sum_{r=1}^{r=n}(-)^{n-r}\frac{rn(n+r)(n^2-1^2)(n^2-2^2)\ldots(n^2-\overline{r-1}^2)}{(r!)^2} = n(n+1).$$

(10.) If n be a positive integer, show that

$$1 - {}_mC_1 + {}_mC_2 - \ldots(-)^n{}_mC_n = (-)^n{}_{m-1}C_n.$$

(11.) If n be an even positive integer,

$${}_mC_n - {}_mC_{n-1}\cdot{}_mC_1 + {}_mC_{n-2}\cdot{}_mC_2 - \ldots + {}_mC_n = (-)^{n/2}{}_mC_{n/2}.$$

(12.) If m and n be positive integers, show that

$${}_mC_0\cdot{}_{m/2}C_n + {}_mC_2\cdot{}_{(m-2)/2}C_{n-1} + {}_mC_4\cdot{}_{(m-4)/2}C_{n-2} + \ldots + {}_mC_{2n}\cdot{}_{(m-2n)/2}C_0 = \frac{m^2(m^2-2^2)\ldots(m^2-2n-2^2)}{(2n)!};$$

$${}_mC_1\cdot{}_{(m-1)/2}C_n + {}_mC_3\cdot{}_{(m-3)/2}C_{n-1} + {}_mC_5\cdot{}_{(m-5)/2}C_{n-2} + \ldots + {}_mC_{2n+1}\cdot{}_{(m-2n-1)/2}C_0 = \frac{m(m^2-1^2)(m^2-3^2)\ldots(m^2-2n-1^2)}{(2n+1)!}.$$

(See Schlömilch, *Handb. d. Alg. Anal.*, § 38.)

(13.) Show, by equating coefficients in the expansion of $(1-x^{-1})^m(1-x)^{-m}$, where m is a positive integer, that

$$1-m^2+\frac{m^2(m^2-1)}{(2!)^2}+\ldots+(-1)^m\frac{m^2(m^2-1^2)\ldots(m^2-\overline{m-1}^2)}{(m!)^2}=0.$$

(14.) If n be a positive multiple of 6, then

$${}_nC_1-{}_nC_3 3+{}_nC_5 3^2-\ldots\qquad=0;$$

$${}_nC_1-{}_nC_3\frac{1}{3}+{}_nC_5\frac{1}{3^2}-\ldots\qquad=0.$$

(15.) If $(1+x)^{-3}=1+a_1x+a_2x^2+\ldots$, sum the series $1-a_1+a_2-a_3+\ldots$ to n terms.

(16.) If $(1+x)^{2n}=1+a_1x+a_2x^2+\ldots$, then $1-a_1^2+a_2^2-\ldots=(-1)^n 2n(2n-1)\ldots(n+1)/n!$.

(17.) $$\frac{r!}{r!\,1!}-\frac{2^2(r+1)!}{(r-1)!\,3!}+\ldots+\frac{(-1)^r 2^{2r}(2r)!}{0!\,(2r+1)!}=\frac{(-1)^r}{2r+1}.$$

(18.) $$\sum_{r=0}^{r=n}1/4^r(r!)^2(2n-2r)!=(4n)!/4^n\{(2n)!\}^3.$$

(19.) Sum to n terms $\Sigma(2n-2)!/2^{2n-1}n\{(n-1)!\}^2$.

(20.) Sum the series

$$n+(n-1)\frac{1}{3}+(n-2)\frac{1.4}{3.6}+(n-3)\frac{1.4.7}{3.6.9}+\ldots+\frac{1.4\ldots(3n-5)}{3.6\ldots(3n-3)}.$$

(21.) Find for what values of n the following series are convergent; and show that when they are convergent their sums are as given below.

$$\frac{1}{m}-\frac{n}{1!}\frac{1}{m+1}+\frac{n(n-1)}{2!}\frac{1}{m+2}-\ldots=\frac{(m-1)!}{(n+1)(n+2)\ldots(n+m)};$$

$$\frac{1}{m}+\frac{n}{1!}\frac{1}{m+1}+\frac{n(n-1)}{2!}\frac{1}{m+2}+\ldots=\frac{(m-1)!}{(n+1)(n+2)\ldots(n+m)}\{{}_{m+n}C_{m-1}2^{n+1}$$
$$-{}_{m+n}C_{m-2}2^{n+2}+\ldots+(-)^{m-1}2^{n+m}+(-)^m 1\},$$

m in both cases being a positive integer.

(22.) $$\sum_{s=0}^{s=n}\frac{(r+s)!\,(m+n-r-s-1)!}{r!\,s!\,(m-r-1)!\,(n-s)!}=\frac{(m+n)!}{m!\,n!}.$$

(23.) $$\sum_{r=0}^{r=m}\sum_{s=0}^{s=n}\frac{(r+s)!\,(m+n-r-s)!}{r!\,s!(m-r)!\,(n-s)!}=\frac{(m+n+1)!}{m!\,n!}.$$

(24.) The number of the r-ary products of three letters, none of which is to be raised to a power $>$ the nth, where $n<r<2n$, is

$$r(3n-r)+1-\tfrac{3}{2}n(n-1).$$

(25.) Prove, for a, b, c, that $\Sigma a^r/(a-b)(a-c)\equiv 0$, if $r=0$, or $r=1$; $\equiv 1$, if $r=2$; and generalise the theorem.

(26.) Show that

$$\frac{a(b-c)(bc-aa')(a^m-a'^m)}{a-a'}+\frac{b(c-a)(ca-bb')(b^m-b'^m)}{b-b'}+\frac{c(a-b)(ab-cc')(c^m-c'^m)}{c-c'}$$
$$=(b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')S_{m-3}/abc,$$

where $aa'=bb'=cc'$, and S_{m-3} is the sum of the $(m-3)$-ary products of a, b, c, a', b', c'. (Math. Trip., 1886.)

(27.) If S_r be the sum of the r-ary products of the roots of the equation $x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n=0$, then

$$0=S_1+a_1,$$
$$0=S_2+S_1a_1+a_2,$$
$$\ldots\quad\ldots\quad\ldots$$
$$0=S_n+S_{n-1}a_1+S_{n-2}a_2+\ldots+a_n,$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$0=S_r+S_{r-1}a_1+S_{r-2}a_2+\ldots+S_{r-n}a_n.$$

(Wronaki.)

(28.) If S_r be the sum of the r-ary products of n letters, P_r the sum of the products r at a time, Σ_r the sum of their rth powers, then

$$\Sigma_r=nS_r-(n-1)P_1S_{r-1}+\ldots+(-1)^r(n-r)P_r, \text{ if } r<n-1.$$
$$=nS_r-(n-1)P_1S_{r-1}+\ldots+(-1)^{n-1}P_{n-1}S_{r-n+1}, \text{ if } r>n-1.$$

(Math. Trip., 1882.)

(29.) If $v=(1-\alpha x)^{-1}(1-\beta x)^{-1}\ldots$, the number of ways of distributing n things, λ of which are of one sort, μ of another sort, . . ., into p boxes placed in a row is the coefficient of $x^n\alpha^\lambda\beta^\mu\ldots$ in the expansion of $(v-1)^p$ in ascending powers of x, namely,

$$u_1-{}_pC_1u_2+{}_pC_2u_3-\ldots,$$

where $$u_s=(p+\lambda-s)!(p+\mu-s)!\ldots/(p-s)!\lambda!(p-s)!\mu!\ldots$$

(Math. Trip., 1888.)

(30.) With the same data as in last question, show that the whole number of ways of distributing the things when the order in which they are arranged inside each box is attended to is

$$n!(n-1)!/(n-p)!(p-1)!\lambda!\mu!\nu!\ldots$$

(Math. Trip., 1888.)

Show that

(31.) $1+1/2+\ldots+1/x={}_xC_1-\frac{1}{2}{}_xC_2+\frac{1}{3}{}_xC_3-\ldots$

(32.) $1-\dfrac{(m+1)m}{3!}2^2+\dfrac{(m+2)(m+1)m(m-1)}{5!}2^4-\ldots=\dfrac{(-1)^m}{2m+1}$.

(33.) $1-\dfrac{m^2}{2!}2^2+\dfrac{m^2(m^2-1^2)}{4!}2^4-\dfrac{m^2(m^2-1^2)(m^2-2^2)}{6!}2^6+\ldots=(-1)^m$.

(34.) If m and n are both positive integers, and $m>n$, then

$$\frac{2^{-n}}{n!}+\frac{(m-n)(m-n-1)}{1!(n+1)!}2^{-n-2}+\frac{(m-n)(m-n-1)(m-n-2)(m-n-3)}{2!(n+2)!}2^{-n-4}+$$
$$\ldots=\frac{1.3.5\ldots(2m-1)}{(m+n)!}.$$

(35.) If r be a positive integer,

$$r\left\{1+\frac{r^2-1^2}{3!}x+\frac{(r^2-1^2)(r^2-2^2)}{5!}x^2+\frac{(r^2-1^2)(r^2-2^2)(r^2-3^2)}{7!}x^3+\ldots\right\}$$
$$=(x+2)^{r-1}-{}_{r-2}C_1(x+2)^{r-3}+{}_{r-3}C_2(x+2)^{r-5}-{}_{r-4}C_3(x+2)^{r-7}+\ldots$$

MULTINOMIAL THEOREM FOR ANY INDEX.

§ 11.] Consider the integral function $a_1x + a_2x^2 + \ldots + a_rx^r$, whose absolute term vanishes, the rest of the coefficients being real quantities positive or negative. Confining ourselves in the meantime to real values of x, we see, since the function vanishes when $x = 0$, that it will in all cases be possible to assign a positive quantity ρ such that for all values of x between $-\rho$ and $+\rho$ we shall have

$$a_1x + a_2x^2 + \ldots + a_rx^r < 1 \qquad (1).$$

In fact, it will be sufficient if ρ be such that

$$a\rho + a\rho^2 + \ldots + a\rho^r < 1$$

where a is the numerical value of the numerically greatest among $a_1, a_2, \ldots, a_r$. That is, it will be sufficient if

$$a\rho(1 - \rho^r)/(1 - \rho) < 1;$$

a fortiori (supposing $\rho < 1$) it will be sufficient if

$$a\rho/(1 - \rho) < 1;$$

that is, if

$$\rho < 1/(a + 1)^* \qquad (2).$$

ρ is, in fact, the numerically least among the roots of the two equations

$$a_rx^r + \ldots + a_1x \pm 1 = 0,$$

as may be seen by considering the graph of $a_rx^r + \ldots + a_1x$.

Therefore, whether m be integral or not, provided $-\rho < x < +\rho$ we can always expand $(1 + a_1x + a_2x^2 + \ldots + a_rx^r)^m$ in the form

$$1 + \Sigma_m C_s(a_1x + a_2x^2 + \ldots + a_rx^r)^s \qquad (3);$$

and the series (3) will be *absolutely* convergent whether m be positive or negative. Hence, since $a_1x + a_2x^2 + \ldots + a_rx^r$ is a terminating series and therefore has a finite value for all values of x positive or negative, it follows from the principle established in chap. XXVI., § 34, that we may arrange (3) according to powers

* This is merely a lower limit for ρ; in any individual case it would in general be much greater.

of x, and the result will be a power series which will converge to the sum $(1 + a_1x + a_2x^2 + \ldots + a_rx^r)^m$ so long as $-\rho < x < +\rho$.

Since s is a positive integer, we can expand ${}_mC_s(a_1x + a_2x^2 + \ldots + a_rx^r)^s$ by the formula of chap. xxiii., § 12. The coefficient of x^n in this expansion will be

$$\Sigma_m C_s s\,!\, a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_r^{\alpha_r} / \alpha_1\,!\, \alpha_2\,! \ldots \alpha_r\,!,$$

that is,

$$\Sigma a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_r^{\alpha_r} m(m-1)\ldots(m-s+1)/\alpha_1!\alpha_2!\ldots\alpha_r! \quad (4),$$

where the summation extends over all positive integral values of $\alpha_1, \alpha_2, \ldots, \alpha_r$, including 0, which are such that

$$\left.\begin{aligned} \alpha_1 + \alpha_2 + \ldots + \alpha_r &= s \\ \alpha_1 + 2\alpha_2 + \ldots + r\alpha_r &= n \end{aligned}\right\} \quad (5).$$

In order, therefore, to find the coefficient of x^n in (3) we have merely to extend the summation in (4) so as to include all values of s; in other words, to drop the first of the two restrictions in (5).

Hence, whether m be integral or not, provided x be small enough, we have

$$(1 + a_1x + a_2x^2 + \ldots + a_rx^r)^m$$
$$= 1 + \Sigma \frac{m(m-1)\ldots(m-\Sigma\alpha_1+1)}{\alpha_1\,!\,\alpha_2\,!\ldots\alpha_r\,!} a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_r^{\alpha_r} x^n \quad (6),$$

the summation to be extended over all positive integral values of $\alpha_1, \alpha_2, \ldots, \alpha_r$, including 0, such that

$$\alpha_1 + 2\alpha_2 + \ldots + r\alpha_r = n.$$

The details of the evaluation of the coefficient in any particular case are much the same as in chap. xxiii., § 12, Example 2, and need not be farther illustrated. It need scarcely be added that when n is very large the calculation is tedious. In some cases it can be avoided by transforming $1 + a_1x + a_2x^2 + \ldots + a_rx^r$ before applying the Binomial Expansion, but in most cases the application of the above formula is in the end both quickest and most conducive to accuracy.

Example. To find the coefficient of x^n in $(1+x+x^2 \ldots +x^r)^m$.

We have

$$(1+x+x^2+\ldots+x^r)^m \equiv \{(1-x^{r+1})/(1-x)\}^m,$$
$$\equiv (1-x^{r+1})^m(1-x)^{-m},$$
$$= (1-x^{r+1})^m(1+\Sigma_m H_n x^n).$$

Hence, if $n<r+1$, the coefficient of x^n is simply

$${}_mH_n = m(m+1)\ldots(m+n-1)/n\,!;$$

but if $n<r+1$, the coefficient of x^n is

$${}_mH_n - {}_mC_1 \cdot {}_mH_{n-r-1} + {}_mC_2 \cdot {}_mH_{n-2r-2} - \ldots$$

NUMERICAL APPROXIMATION BY MEANS OF THE BINOMIAL THEOREM.

§ 12.] The Binomial Expansion may be used for the purpose of approximating to the numerical value of $(1+x)^m$. According as we retain the first two, the first three, . . ., the first $n+1$ terms of the series $1 + {}_nC_1x + {}_nC_2x^2 + \ldots$, we may be said to take a first, a second, . . . an nth approximation to $(1+x)^m$.

The principal points to be attended to, are—

1st, To include in our approximation the terms of greatest numerical value; in other words, to take n so great that the numerically greatest term, at least, is included.

2nd, To take n so great that the residue of the series is certainly less than half a unit in the decimal place next after that to which absolute accuracy is required.

3rd, To calculate each of the terms retained to such a degree of accuracy that the accumulated error from the neglected digits in all the terms retained is less a unit in the place next after that to which absolute accuracy is required.

The last condition is easily secured by a little attention in each particular case. We proceed to discuss the other two.

§ 13.] *The order of the numerically greatest term.*

In the case of the Binomial Series $(1+x)^m$, if ξ denote the numerical value of x, so that $0<\xi<1$, we have for the numerical value of the convergency-ratio u_{n+1}/u_n

$$\sigma_n = \frac{m-n}{n+1}\xi, \text{ or } = \frac{n-m}{n+1}\xi, \qquad (1),$$

according as $m-n$ is positive or negative.

Hence it is obvious, in the first place, that if $-1 \leqq m < +1$, that is, if m be a positive or negative proper fraction, the condition $\sigma_n < 1$ is satisfied from the very beginning, and the first term will be the greatest.

If $m > +1$, the condition $\sigma_n < 1$ is obviously satisfied for any value of n which exceeds m; in fact, the condition will be satisfied as soon as

$$(m-n)\xi < n+1,$$

that is,
$$n > (m\xi - 1)/(1+\xi) \qquad (2),$$

the right-hand side of which is obviously less than m. This condition is satisfied from the beginning if $\xi < 2/(m-1)$.

If m be $< -1 = -\mu$, say, where $\mu > 1$, the condition $\sigma_n < 1$ will be satisfied as soon as

$$(\mu+n)\xi < n+1,$$

that is,
$$n > (\mu\xi - 1)/(1-\xi) \qquad (3).$$

This condition is satisfied from the beginning if $\xi < 2/(\mu+1)$.

§ 14.] *Upper limit of the residue.* We have seen that, ultimately, the terms of a Binomial Series either (1) alternate in sign or (2) are of constant sign.

To the first of these classes belong the expansions of $(1+x)^m$ and $(1+x)^{-m}$, where x and m are positive.

If n be greater than the order of the numerically greatest term, and in the case of $(1+x)^m$ (see § 4) also $> m$, then the residue may be written in the form

$$R_n = \pm (u_{n+1} - u_{n+2} + u_{n+3} - \ldots) \qquad (1),$$

where u_{n+1}, u_{n+2}, u_{n+3} . . . are the numerical values of the various terms, and we have $u_{n+1} > u_{n+2} > u_{n+3} > \ldots$

Hence, in the present case, the error committed by taking an nth approximation is numerically less than u_{n+1}. In other words,

if we stop at the term of the nth order, the following term is an upper limit for the error of the approximation.

Cor. *A lower limit for the error is obviously* $u_{n+1} - u_{n+2}$.

The expansions of $(1-x)^m$ and $(1-x)^{-m}$ belong to the second class of series, in which the terms are all ultimately of the same sign. It will be convenient to consider these two expansions separately.

In the case of $(1-x)^m$, if we take $n > m$, then we shall certainly include the numerically greatest term; and σ_n, the numerical value of the convergency-ratio, will be $(n-m)x/(n+1)$, that is, $\{1-(m+1)/(n+1)\}x$. This continually increases as n increases, and has for its limit x, when $n = \infty$. Hence

$$\sigma_{n+1} < \sigma_{n+2} < \ldots < x < 1.$$

Therefore, u_{n+1}, u_{n+2}, . . . having the same meaning as before,

$$\begin{aligned} R_n &= \pm (u_{n+1} + u_{n+2} + u_{n+3} + \ldots), \\ &= \pm u_{n+1}(1 + \sigma_{n+1} + \sigma_{n+1}\sigma_{n+2} + \sigma_{n+1}\sigma_{n+2}\sigma_{n+3} + \ldots). \end{aligned}$$

Therefore

$$\begin{aligned} \text{Mod } R_n &< u_{n+1}(1 + x + x^2 + x^3 + \ldots), \\ &< u_{n+1}/(1-x) \qquad (2). \end{aligned}$$

Hence the error in this case *is numerically less than* $u_{n+1}/(1-x)$, *and it is in excess or in defect according as the least integer which exceeds m is even or odd* (see § 4).

Cor. *A lower limit for the error is obviously* $u_{n+1}/(1-\sigma_{n+1})$, *that is,* ${}_mC_{n+1}x^{n+1}/\{1-(n+1-m)x/(n+2)\}$.

In the expansion of $(1-x)^{-m}$, all the terms are positive; and, in order to include the greatest term, we have merely to take $n > (mx-1)/(1-x)$.

We have, in this case,

$$\begin{aligned} \sigma_n = (n+m)x/(n+1) &= \{1-(1-m)/(n+1)\}x, \\ &= \{1+(m-1)/(n+1)\}x. \end{aligned}$$

Hence, if $m < 1$,

$$\sigma_{n+1} < \sigma_{n+2} < \ldots < x < 1,$$

and *an upper limit of* R_n *will be* $u_{n+1}/(1-x)$ *as in last case, a lower limit being* $u_{n+1}/(1-\sigma_{n+1})$, *that is,* ${}_mH_{n+1}x^{n+1}/\{1-(n+1+m)x/(n+2)\}$.

If $m>1$,

$$1>\sigma_{n+1}>\sigma_{n+2}>\ldots>x,$$

and *an upper limit of* R_n *will be* $u_{n+1}/(1-\sigma_{n+1})$, *that is,* ${}_mH_{n+1}x^{n+1}/\{1-(n+1+m)x/(n+2)\}$, *a lower limit being* $u_{n+1}/(1-x)$.

The error for $(1-x)^{-m}$ is, of course, always in defect.

Example 1. To calculate the cube root of 29 to 6 places of decimals. The nearest cube to 29 is 27. We therefore write

$$\sqrt[3]{29}=(3^3+2)^{1/3}=3(1+2/3^3)^{1/3},$$
$$=u_0+u_1-u_2+u_3-u_4\ldots.$$

The first term is here the greatest; and the terms alternate in sign after u_1. Also u_r, written in the most convenient form for calculating successive terms, is

$$u_r=3\left(\tfrac{2}{81}\right)\left(\tfrac{4}{162}\right)\left(\tfrac{10}{243}\right)\left(\tfrac{16}{324}\right)\ldots\left(\frac{6r-8}{81r}\right).$$

Therefore

	+	−
$u_0=$	3·000,000,00	
$u_1=u_0 2/81=$	74,074,07	
$u_2=u_1 4/162=$		·001,828,99
$u_3=u_2 10/243=$	75,27	
$u_4=u_3 16/324=$		3,72
	3·074,149,34	·001,832,71
	·001,832,71	
	3·072,316,63	
$u_5=u_4 22/405$	20	

Hence the error in defect, due to neglect of the residue, amounts to less than 2 in the seventh place. The error for neglect of digits does not exceed 1 in the seventh place. Therefore, the best 6-place approximation to $\sqrt[3]{29}$ is 3·072,317. In Barlow's Tables we find 3·072,316,8 given as the value to 7 places.

Example 2. To calculate $(1-x)^m/(1+x+x^2)^m$ to a second approximation, x being small.

$$(1-x)^m(1+x+x^2)^{-m}$$
$$=\left\{1-mx+\frac{m(m-1)}{2}x^2\right\}\times\left\{1-m(x+x^2)+\frac{m(m+1)}{2}x^2\right\},$$

where we have already neglected all powers of x above the second in each of the two series;

$$= \left\{1 - mx + \frac{m(m-1)}{2}x^2\right\}\left\{1 - mx + \frac{m(m-1)}{2}x^2\right\},$$

$$= 1 + (-m-m)x + \left\{\frac{m(m-1)}{2} + m^2 + \frac{m(m-1)}{2}\right\}x^2,$$

where higher powers of x than x^2 have again been neglected in distributing the product;

$$= 1 - 2mx + m(2m-1)x^2.$$

EXERCISES XI.

(1.) The general term in the expansion of $(1+x+y+xy)/(1+x+y)$ is $(-1)^{m+n}(m+n-2)!x^m y^n/(m-1)!(n-1)!$.

Determine limits for x within which the following multinomials can be expanded in convergent series of ascending powers of x; and find the coefficients of

(2.) x^4 in $(1-2x+x^2-3x^3)^{-\frac{1}{2}}$. (3.) x^5 in $(1-3x-7x^2+x^3)^{-\frac{1}{3}}$.

(4.) x^6 and x^7 in $(x+3x^3+5x^5+7x^7+\ldots)^{-2}$.

(5.) x^7 in $(1-3x+x^3-x^5)^{-\frac{1}{3}}$. (6.) x^r in $(2+3x+x^2)^{-2}$.

(7.) Show that in $(9a^2+6ax+4x^2)^{-1}$ the coefficient of x^{3r} is $2^{3r}(3a)^{-3r-2}$; and that the coefficient of every third term vanishes.

(8.) The coefficient of x^m in $(1+x+x^2)^m$ (m a positive integer) is

$$1 + \frac{m(m-1)}{(1!)^2} + \frac{m(m-1)(m-2)(m-3)}{(2!)^2} + \ldots .$$

(9.) The coefficient of x^{3r+1} in $(1+x)/(1+x+x^2)^2$ is $-(r+1)$.

(10.) Evaluate $\sqrt[100]{(100/99)}$, and $\sqrt[10]{(1002/998)}$, each to 10 places of decimals; and demonstrate in each case the accuracy of your approximation.

Find a first approximation to each of the following, x being small:—

(11.) $$\frac{\{x+\sqrt{(x^2+1)}\}^{2m} - \{x-\sqrt{(x^2+1)}\}^{2m}}{\{x+\sqrt{(x^2+1)}\}^{2m+1} - \{x-\sqrt{(x^2+1)}\}^{2m+1}}.$$

(12.) $(1+x)(1+rx)(1+r^2x)\ldots/(1-x)(1-x)^r(1-x)^{r^2}\ldots$.

(13.) $\sqrt{(2-\sqrt{(2-\sqrt{(2-\ldots-\sqrt{(1+x)}\ldots)})})}$; where $\sqrt{}$ is repeated n times.

(14.) If x be small compared with N^2, then $\sqrt{(N^2+x)} = N + x/4N + Nx/2(2N^2+x)$, the error being of the order x^4/N^7. For example, show that $\sqrt{(101)} = 10\frac{4}{5}\frac{0}{0}\frac{1}{0}$, to 8 places of decimals.

(15.) If p differ from N^3 by less than 1 per cent of either, then $\sqrt[3]{p}$ differs from $\frac{2}{3}N + \frac{1}{3}p/N^2$ by less than N/90000. (Math. Trip., 1882.)

(16.) If $p=N^4+x$ where x is small, then approximately

$$\sqrt[4]{p}=\frac{51}{56}N+\frac{5}{56}p/N^3+\frac{27}{14}Nx/(7p+5N^4);$$

show that when $N=10$, $x=1$, this approximation is accurate to 16 places of decimals. (Math. Trip., 1886.)

(17.) Show that $\lim_{n=\infty}\{1/\sqrt{n^2}+1/\sqrt{(n^2+1)}+\ldots+1/\sqrt{(n^2+2n)}\}=2$.

(Catalan, *Nouv. Ann.*, sec. i., t. 17).

(18.) Find an upper limit for the residue in the expansion of $(1+x)^m$ when m is a positive integer.

CHAPTER XXVIII.

Exponential and Logarithmic Series.

EXPONENTIAL SERIES.

§ 1.] We have already attached a definite meaning to the symbol a^x when a is a positive real quantity, and x any positive or negative commensurable quantity. We propose now to discuss the possibility of expanding a^x in a series of ascending powers of x.

If we assume that a convergent expansion of a^x in ascending powers of x exists, then we can easily determine its coefficients.

For, let

$$a^x = A_0 + A_1x + A_2x^2 + \ldots + A_nx^n + \ldots \qquad (1),$$

then, proceeding exactly as in chap. xxvii., § 2, we have

$$L(a^{x+h} - a^x)/h = A_1 + 2A_2x + \ldots + nA_nx^{n-1} + \ldots;$$

and the series on the right will be convergent so long as x lies within limits for which (1) is convergent. Now (by chap. xxv., § 13)

$$\begin{aligned} L(a^{x+h} - a^x)/h &= a^x \lambda L(e^{\lambda h} - 1)/\lambda h, \\ &= \lambda a^x, \end{aligned}$$

where $\lambda = \log_e a$, and e is Napier's Base, namely, the finite quantity $\underset{n=\infty}{L}\,(1 + 1/n)^n$. Hence

$$\lambda a^x = 1A_1 + 2A_2x + \ldots + nA_nx^{n-1} + \ldots \qquad (2).$$

Therefore, by (1),

$$\begin{aligned} \lambda(A_0 + A_1x + \ldots + A_{n-1}x^{n-1} + \ldots) \\ = 1A_1 + 2A_2x + \ldots + nA_nx^{n-1} \ldots \quad (3). \end{aligned}$$

Since both the series in (3) are convergent, we must have

$$1A_1 = \lambda A_0, \quad 2A_2 = \lambda A_1, \quad \ldots, \quad nA_n = \lambda A_{n-1}.$$

Using these equations, we find, successively,

$$A_1 = A_0\lambda/1!, \quad A_2 = A_0\lambda^2/2!, \quad \ldots, \quad A_n = A_0\lambda^n/n! \quad (4).$$

Also, since, by the meaning attached to a^x, $a^0 = +1$, putting $x = 0$ on both sides of (1), we have

$$+1 = A_0 \quad (5).$$

Hence, finally,

$$a^x = 1 + \lambda x/1! + (\lambda x)^2/2! + \ldots + (\lambda x)^n/n! + \ldots \quad (6).$$

We see, *a posteriori*, that the expansion found is really convergent for all values of x (chap. xxvi., § 5), and also that the series in (2) is convergent for all values of x. Our hypotheses are therefore justified.

This demonstration is subject to the same objection as the corresponding one for the Binomial Series: it is, however, interesting, because it shows what the expansion of a^x must be, provided it exist at all. We shall next give two other demonstrations, each of which supplies the deficiency of that just given, and each of which has an interest of its own.

§ 2.] *Deduction of the Exponential from the Binomial Expansion.*

By the binomial theorem,* we have, provided z be numerically greater than 1,

$$\left(1 + \frac{1}{z}\right)^{zx} = 1 + zx\frac{1}{z} + \frac{zx(zx-1)}{2!}\frac{1}{z^2} + \ldots + \frac{zx(zx-1)\ldots(zx-n+1)}{n!}\frac{1}{z^n} + \ldots$$

$$= 1 + x + \frac{x^2(1-1/zx)}{2!} + \ldots + \frac{x^n(1-1/zx)\ldots(1-\overline{n-1}/zx)}{n!} + R_n \quad (1),$$

where

$$R_n = \frac{x^{n+1}(1-1/zx)\ldots(1-n/zx)}{(n+1)!} + \frac{x^{n+2}(1-1/zx)\ldots(1-\overline{n+1}/zx)}{(n+2)!} + \ldots \quad (2).$$

* In what follows we have restricted the value of the index zx. Since z is to be ultimately made infinite, there is no objection to our supposing it always so chosen that zx is a positive integer. We then depend merely on the binomial expansion for positive integral indices. This will not affect the value of $L(1+1/z)^{zx}$, for it has been shown (chap. xxv., § 13) that this has the same value when z becomes $+\infty$ or $-\infty$, and whether z increases by integral or by fractional increments.

Suppose now x to be a *given quantity*; and give to n any fixed integral value whatever. Then, no matter what positive or negative commensurable value x may have, we can always choose z as large as we please, and at the same time such that zx is a positive integer, p say, where $p > n$. The series (2) will then terminate; and we shall have $1/zx < 2/zx < \ldots < n/zx \ldots < (p-1)/zx < 1$. With this understanding, it follows that

$$\begin{aligned} R_n &< \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \ldots + \frac{x^p}{p!}, \\ &< \frac{x^{n+1}}{(n+1)!}\left\{1 + \frac{x}{n+2} + \frac{x^2}{(n+2)^2} + \ldots \text{ ad } \infty\right\}, \\ &< x^{n+1}/(n+1)!\{1 - x/(n+2)\} \qquad (3); \end{aligned}$$

and we have

$$\left(1+\frac{1}{z}\right)^{zx} = 1 + x + \frac{x^2(1-1/p)}{2!} + \ldots + \frac{x^n(1-1/p)\ldots(1-\overline{n-1}/p)}{n!} + R_n \qquad (4),$$

where R_n satisfies the condition (3).

Now let z, and therefore also p, increase without limit (n remaining fixed as before). Then, since

$$\mathop{L}_{p=\infty} (1-1/p)\ldots(1-\overline{n-1}/p) = 1,$$

we have

$$\mathop{L}_{z=\infty}\left(1+\frac{1}{z}\right)^{zx} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n \qquad (5),$$

R_n being still subject to (3).

We may now, if we choose, consider the effect of increasing n. When this is done, $x^{n+1}/(n+1)!\{1 - x/(n+2)\}$ (see chap. XXV., § 15) continually diminishes, having zero for its limit when $n = \infty$; we may therefore write

$$\mathop{L}_{z=\infty}\left(1+\frac{1}{z}\right)^{zx} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \text{ ad } \infty \qquad (6).$$

Thus the value of $L(1 + 1/z)^{zx}$ is obtained in the form of an infinite series, which converges for all values of x. For most purposes the form (5) is, however, the most convenient, since it gives an upper limit for the residue of the series.

§ 3.] The conditions of the demonstration of last paragraph will not be violated if we put $x = 1$. Hence, using e, as in chap. XXV., to denote $\mathrm{L}_{z=\infty} (1 + 1/z)^z$, we have

$$e = \mathrm{L}_{z=\infty} \left(1 + \frac{1}{z}\right)^z = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \mathrm{R}_n \qquad (7),$$

where $$\mathrm{R}_n < (n+2)/(n+1)(n+1)! \qquad (8).$$

This formula enables us to calculate e with comparative rapidity to a large number of decimal places. We have merely to divide 1 by 2, then the quotient by 3; and so on. Proceeding as far as $n = 12$, we have

$1 + 1$	$= 2{\cdot}000000000$
$1/2!$	$= \cdot500000000$
$1/3!$	$= 166666667$
$1/4!$	$= 41666667$
$1/5!$	$= 8333333$
$1/6!$	$= 1388889$
$1/7!$	$= 198413$
$1/8!$	$= 24801$
$1/9!$	$= 2756$
$1/10!$	$= 276$
$1/11!$	$= 25$
$1/12!$	$= 2$
	$2{\cdot}718281829$

Here the error in the last figure owing to figures neglected in the arithmetical calculation could not exceed the carriage from 10×5, that is, 5. Also the residue $\mathrm{R}_{12} < \frac{14}{13}(1/13!) < \frac{14}{13} \cdot 0000000002 < \cdot0000000003$, so that the neglect of R_{12} would certainly not affect the eighth place. Hence we have as the nearest 7-place approximation for e

$$e = 2{\cdot}7182818.$$

It is usual to give a demonstration that the numerical constant e is incommensurable. The ordinary demonstration is as follows:—

Let us suppose that e is commensurable, say $= p/q$, where p and q are finite positive integers. Then we have by (7)

$$p/q = 2 + 1/2! + \ldots + 1/q! + \mathrm{R}_q,$$

where $$\mathrm{R}_q < (q+2)/(q+1)^2 q!.$$

Hence, multiplying by $q!$, we get

$$p.(q-1)! = \mathrm{I} + q!\mathrm{R}_q,$$

where $p(q-1)!$ and I are obviously integral numbers. Hence $q!\mathrm{R}_q$ must be integral.

Now
$$q!\mathrm{R}_q < (q+2)/(q+1)^2,$$
$$< (q+2)/\{q(q+2)+1\},$$

that is, $q!\mathrm{R}_q$ is a positive proper fraction.

The assumption that e is commensurable therefore leads to an arithmetical absurdity, and is inadmissible.

Another demonstration which gives more insight into the nature of this and some other similar cases of incommensurability in the value of an infinite series is as follows:—

If $r_1, r_2, \ldots, r_n, \ldots$ be an infinite series of integers *given in magnitude and in order*, then it can be shown (see chap. ix., § 2) that any commensurable number p/q (where p and q are prime to each other, and $p<q$) can be expanded, and that in one way only, in the form

$$\frac{p}{q} = \frac{p_1}{r_1} + \frac{p_2}{r_1 r_2} + \frac{p_3}{r_1 r_2 r_3} + \ldots + \frac{p_n}{r_1 r_2 \ldots r_n} + \ldots \qquad (9),$$

where $p_1 < r_1$, $p_2 < r_2$, $\ldots$, $p_n < r_n$, $\ldots$; and that the series will always terminate when either q or all its factors occur among the factors of the integers $r_1, r_2, \ldots, r_n, \ldots$ Hence no *infinite* series of the form (9) can represent any vulgar fraction whose denominator consists of factors which occur among $r_1, r_2, \ldots, r_n, \ldots$.

In particular, if $r_1, r_2, \ldots, r_n, \ldots$ *contain all the natural primes, and, a fortiori, if they be the succession of natural numbers (excepting* 1*), namely,* 2, 3, 4, 5, $\ldots, n+1, \ldots$, *then the series in* (9) *cannot represent any commensurable number at all.**

The incommensurability of e is a mere particular case of the last conclusion; for we have in the series representing $e-2$

$$r_1 = 2, \quad r_2 = 3, \quad \ldots, \quad r_n = n+1, \quad \ldots;$$
$$p_1 = 1, \quad p_2 = 1, \quad \ldots, \quad p_n = 1, \quad \ldots.$$

Hence $e-2$ is incommensurable, and therefore e also.

§ 4.] Returning to equation (5) of § 2, since $\mathrm{L}(1+1/z)^z$ has a finite value e, we have $\mathrm{L}(1+1/z)^{zx} = \{\mathrm{L}(1+1/z)^z\}^x = e^x$, therefore

---

* It should be noticed that an infinite series of the form (9) may represent a fraction whose denominator contains a factor not occurring among $r_1, r_2, \ldots, r_n, \ldots$, for example,

$$\frac{1}{2} = \frac{1}{3} + \frac{2}{3.5} + \frac{3}{3.5.7} + \frac{4}{3.5.7.9} + \ldots \text{ ad } \infty.$$

This point seems to have been overlooked by some mathematical writers.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + R_n \qquad (10),$$

where R_n is subject to the inequality (3).

Finally, since $a^x = e^{\lambda x}$, where $\lambda = \log_e a$, we have

$$a^x = 1 + \frac{(\lambda x)}{1!} + \frac{(\lambda x)^2}{2!} + \ldots + \frac{(\lambda x)^n}{n!} + R_n \qquad (11),$$

where $$R_n < (\lambda x)^{n+1}/(n+1)!\{1 - \lambda x/(n+2)\} \qquad (12).$$

Since $LR_n = 0$ when $n = \infty$, the series (10) and (11) may of course each be continued to infinity.

This completes our second demonstration of the exponential theorem.

§ 5.] *Summation of the Exponential Series for real values of x.*

A third demonstration was given by Cauchy in his *Analyse Algébrique*. It follows closely the lines of the second demonstration of the binomial theorem; and no doubt it was suggested by the elegant process, due to Euler, on which that demonstration is founded. This third demonstration is of great importance, because we shall (in chap. xxix.) use the process involved in it to settle the more general question regarding the summation of the Exponential Series when x is a complex number.

Denote the infinite series

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

by the symbol $f(x)$. Then, since the series is convergent for all values of x, $f(x)$ is a single valued, finite, continuous function of x (chap. xxvi., § 19).

Also, since $f(x)$ and $f(y)$ are both absolutely convergent series, we have, by the rule for the multiplication of series (chap. xxvi., § 14),

$$f(x)f(y) = 1 + (x+y) + \left(\frac{x^2}{2!} + \frac{xy}{1!1!} + \frac{y^2}{2!}\right) + \ldots$$
$$+ \left(\frac{x^n}{n!} + \frac{x^{n-1}y}{(n-1)!1!} + \frac{x^{n-2}y^2}{(n-2)!2!} + \ldots + \frac{y^n}{n!}\right) + \ldots .$$

Now

$$\frac{x^n}{n!}+\frac{x^{n-1}y}{(n-1)!\,1!}+\frac{x^{n-2}y^2}{(n-2)!\,2!}+\ldots+\frac{y^n}{n!}$$
$$=(x^n+{}_nC_1\,x^{n-1}y+{}_nC_2\,x^{n-2}y^2+\ldots+y^n)/n!,$$
$$=(x+y)^n/n!,$$

by the binomial theorem for positive integral exponents.

Hence
$$f(x)\,f(y)=1+\Sigma(x+y)^n/n!,$$
$$=f(x+y) \qquad (1).$$

Hence
$$f(x)\,f(y)\,f(z)=f(x+y)\,f(z),$$
$$=f(x+y+z);$$

and, in general, $x, y, z, \ldots$ being any real quantities positive or negative,

$$f(x)\,f(y)\,f(z)\ldots=f(x+y+z+\ldots) \qquad (2).$$

This *last result* is called the *Addition Theorem for the Exponential Series.*

From (2), putting $x=y=z, \ldots, =1$, and supposing the number of letters to be n, we deduce

$$\{f(1)\}^n=f(n) \qquad (3).$$

Also, taking the number of the letters to be q, and each to be p/q, we deduce

$$\{f(p/q)\}^q=f(p) \qquad (4),$$

where p and q are any positive integers. From (4), by means of (3), we deduce

$$\{f(p/q)\}^q=\{f(1)\}^p \qquad (5).$$

Finally, from 1, putting $y=-x$, we deduce

$$f(x)\,f(-x)=f(0) \qquad (6).$$

The equations (5) and (6) enable us to sum the series $f(x)$ for all commensurable values of x.

From (5) we see that $f(p/q)$ is a qth root of $\{f(1)\}^p$. Now, since p/q is positive, the value of $f(p/q)$ is obviously real and positive. Also $f(1)$, that is, $1+1/1!+1/2!+\ldots$, is a finite positive quantity, which we may call e. Therefore $\{f(1)\}^p$, or e^p, is real and positive. Hence $f(p/q)$ must be the real positive qth root of e^p, that is, $e^{p/q}$. Hence

$$1 + \frac{p/q}{1!} + \frac{(p/q)^2}{2!} + \ldots = e^{p/q} \qquad (7),$$

p and q being any positive integers.

Finally, since $f(0) = 1$, we see from (6) that

$$\begin{aligned} f(-p/q) &= 1/f(p/q), \\ &= 1/e^{p/q}, \\ &= e^{-p/q}. \end{aligned}$$

Hence

$$1 + \frac{(-p/q)}{1!} + \frac{(-p/q)^2}{2!} + \ldots = e^{-p/q} \qquad (8),$$

where p/q is any positive commensurable number.

By combining (7) and (8) we complete the demonstration of the theorem, that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots,$$

for all commensurable values of x, e being given by

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots .$$

The student will not fail to observe that e is introduced and defined in the course of the demonstration.

The extension of the theorem to the case where the base is any positive quantity a is at once effected by the transformation $a^x = e^{\lambda x}$, as in last demonstration.

§ 6.] From the Exponential Series we may derive a large number of others; and, conversely, by means of it a variety of series can be summed.

Bernoulli's Numbers.—One of the most important among the series which can be deduced from the exponential theorem is the expansion of $x/(1 - e^{-x})$, the coefficients in the even terms of which are closely connected with the famous numbers of Bernoulli.

We shall first give Cauchy's demonstration, which shows, *a priori*, that $x/(1 - e^{-x})$ *can be expanded in an ascending series of powers of* x, *provided* x *lie within certain limits.*

We have

$$\frac{x}{1-e^{-x}} = \frac{1}{(1-e^{-x})/x} = \frac{1}{1-y} \qquad (1),$$

where

$$y = 1-(1-e^{-x})/x \qquad (2).$$

Now, from (1), we have

$$x/(1-e^{-x}) = 1+y+y^2+\ldots \text{ ad } \infty \qquad (3);$$

and this series will be absolutely convergent provided $-1<y<+1$.

Also, from (2), using the exponential theorem, we have

$$y = x/2!-x^2/3!+x^3/4!-\ldots \text{ ad } \infty \qquad (4);$$

and this series is absolutely convergent for all values of x, and therefore remains convergent when all the signs are taken alike.

If, therefore, we can find a value of ρ such that

$$\rho/2!+\rho^2/3!+\rho^3/4!+\ldots \text{ ad } \infty < 1 \qquad (A),$$

then, for all values of x between $-\rho$ and $+\rho$, Cauchy's conditions of absolute convergency (chap. xxvi., § 34) will be fulfilled for the double series which results, when we substitute in (3) the value of y given by (4). This double series may therefore be arranged according to powers of x, and the result will be a convergent expansion for $x/(1-e^{-x})$.

It is easy to show that a value of ρ can be found to satisfy the condition (A); for we have

$$\rho/2!+\rho^2/3!+\ldots = (e^\rho-1)/\rho-1.$$

We have, therefore, merely to choose ρ so that

$$e^\rho-1<2\rho \qquad (5).$$

If the graphs of e^x-1 and of $2x$ be drawn, it will be seen that both pass through the origin, the former being inclined to the x-axis at an angle whose tangent is 1, the latter at an angle whose tangent is 2, that is to say, at a greater angle. Therefore, since e^x-1 increases as x increases, and that ultimately much faster than $2x$, the graph of e^x-1 will cross the graph of $2x$ just once. Therefore the inequality (5) will be satisfied provided ρ be less than the unique positive root of the equation $e^x-1=2x$. Since $e^1-1<2\times 1$, and $e^3-1>2\times 2$, this root lies

between 1 and 2.* It will, therefore, certainly be possible to expand $x/(1-e^{-x})$ in a convergent series of powers of x if $-1<x<+1$.

If we make the substitution for y, and calculate the coefficients of the first few terms, we find that

$$\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} - \ldots \qquad (6).$$

Knowing, *a priori*, that the expansion exists, we can easily find a recurrence formula for calculating the successive coefficients. Let

$$x/(1-e^{-x}) = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots \qquad (7).$$

Then, putting $-x$ in place of x, we must have, since $-x/(1-e^{x}) = e^{-x}x/(1-e^{-x})$,

$$e^{-x}x/(1-e^{-x}) = A_0 - A_1x + A_2x^2 - A_3x^3 + \ldots \qquad (8).$$

Since both the series are convergent, we have, by subtracting,

$$x = 2A_1x + 2A_3x^3 + \ldots \qquad (9).$$

Hence $A_1 = \frac{1}{2}$; and all the other coefficients of odd order must vanish.

Therefore, from (7), we have

$$\begin{aligned} x &= (A_0 + \tfrac{1}{2}x + A_2x^2 + A_4x^4 + \ldots)(1-e^{-x}), \\ &= (A_0 + \tfrac{1}{2}x + A_2x^2 + A_4x^4 + \ldots + A_{2n}x^{2n} + \ldots) \\ &\qquad \times \left(\frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots - \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} - \ldots\right). \end{aligned}$$

The product of these two convergent series will be another convergent series, all of whose coefficients, except the coefficient of x, must vanish. Hence, equating coefficients of odd powers of x, we deduce $A_0 = 1$, and

$$\frac{A_{2n}}{1!} + \frac{A_{2n-2}}{3!} + \ldots + \frac{A_2}{(2n-1)!} - \frac{1}{2(2n)!} + \frac{1}{(2n+1)!} = 0,$$

* More nearly, the root is 1.250 . . .; but the actual value, as will be seen presently, is not of much importance.

that is,

$$\frac{A_{2n}}{1!}+\frac{A_{2n-2}}{3!}+\ldots+\frac{A_2}{(2n-1)!}=\frac{2n-1}{2(2n+1)!} \qquad (10).$$

In like manner, if we equate the coefficients of even powers of x, we deduce

$$\frac{A_{2n}}{2!}+\frac{A_{2n-2}}{4!}+\ldots+\frac{A_2}{(2n)!}=\frac{2n}{2(2n+2)!} \qquad (11).$$

If, as is usual, we put $A_{2n}=(-)^{n-1}B_n/(2n)!$, our expansion becomes

$$\frac{x}{1-e^{-x}}=1+\frac{1}{2}x+\frac{B_1}{2!}x^2-\frac{B_2}{4!}x^4+\frac{B_3}{6!}x^6-\ldots \qquad (12);$$

and the equations (10) and (11) may be written

$$_{n+1}C_2\,B_n-{}_{n+1}C_{2n-2}\,B_{n-1}+\ldots(-)^{n-1}{}_{n+1}C_2\,B_1=(-)^{n-1}(n-\tfrac{1}{2}) \qquad (10')$$

and

$$_{n+2}C_{2n}\,B_n-{}_{n+2}C_{2n-2}\,B_{n-1}+\ldots(-)^{n-1}{}_{n+2}C_2\,B_1=(-)^{n-1}n \qquad (11')$$

respectively.

If we put $n=1$, $n=2$, $n=3$, . . ., successively, either in (10′) or in (11′), we can calculate, one after the other, the numbers B_1, B_2, . . ., B_n, . . ., which are called Bernoulli's numbers.* Since we know, *a priori*, that the expansion exists, the two equations (10′) and (11′) must of necessity be consistent. Neither of them furnishes the most convenient method for calculating the numbers rapidly to a large number of decimal places; but it is easy to deduce from them exact values for a few of the earlier in the series, namely,

$$B_1=\frac{1}{6},\ B_2=\frac{1}{30},\ B_3=\frac{1}{42},\ B_4=\frac{1}{30},$$

$$B_5=\frac{5}{66},\ B_6=\frac{691}{2730},\ B_7=\frac{7}{6},\ B_8=\frac{3617}{510},$$

$$B_9=\frac{43867}{798},\ B_{10}=\frac{1222277}{2310},\ \&c.$$

* There is considerable divergence among mathematical writers as to the notation for Bernoulli's numbers. What we have denoted by B_n is often denoted by B_{2n}, or by B_{2n-1}.

We shall return to the properties of these numbers in chap. XXX.

Remark regarding the limits within which the expansion of $x/(1-e^{-x})$ *is valid.*—If we denote the series

$$1+\frac{1}{2}x+\frac{B_1}{2!}x^2-\frac{B_2}{4!}x^4+\ldots$$

by $\phi(x)$, we may state the problem we have just solved as follows:—*To find a convergent series* $\phi(x)$ *such that* $(1-e^{-x})\phi(x)=x$, *that is, such that* $(x-x^2/2!+x^3/3!-\ldots)\phi(x)=x$.

Now, since $x-x^2/2!+x^3/3!-$ is absolutely convergent for all values of x, and the coefficients of $\phi(x)$ satisfy (10′) and (11′), $\phi(x)$ will satisfy the condition $(x-x^2/2!+x^3/3!-\ldots)\phi(x)=x$ so long as $\phi(x)$ is convergent. Hence, so long as $\phi(x)$ is convergent, it will be the expansion of $x/(1-e^{-x})$. As a matter of fact, it follows from an expression for Bernoulli's numbers given in chap. XXX. that $\phi(x)$ is convergent so long as $-2\pi<x<+2\pi$. The actual limits of the validity of the expansion are therefore much wider than those originally assigned in the *a priori* proof of its existence.

Cor. 1. *Since* $x(e^x + e^{-x})/(e^x - e^{-x}) \equiv x/(1 - e^{-2x}) - x/(1 - e^{2x})$, *we deduce from* (12)

$$x\frac{e^x+e^{-x}}{e^x-e^{-x}} = 1+\frac{B_1}{2!}2^2x^2-\frac{B_2}{4!}2^4x^4+\frac{B_3}{6!}2^6x^6-\ldots \quad (13).$$

Cor. 2. *Since* $x/(1 + e^{-x}) \equiv 2x/(1 - e^{-2x}) - x/(1 - e^{-x})$,

$$\frac{x}{1+e^{-x}} = \frac{1}{2}(2^1-1)x+\frac{B_1}{2!}(2^2-1)x^2-\frac{B_2}{4!}(2^4-1)x^4+\ldots \quad (14).$$

§ 7.] *Bernoulli's Theorem.*—We have already seen that the sum of the rth powers of the first n integers $({}_nS_r)$ is an integral function of n of the $r+1$th degree (see chap. XX., § 9).

We shall now show that the coefficients of this function can be expressed by means of Bernoulli's numbers.

From the identity

$$(e^{nx}-1)/(e^x-1) \equiv 1 + e^x + e^{2x} + \ldots + e^{(n-1)x},$$

that is,

$$(e^{nx}-1)/(1-e^{-x}) \equiv e^x + e^{2x} + e^{3x} + \ldots + e^{nx},$$

we deduce at once

$$\left\{\frac{nx}{1!}+\frac{n^2x^2}{2!}+\ldots+\frac{n^rx^r}{r!}+\ldots\right\}\left\{1+\frac{1}{2}x+\frac{B_1}{2!}x^2-\frac{B_2}{4!}x^4+\ldots\right\}$$
$$= nx+\frac{{}^nS_1\,x^2}{1!}+\ldots+\frac{{}^nS_r\,x^{r+1}}{r!}+\ldots \quad (1),$$

wherein all the series are absolutely convergent, so long as n is finite, provided x do not exceed the limits within which $1 + \frac{1}{2}x + B_1x^2/2\,! - B_2x^4/4\,! + \ldots$ is convergent. The coefficient of x^{r+1} on the right of (1) must therefore be equal to the coefficient of x^{r+1} in the convergent series which is the product of the factors on the left. Hence

$$\frac{{}_nS_r}{r\,!} = \frac{n^{r+1}}{(r+1)\,!} + \frac{n^r}{2.r\,!} + \frac{B_1n^{r-1}}{2\,!(r-1)\,!} - \frac{B_2n^{r-3}}{4\,!(r-3)\,!} + \frac{B_3n^{r-5}}{6\,!(r-5)\,!} \cdot \ldots$$

Therefore

$${}_nS_r = \frac{n^{r+1}}{r+1} + \frac{1}{2}n^r + \frac{r}{2\,!}B_1n^{r-1} - \frac{r(r-1)(r-2)}{4\,!}B_2n^{r-3} + \frac{r(r-1)(r-2)(r-3)(r-4)}{6\,!}B_3n^{r-5} - \ldots \quad (2),$$

the last term being $(-)^{\frac{1}{2}(r-2)}B_{\frac{1}{2}r}n$, or $\frac{1}{2}(-)^{\frac{1}{2}(r-3)}rB_{\frac{1}{2}(r-1)}n^2$, according as r is even or odd.

This formula was first given by James Bernoulli (*Ars Conjectandi*, p. 97, published posthumously at Basel in 1713). He gave no general demonstration; but was quite aware of the importance of his theorem, for he boasts that by means of it he calculated *intra semi-quadrantem horæ!* the sum of the 10th powers of the first thousand integers, and found it to be

91,409,924,241,424,243,424,241,924,242,500.

It will be a good exercise for the reader to check Bernoulli's result.*

SUMMATION OF SERIES BY MEANS OF THE EXPONENTIAL THEOREM.

§ 8.] Among the series which can be summed by means of the Exponential Series, two, related to it in the same way as the series of chap. xxvii., § 5, are related to the Binomial Series, deserve special mention.

We can always sum the series $\sum^{\infty}\phi_r(n)x^n/n\,!$, *where* $\phi_r(n)$ *is an integral function of* n *of the* r*th degree.* (*Integro-Exponential Series.*)

* For farther information regarding Bernoulli's numbers, see Boole's *Finite Differences* (ed. by Moulton); and, for a useful bibliography of the relative literature, Ely, *Am. Jour. Math.* (1882.)

For, as in chap. xxvii., § 5, we can always establish an identity of the form

$$\phi_r(n) = A_0 + A_1 n + A_2 n(n-1) + \ldots + A_r n(n-1)\ldots(n-r+1).$$

Then we have, taking, for simplicity of illustration, the lower limit of summation to be 0,

$$\sum_0^\infty \frac{\phi_r(n)x^n}{n!} = A_0 \sum_0^\infty \frac{x^n}{n!} + A_1 x \sum_1^\infty \frac{x^{n-1}}{(n-1)!} + A_2 x^2 \sum_2^\infty \frac{x^{n-2}}{(n-2)!} + \ldots + A_r x^r \sum_r^\infty \frac{x^{n-r}}{(n-r)!},$$

$$= (A_0 + A_1 x + A_2 x^2 + \ldots + A_r x^r)e^x.$$

Cor. *We can in general sum the series* $\sum_0^\infty \phi_r(n)x^n/n!(n+a)(n+b)\ldots(n+k)$, *where* $a, b, \ldots, k$ *are unequal positive integers.*

The process is the same as that used in the corollary of chap. xxvii., § 5, only the details are a little simpler. (See Example 5, below.)

Example 1. To deduce the formulæ (3), (4), (5) of chap. xxvii., § 9, by means of the exponential theorem.

$$(x+n)^s - {}_nC_1(x+n-1)^s + \ldots (-)^r {}_nC_r (x+n-r)^s + \ldots (-)^n x^s$$

is evidently the coefficient of z^s in

$$s!\{e^{(x+n)z} - {}_nC_1 e^{(x+n-1)z} + \ldots (-)^r {}_nC_r e^{(x+n-r)z} + \ldots (-)^n e^{xz}\}$$

$$= s!\, e^{xz}(e^z - 1)^n,$$

$$= s!\left\{1 + xz + \frac{x^2z^2}{2!} + \ldots\right\}\left\{1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\right\}^n z^n,$$

$$= s!\, z^n \left\{1 + xz + \frac{x^2z^2}{2!} + \ldots\right\}\left\{1 + \frac{n}{2}z + \frac{n(3n+1)}{24}z^2 + \ldots\right\}.$$

The lowest power of z in the product last written is z^n, and the coefficients of z^n, z^{n+1}, z^{n+2} are $s!$, $s!(x+\frac{1}{2}n)$, $\frac{1}{2}s!\{x^2 + nx + \frac{1}{12}n(3n+1)\}$ respectively.

Hence

$$(x+n)^s - {}_nC_1(x+n-1)^s + \ldots (-)^r {}_nC_r(x+n-r)^s + \ldots (-)^n x^s$$
$$= 0, \text{ if } s < n;$$
$$= n!, \text{ if } s = n;$$
$$= (n+1)!(x+\tfrac{1}{2}n), \text{ if } s = n+1;$$
$$= \tfrac{1}{2}(n+2)!\{x^2 + nx + \tfrac{1}{12}n(3n+1)\}, \text{ if } s = n+2.$$

Example 2. If n and r be positive integers, show that

$$e^x\left\{\frac{1}{r!} + \frac{n}{1!(r+1)!}x + \ldots + \frac{n(n-1)\ldots(n-s+1)}{s!(r+s)!}x^s + \ldots + \frac{n(n-1)\ldots 1}{n!(r+n)!}x^n\right\}$$

$$= \frac{1}{r!} + \frac{n+r+1}{1!(r+1)!}x + \ldots + \frac{(n+r+1)(n+r+2)\ldots(n+r+s)}{s!(r+s)!}x^s + \ldots \text{ ad } \infty.$$

The right-hand side is the coefficient of z^{n+r} in

$$(z+x)^n+\frac{(z+x)^{n+1}}{1!}+\ldots+\frac{(z+x)^{n+r}}{r!}+\ldots+\frac{(z+x)^{n+r+s}}{(r+s)!}+\ldots$$
$$=(z+x)^n e^{z+x},$$
$$=e^x\{z^n+{}_nC_1 z^{n-1}x+\ldots+{}_nC_n x^n\}\times\left\{1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots\right\}.$$

Now the coefficient of z^{n+r} in this product is

$$e^x\left\{\frac{1}{r!}+\frac{n}{1!(r+1)!}x+\ldots+\frac{n(n-1)\ldots1}{n!(r+n)!}x^n\right\}.$$

Hence the theorem.

If we put $r=0$, and $x=1$, we have

$$1+\frac{n+1}{(1!)^2}+\frac{(n+1)(n+2)}{(2!)^2}+\ldots \text{ ad } \infty$$
$$=e\left\{1+\frac{n}{(1!)^2}+\frac{n(n-1)}{(2!)^2}+\ldots+\frac{n(n-1)\ldots1}{(n!)^2}\right\}.$$

Example 3. Sum the series

$$\frac{1^3}{1!}x+\frac{1^3+2^3}{2!}x^2+\ldots+\frac{1^3+2^3+\ldots+n^3}{n!}x^n+\ldots \text{ ad } \infty.$$

We have (by chap. XX., § 7)

$$1^3+2^3+\ldots+n^3=(n^4+2n^3+n^2)/4,$$
$$=\tfrac{1}{4}\{A_0+A_1n+A_2n(n-1)+A_3n(n-1)(n-2)+A_4n(n-1)(n-2)(n-3)\},$$

where $A_0, A_1, \ldots, A_4$ may be calculated as follows:—

	1+ 2+ 1+ 0+|0	$A_0=0$,
+1	0+ 1+ 3+ 4	
	1+ 3+ 4+|4	$A_1=4$,
+2	0+ 2+ 10	
	1+ 5+|14	$A_2=14$,
+3	0+ 3	
	1+|8	$A_3=8$, $A_4=1$.

Hence

$$\Sigma\frac{1^3+2^3+\ldots+n^3}{n!}x^n=x\Sigma\frac{x^{n-1}}{(n-1)!}+\frac{7}{2}x^2\Sigma\frac{x^{n-2}}{(n-2)!}+2x^3\Sigma\frac{x^{n-3}}{(n-3)!}+\frac{1}{4}x^4\Sigma\frac{x^{n-4}}{(n-4)!},$$
$$=(x+\tfrac{7}{2}x^2+2x^3+\tfrac{1}{4}x^4)e^x.$$

If we put $x=1$, we have

$$\Sigma(1^3+2^3+\ldots+n^3)/n!=27e/4.$$

Example 4. Show that $\sum_{n=1}^{n=\infty} n^3/n!=5e$.

Since $n^3\equiv n+3n(n-1)+n(n-1)(n-2)$,

$$\Sigma n^3/n!=\Sigma 1/(n-1)!+3\Sigma 1/(n-2)!+\Sigma 1/(n-3)!,$$
$$=5e.$$

Example 5. Evaluate $\sum_1^\infty (n-1)x^n/(n+2)n!$.

$$\Sigma\frac{(n-1)x^n}{(n+2)n!}=\frac{1}{x^2}\Sigma\frac{(n^2-1)x^{n+2}}{(n+2)!}.$$

Now $$n^2-1\equiv 3-3(n+2)+(n+2)(n+1).$$

Therefore

$$\sum_1^\infty\frac{(n-1)x^n}{(n+2)n!}=\frac{1}{x^2}\left\{3\sum_1^\infty\frac{x^{n+2}}{(n+2)!}-3x\sum_1^\infty\frac{x^{n+1}}{(n+1)!}+x^2\sum_1^\infty\frac{x^n}{n!}\right\},$$

$$=\{3(e^x-1-x-\tfrac{1}{2}x^2)-3x(e^x-1-x)+x^2(e^x-1)\}/x^2,$$

$$=\{(x^2-3x+3)e^x+(\tfrac{1}{2}x^2-3)\}/x^2.$$

EXERCISES XII.

(1.) Evaluate $1/e$ to six places of decimals.

(2.) Calculate $\dot{x}$ to a second approximation from the equation $50\log_e(1+x)=49x$.

(3.) If $e^x=1+xe^{hx}$, and x^4 be negligible, show that

$$h=1/2!+x/4!-x^3/4!\,5!.$$

(4.) Show that, if n be any positive integer,

$$(1-1/n)^{-n}>1+1/1!+1/2!+\ldots+1/n!>(1+1/n)^n.$$

(5.) Sum from 0 to ∞ $\Sigma(1-3n+n^2)x^n/n!$.

Sum to infinity

(6.) $1^2/2!+2^2/3!+3^2/4!+\ldots$.

(7.) $1^3/2!+2^3/3!+3^3/4!+\ldots$.

(8.) $1-2^3/1!+3^3/2!-4^3/3!+\ldots$.

(9.) $1^4+2^4/2!+3^4/3!+\ldots$.

Show that

(10.) $1/(2n)!-1/1!(2n-1)!+1/2!(2n-2)!-\ldots-1/1!(2n-1)!+1/(2n)!=0$.

(11.) If $n>3$, $n^3+{}_nC_2(n-2)^3+{}_nC_4(n-4)^3+\ldots=n^2(n+3)2^{n-4}$.

(12.) $n^n-{}_nC_1(n-2)^n+{}_nC_2(n-4)^n-\ldots=2^n n!$.

(13.) By expanding $e^{1/(1-x)}$, or otherwise, show that, if

$A_r=\sum_{n=1}^{n=\infty}(n+r-1)!/n!(n-1)!$, then $A_{r+1}-(2r+1)A_r+r(r-1)A_{r-1}=0$. (Math. Trip., 1882.)

(14.) Prove that

$$(x-x^3/3!+x^5/5!-\ldots)(1-x^2/2!+x^4/4!-\ldots)=\Sigma(-)^r 2^{2r}x^{2r+1}/(2r+1)!.$$

(15.) Solve the equation $x^3-x-1/n=0$; and show that the nth power of its greater root has e for its limit when $n=\infty$.

(16.) For all positive integral values of n

$$n^{n-1}\left(\frac{n-1}{2}\right)^{n-2}\left(\frac{n-2}{3}\right)^{n-3}\ldots\left(\frac{2}{n-1}\right)<e^{n(n-1)/2}.$$

(17.) If

$$x^n=A_0+\frac{A_1}{1!}(x-1)+\frac{A_2}{2!}(x-1)(x-2)+\ldots+\frac{A_n}{n!}(x-1)(x-2)\ldots(x-n),$$

show that $A_s=(s+1)^n-{}_nC_1 s^n+{}_nC_2(s-1)^n-\ldots(-)^s{}_nC_s 1^n$.

(18.) Show that $\sum_1^\infty (n^3+2n^2+n-1)/n! = 9e+1$.

(19.) Sum $\Sigma(n+a)(n+b)(n+c)x^n/n!$ from $n=0$ to $n=\infty$.

(20.) Show that e cannot be a root of a quadratic equation having finite rational coefficients.

(21.) Sum the series $\Sigma x^n/(n+3)n!$ from $n=0$ to $n=\infty$.

(22.) Sum to infinity the series $1^3/3.1! + 3^3/4.2! + 5^3/5.3! + \ldots$.

If $B_1, B_2, \ldots, B_n$ denote Bernoulli's numbers, show that

(23.) ${}_{2n+1}C_{2n-1} B_n - {}_{2n+1}C_{2n-3} B_{n-1} + \ldots (-)^{n-1} {}_{2n+1}C_1 B_1 = (-1)^{n-1}$.

(24.) ${}_{2n+1}C_{2n} B_n - \frac{{}_{2n+1}C_{2n-2} B_{n-1}}{2^2} + \ldots (-)^{n-1} \frac{{}_{2n+1}C_2 B_1}{2^{2n}} = (-)^{n-1} \frac{2n}{2^{2n}}$.

(25.) $\frac{1}{2}{}_nC_1 B_1 - \frac{1}{4}{}_nC_3 B_2 + \frac{1}{6}{}_nC_5 B_3 - \ldots = (n-1)/2(n+1)$, the last term on the left being $(-)^{\frac{1}{2}(n-2)} B_{n/2}$, or $\frac{1}{2}(-)^{\frac{1}{2}(n-2)} n B_{(n-1)/2}$, according as n is even or odd.

(26.) By comparing Bernoulli's expression for $1^r + 2^r + \ldots + n^r$ with the expressions deducible from Lagrange's Interpolation Formula, show that

$$\sum_1^{t=2p+1} (-)^{t-1} {}_{2p+1}C_t \frac{S_{2p}}{t} = (-)^{p-1} B_p;$$

$$\sum_1^{t=2p+2} (-)^{t-1} {}_{2p+1}C_t \frac{S_{2p+1}}{t} = 0.$$

Also that

$$\sum_1^{t=2p} (-)^{t-1} {}_{2p}C_t \frac{S_{2p}}{t(t+1)} = (-)^{p-1} B_p;$$

$$\sum_1^{t=2p+1} (-)^{t-1} {}_{2p+1}C_t \frac{S_{2p+1}}{t(t+1)} = 0.$$

(Kronecker, *Crelle's Jour.*, Bd. lxxxiv.; 1887.)

(27.) $x(e^x - e^{-x})/(e^x + e^{-x}) = \frac{B_1}{2!}(2^2-1)2^2x^2 + \frac{B_2}{4!}(2^4-1)2^4x^4 + \frac{B_3}{6!}(2^6-1)2^6x^6 + \ldots$

LOGARITHMIC SERIES.

§ 9.] *Expansion of* $\log(1+x)$.—It is obvious that no function of x which becomes infinite in value when $x=0$ can be expanded in a convergent series of ascending powers of x. For, if we suppose

$$f(x) = A_0 + A_1x + A_2x^2 + \ldots,$$

then on putting $x=0$ we have $\infty = A_0$; and the attempt to determine even the first coefficient fails.

There can therefore be no expansion of $\log x$ of the kind mentioned.

We can, however, expand $\log(1+x)$ *in a series of ascending powers of* x *provided* x *be numerically less than unity.*

The base in the first instance is understood to be e as usual.

By § 4, we have

$$(1+x)^z = 1 + z\{\log(1+x)\} + z^2\{\log(1+x)\}^2/2! + \ldots \quad (1);$$

and this series is convergent for all values of z.

Again, by the binomial theorem, we have, provided the numerical value of x be less than 1,

$$(1+x)^z = 1 + zx + z(z-1)x^2/2! + z(z-1)(z-2)x^3/3! + \ldots,$$
$$= 1 + zx - z(1-z/1)x^2/2 + z(1-z/1)(1-z/2)x^3/3 + \ldots \quad (2).$$

If we arrange this as a double series, we have

$$(1+x)^z = 1 + zx - \{zx^2/2 - z^2x^2/2\} + \{zx^3/3 - (1+\tfrac{1}{2})z^2x^3/3 + \tfrac{1}{2}z^3x^3/3\} +$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(-)^{n-1}\{zx^n/n - {}_{n-1}\mathrm{P}_1 z^2x^n/n + {}_{n-1}\mathrm{P}_2 z^3x^n/n - \ldots$$
$$(-)^{n-1}{}_{n-1}\mathrm{P}_{n-1} z^n x^n/n\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

where ${}_{n-1}\mathrm{P}_r$ stands for the sum of all the r-products of $1/1$, $1/2, \ldots, 1/(n-1)$, without repetition.

In order that Cauchy's criterion for the absolute convergency of the double series (3) may be satisfied, it will be sufficient if the series

$$zx^n/n + {}_{n-1}\mathrm{P}_1 z^2x^n/n + \ldots + {}_{n-1}\mathrm{P}_{n-1} z^n x^n/n \qquad (4)$$

and

$$1 + zx + z(1+z/1)x^2/2 + z(1+z/1)(1+z/2)x^3/3 + \ldots \qquad (5)$$

be both convergent when z and x are positive.

Now the sum of (4) is always $z(z+1)\ldots(z+n-1)x^n/n!$; and this has 0 for its limit when $n=\infty$, provided $x<1$. Also, the series (5) is absolutely convergent when $x<1$.

Hence, by chap. xxvi., § 34, we may rearrange the series (3) according to powers of z, and it will still converge to $(1+x)^z$.

Confining our attention to the first power of z, for the present, we thus find

$$(1+x)^z = 1 + \{x/1 - x^2/2 + x^3/3 - \ldots\}z + \ldots \qquad (5).$$

Now, since there can only be one convergent expansion of

$(1+x)^z$ in powers of z, the series in (1) and (5) must be identical. Therefore

$$\log(1+x) = x/1 - x^2/2 + x^3/3 - \ldots (-)^{n-1}x^n/n + \ldots \quad (6).$$

The series on the right of (6) is usually called the logarithmic series. It is absolutely convergent so long as $-1 < x < 1$, and it is precisely under this restriction that the above demonstration is valid.

If we put $x = 1$ on the right of (6), we get the series $1/1 - 1/2 + 1/3 - \ldots (-1)^{n-1}/n + \ldots$, which is semi-convergent. Hence, by Abel's Theorem (chap. xxvi., § 20), equation (6) will still hold in this case; and we have

$$\log 2 = 1/1 - 1/2 + 1/3 - \ldots + (-1)^{n-1}/n + \ldots \quad (7),$$

provided the order of the terms as written be adhered to.

If we put $x = -1$ in (6), the series becomes divergent. It diverges, however, to $-\infty$; so that, since $\log 0 = -\infty$, the theorem still holds in a certain sense.

Cor. *If we arrange the coefficients of the remaining powers of z in* (5), *and compare with* (1), *we find*

$$\{\log(1+x)\}^2 = 2!\{{}_1\text{P}_1 x^2/2 - {}_2\text{P}_1 x^3/3 + {}_3\text{P}_1 x^4/4 - \ldots\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\{\log(1+x)\}^n = n!\{{}_{n-1}\text{P}_{n-1} x^n/n - {}_n\text{P}_{n-1} x^{n+1}/(n+1) + {}_{n+1}\text{P}_{n-1} x^{n+2}/(n+2) - \ldots\} \quad (8).$$

These formulæ and the above demonstration are given by Cauchy in his *Analyse Algebrique.*

§ 10.] A variety of expansions can be deduced from the logarithmic theorem. The following are some of those that are most commonly met with :—

We have

$$\log(1+x) = x/1 - x^2/2 + x^3/3 - \ldots (-)^{n-1}x^n/n + \ldots;$$

also

$$\log(1-x) = -x/1 - x^2/2 - x^3/3 - \ldots - x^n/n - \ldots$$

Hence, by subtraction, since $\log(1+x) - \log(1-x) \equiv \log\{(1+x)/(1-x)\}$, we deduce

$$\log\{(1+x)/(1-x)\} = 2\{x/1 + x^3/3 + \ldots + x^{2n-1}/(2n-1) + \ldots\} \quad (9).$$

Putting in (9) $y = (1+x)/(1-x)$, and therefore $x = (y-1)/(y+1)$, we get

$$\log y = 2\left\{\frac{1}{1}\left(\frac{y-1}{y+1}\right) + \frac{1}{3}\left(\frac{y-1}{y+1}\right)^3 + \ldots + \frac{1}{2n-1}\left(\frac{y-1}{y+1}\right)^{2n-1} + \ldots\right\} \quad (10),$$

an expansion for $\log y$ (but not, be it observed, in powers of y) which will be convergent if y be positive—the only case at present in question.

Again, since $1 + x \equiv x(1 + 1/x)$, and $\log(1+x) \equiv \log x + \log(1 + 1/x)$, putting in (10) $y = 1 + 1/x$, so that $(y-1)/(y+1) = 1/(2x+1)$, we have

$$\log(1+x) = \log x + 2\{1/1(2x+1) + 1/3(2x+1)^3 + \ldots\} \quad (11).$$

Finally, since $x + 1 \equiv x^2(1 - 1/x^2)/(x-1)$,

$$\log(x+1) = 2\log x - \log(x-1) - 2\{1/1(2x^2-1) + 1/3(2x^2-1)^3 + \ldots\} \quad (12).$$

If, in any of the above formulæ, we wish to use a base a different from e, we have simply to multiply by the "modulus" $1/\log_e a$ (see chap. xxi., § 9). Thus, for example, from (10) we derive

$$\log_a y = \frac{2}{\log_e a}\left\{\left(\frac{y-1}{y+1}\right) + \frac{1}{3}\left(\frac{y-1}{y+1}\right)^3 + \ldots\right\} \quad (13).$$

ON THE CALCULATION OF LOGARITHMS.

§ 11.] The early calculators of logarithms largely used methods depending on the repeated extraction of the square root. This process was combined with the Method of Differences, which seems to have arisen out of the practical necessities of the Logarithmic Calculator.*

* See Glaisher, Art. "Logarithms," *Encyclopædia Britannica*, 9th ed., from which much of what follows is taken.

Thus, Briggs used the approximate formula

$$\log_{10}2 = (2^{10/2^{47}} - 1)2^{47}/10 \log_e 10,$$

depending on the accurate formula

$$\underset{z=0}{\mathrm{L}}\,(x^z - 1)/z = \log_e x,$$

which we have already established in the chapter on Limits, and which might readily be deduced from the exponential theorem. The calculation of $\log_{10}2$ in this way, therefore, involved the raising of 2 to the tenth power and the subsequent extraction of the square root 47 times!

Calculations of this kind were infinitely laborious, and nothing but the enthusiasm of pioneers could have sustained the calculators. If it were necessary nowadays to calculate a logarithmic table afresh, or to calculate the logarithm of a single number to a large number of places, some method involving the use of logarithmic series would probably be adopted.

The series in § 10 enable us to calculate fairly rapidly the Napierian Logarithms of the small primes, 2, 3, 5, 7.

Thus, putting $y = 2$ in (10) we have

$$\log 2 = 2\{1/1 \,.\, 3 + 1/3 \,.\, 3^3 + 1/5 \,.\, 3^5 + \ldots\}.$$

The calculation to nine places may be arranged thus:—

$1/3$	·333,333,333	$1/1 \,.\, 3$	·333,333,333	
$1/3^3$	37,037,037	$1/3 \,.\, 3^3$	12,345,679	
$1/3^5$	4,115,226	$1/5 \,.\, 3^5$	823,045	
$1/3^7$	457,247	$1/7 \,.\, 3^7$	65,321	
$1/3^9$	50,805	$1/9 \,.\, 3^9$	5,645	
$1/3^{11}$	5,645	$1/11 \,.\, 3^{11}$	513	
$1/3^{13}$	627	$1/13 \,.\, 3^{13}$	48	
$1/3^{15}$	70	$1/15 \,.\, 3^{15}$	5	
$1/3^{17}$	8	$1/17 \,.\, 3^{17}$	0	
			·346,573,589	± 4
			2	
			·693,147,178	± 8

By the principle of chap. xxvi., § 30, the residue of the series is less than

$$\{1/19 \,.\, 3^{19}\}/(1 - \tfrac{1}{9}),$$

that is, less than ·000,000,000,06; and the utmost error from the carriage to the last line is ± 4. The utmost error in our calculation is ± 8. Hence, subject to an error of 1 at the utmost in the last place, we have

$$\log 2 = \cdot 693,147,18.$$

Having thus calculated log 2, we can obtain log 3 more rapidly by putting $x = 2$ in (11). Thus

$$\log 3 = \log 2 + 2\{1/1 \,.\, 5 + 1/3 \,.\, 5^3 + 1/5 \,.\, 5^5 + \ldots\}.$$

Knowing log 2 and log 3, we can deduce $\log 4 = 2 \log 2$, and $\log 6 = \log 3 + \log 2$. Then, putting $x = 4$ in (12), we have

$$\log 5 = 2 \log 4 - \log 3 - 2\{1/31 + 1/3 \,.\, 31^3 + \ldots\}.$$

Also, putting $x = 6$ in (12), we have

$$\log 7 = 2 \log 6 - \log 5 - 2\{1/71 + 1/3 \,.\, 71^3 + \ldots\}.$$

It will be a good exercise in computation for the student to calculate by means of these formulæ the Napierian Logarithms of the first 10 integers. The following table of the results to ten places will serve for verification :—

No.	Logarithm.
1	0·000,000,000,0
2	0·693,147,180,$\bar{6}$*
3	1·098,612,288,$\bar{7}$
4	1·386,294,361,1
5	1·609,437,912,4
6	1·791,759,469,2
7	1·945,910,149,$\bar{1}$
8	2·079,441,541,$\bar{7}$
9	2·197,224,577,3
10	2·302,585,093,$\bar{0}$

From the value of $\log_e 10$ we deduce the value of its reciprocal, namely, M = ·434,294,481,903,251; and, by multiplying by this number, we can convert the Napierian Logarithm of

* $\bar{6}$ means that the 10th digit has been increased by a unit, because the 11th exceeds 4.

any number into the ordinary or Briggian Logarithm, whose base is 10.

Much more powerful methods than the above can be found for calculating log 2, log 3, log 5, log 7, and M.

By one of these (see Exercises XIII., 2, below) Professor J. C. Adams has calculated these numbers to 260 places of decimals.

§ 12.] *The Factor Method of calculating Logarithms* * is one of the most powerful, and at the same time one of the most instructive, from an arithmetical point of view, of all the methods that have been proposed for readily finding the logarithm of a given number to a large number of decimals.

This method depends on the fact that every number may, to any desired degree of accuracy, be expressed in the form

$$10^m p_0/(1 - p_1/10)(1 - p_2/10^2)(1 - p_3/10^3) \ldots \qquad (1),$$

where p_0, p_1, p_2, . . . each denote one of the 10 digits, 0, 1, 2, . . ., 9, p_0 being of course not 0.

Take, for example, 314159 as the given number. First divide by 10^5. 3, and we have

$$3{\cdot}14159 = 10^5 . 3 . 1{\cdot}047,196,666,666 \ldots .$$

Next multiply 1·047,196,666,666 by $1 - 4/10^5$, that is, cut off two digits from the end of the number, then multiply by 4 and subtract the result from the number itself. The effect of this will be to destroy the first significant figure after the decimal point. We have in fact

$$1{\cdot}047,196,666,666 \times (1 - 4/10^5) = 1{\cdot}005,308,800,000.$$

Next multiply 1·005,308,800,000 by $1 - 5/10^3$, and so on till the twelve figures after the point are all reduced to zero. The actual calculation can be performed very quickly, as follows :—

* For a full history of this method see Glaisher's article above quoted; or the Introduction to Gray's *Tables for the Formation of Logarithms and Anti-Logarithms to Twenty-four Places* (1876).

1·0 4 7, 1 9 6, 6 6 6, 6|6 6	$4/10^{2}$
4 1, 8 8 7, 8 6 6, 6 6 6	
5, 3 0 8, 8 0 0,|0 0 0	$5/10^{5}$
5, 0 2 6, 5 4 4, 0 0 0	
2 8 2, 2 5|6, 0 0 0	$2/10^{4}$
2 0 0, 0 5 6, 4 5 1	
8 2, 1|9 9, 5 4 9	$8/10^{5}$
8 0, 0 0 6, 5 7 6	
2,|1 9 2, 9 7 3	$2/10^{6}$
2, 0 0 0, 0 0 4	
|1 9 2, 9 6 9	$1/10^{7}$
1 0 0, 0 0 0	
9 2, 9 6 9	$9/10^{8}$, $2/10^{9}$, $9/10^{10}$, $6/10^{11}$, $9/10^{12}$.

The remaining factors being obvious without farther calculation. Hence we have

$$314159 \times (1 - 4/10^{2})(1 - 5/10^{5}) \ldots (1 - 9/10^{12}) = 10^{6}.3(1 + x/10^{13}),\ x > 9.$$

Therefore

$$314159 = 10^{6}.3(1 + x/10^{13})/(1 - 4/10^{2})(1 - 5/10^{5}) \ldots (1 - 9/10^{12}) \quad (2).$$

Since $\log(1 + x/10^{13}) < x/10^{13}$, it follows from (2) that, as far as the twelfth place of decimals,

$$\begin{aligned}\log 314159 = {} & 5 \log 10 + \log 3 - \log(1 - 4/10^{2}) - \log(1 - 5/10^{5}) \\ & - \log(1 - 2/10^{4}) - \log(1 - 8/10^{6}) - \log(1 - 2/10^{9}) \\ & - \log(1 - 1/10^{7}) - \log(1 - 9/10^{8}) - \log(1 - 2/10^{9}) \\ & - \log(1 - 9/10^{10}) - \log(1 - 6/10^{11}) - \log(1 - 9/10^{12}).\end{aligned}$$

All, therefore, that is required to enable us to calculate $\log 314159$ to twelve places is an auxiliary table containing the logarithms of the first 10 integers, and the logarithms of $1 - p/10^{r}$ for all integral values of p from 1 to 9, and for all integral values of r from 1 to 12. To make quite sure of the last figure this auxiliary table should go to at least thirteen places.

§ 13.] It should be noticed that a method like the above is suitable when only solitary logarithms are required. If a complete table were required, the Method of Differences would be employed to find the great majority of the numbers to be entered.

A full discussion of this method would be out of place here;* but we may, before leaving this part of the subject, give an analytical view of the method of interpolation by *First* Differences, already discussed graphically in chap. xxi.

We have

$$\log_{10}(x+h) - \log_{10}x = \log_{10}(1 + h/x)$$
$$= \mathrm{M}\{h/x - \tfrac{1}{2}(h/x)^2 + \tfrac{1}{3}(h/x)^3 - \ldots\} \qquad (1).$$

Hence, if $h < x$, we have approximately

$$\log_{10}(x+h) - \log_{10}x = \mathrm{M}h/x \qquad (2),$$

the error being less than $\frac{1}{2}\mathrm{M}(h/x)^2$.

The equation (2) shows that, if $\frac{1}{2}\mathrm{M}(k/x)^2$ do not affect the nth place of decimals, then, so long as $h \not> k$, the differences of the values of the function are proportional to the differences of the values of the argument, provided we do not tabulate beyond the nth place of decimals.

Take, for example, the table sampled in chap. xxi., where the numbers are entered to five and the logarithms to seven places. Suppose $x = 30000$; and let us inquire within what limits it would certainly be safe to apply the rule of proportional parts. We must have

$$\tfrac{1}{2} \times \cdot 4343(h/30000)^2 < 5/10^8,$$

if the interpolated logarithm is to be correct to the last figure, that is,

$$h < 3\sqrt{23 \cdot 04},$$
$$< 14.$$

It would therefore certainly be safe to apply the rule and interpolate to seven places the logarithms of all numbers lying between 30000 and 30014. This agrees with the fact that in the table the tabular difference has the constant value 144 within, and indeed beyond, the limits mentioned.

SUMMATION OF SERIES BY MEANS OF THE LOGARITHMIC SERIES.

§ 14.] A great variety of series may, of course, be summed by means of the Logarithmic Series. Of the simple power series that can be so summed many are included directly or indirectly under the following theorem, which stands in the same relation

* For sources of information, see Glaisher, *l.c.*

to the logarithmic theorem as do the theorems of chap. xxvii., § 5, and chap. xxviii., § 8, to the binomial and exponential theorems:—

The series whose general term is $\phi(n)x^n/(n+a)(n+b)\ldots(n+k)$, *where* $\phi(n)$ *is an integral function of* n, *and* $a, b, \ldots, k$ *are positive or negative* unequal integers, can always be summed to infinity provided the series is convergent.*

It can easily be shown that the series is convergent provided x be numerically less than unity, and divergent if x be numerically greater than unity.

If the degree of $\phi(n)$ be greater than the degree of $(n+a)(n+b)\ldots(n+k)$, the general term can be split into

$$\psi(n)x^n + \chi(n)x^n/(n+a)(n+b)\ldots(n+k) \qquad (1),$$

where $\psi(n)$ and $\chi(n)$ are integral functions of n, the degree of the latter being less than the degree of $(n+a)(n+b)\ldots(n+k)$.

Now $\Sigma\psi(n)x^n$ is an integro-geometric series, and can be summed by the method of chap. xx., § 13.

By the method of Partial Fractions (chap. viii.) we can express $\chi(n)/(n+a)(n+b)\ldots(n+k)$ in the form

$$A/(n+a) + B/(n+b) + \ldots + K/(n+k),$$

where A, B, . . ., K are independent of n. Hence the second part of (1) can be split up into

$$Ax^n/(n+a) + Bx^n/(n+b) + \ldots + Kx^n/(n+k) \qquad (2);$$

and we have merely to sum the series

$$A\Sigma x^n/(n+a), \quad B\Sigma x^n/(n+b), \quad \ldots, \quad K\Sigma x^n/(n+k) \quad (3).$$

Now, supposing, for simplicity of illustration, that the summation extends from $n=1$ to $n=\infty$, we have

$$\begin{aligned} A\sum_1^{\infty} x^n/(n+a) &= Ax^{-a}\sum_1^{\infty} x^{n+a}/(n+a), \\ &= -Ax^{-a}\{x/1 + x^2/2 + \ldots + x^a/a + \log(1-x)\} \quad (4). \end{aligned}$$

Each of the other series (3) may be summed in like manner. Hence the summation can be completely effected.

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* When any of the integers $a, b, \ldots, k$ are negative, the method requires the evaluation of limits in certain cases.

If $x = 1$, the series under consideration will not be convergent unless the degree of $\phi(n)$ be less than the degree of $(n+a)(n+b)\ldots(n+k)$. It will be absolutely convergent if the degree of $\phi(n)$ be less than that of $(n+a)(n+b)\ldots(n+k)$ by two units. If the degree of $\phi(n)$ be less than that of $(n+a)(n+b)\ldots(n+k)$ by only one unit, then the series is semi-convergent if the terms ultimately alternate in sign, and divergent if they have ultimately all the same sign.

In all cases, however, where the series is convergent we can, by Abel's Theorem, find the sum for $x = 1$ by first summing for $x < 1$, and then taking the limit of this sum when $x = 1$.

In the special case where $\phi(n)$ is lower in degree by two units than $(n+a)(n+b)\ldots(n+k)$, and $a, b, \ldots, k$ are all positive, an elegant general form can be given for $\sum_1^\infty \phi(n)/(n+a)(n+b)\ldots(n+k)$.

From the identity

$$\phi(n)/(n+a)(n+b)\ldots(n+k) \equiv A/(n+a) + B/(n+b) + \ldots + K/(n+k),$$

we have

$$\phi(n) \equiv A(n+b)(n+c)\ldots(n+k) + B(n+a)(n+c)\ldots(n+k) + \ldots + K(n+a)(n+b)\ldots(n+j) \quad (5),$$

and, bearing in mind the degree of $\phi(n)$, we have

$$A + B + \ldots + K = 0 \quad (6).$$

Also, putting in succession $n = -a, n = -b, \ldots, n = -k$, we have

$$\left.\begin{aligned} A &= \phi(-a)/(b-a)(c-a)\ldots(k-a) \\ B &= \phi(-b)/(a-b)(c-b)\ldots(k-b) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K &= \phi(-k)/(a-k)(b-k)\ldots(j-k) \end{aligned}\right\} \quad (7).$$

Reverting to the general result, we see from (4) that

$$\sum_1^\infty \phi(n)x^n/(n+a)(n+b)\ldots(n+k) = -\Sigma A x^{-a}(x/1 + x^2/2 + \ldots + x^a/a) - \log(1-x).\Sigma A x^{-a} \quad (8),$$

where the Σ on the right hand indicates summation with respect to $a, b, \ldots, k$.

Now, since $A+B+\ldots+K=0$, ΣAx^{-a} is an algebraical function of x which vanishes when $x=1$. Also $1-x$ is an algebraical function of x having the same property. Therefore, by chap. XXV., § 17, we have

$$\begin{aligned}\operatorname*{L}_{x=1}\log(1-x).\Sigma Ax^{-a} &= \operatorname*{L}_{x=1}\log\left\{(1-x)^{\Sigma Ax^{-a}}\right\},\\ &= \log 1,\\ &= 0.\end{aligned}$$

Hence, taking the limit on both sides of (8), we have, by Abel's Theorem,

$$\begin{aligned}\sum_1^\infty \phi(n)/(n+a)(n+b)\ldots(n+k) &= -\Sigma A(1/1+1/2+\ldots+1/a),\\ &= -\Sigma\frac{\phi(-a)(1/1+1/2+\ldots+1/a)}{(b-a)(c-a)\ldots(c-k)} \quad (9),\end{aligned}$$

the Σ on the right denoting summation with respect to $a, b, c, \ldots, k$.

Example 1. Evaluate $\sum_2^\infty n^3x^n/(n-1)(n+2)$.

We have $n^3x^n/(n-1)(n+2) \equiv (n-1)x^n + \frac{1}{3}x^n/(n-1) + \frac{8}{3}x^n/(n+2)$.

Now $$\sum_2^\infty (n-1)x^n = 1x^2+2x^3+3x^4+\ldots,$$

$$\begin{aligned}(1-x)^2\sum_2^\infty (n-1)x^n &= 1x^2+2x^3+3x^4+\ldots\\ &\quad -2.1x^3-2.2x^4-\ldots\\ &\quad +1x^4+\ldots,\\ &= x^2.\end{aligned}$$

Hence $$\sum_2^\infty (n-1)x^n = x^2/(1-x)^2.$$

Also $$\begin{aligned}\tfrac{1}{3}\sum_2^\infty x^n/(n-1) &= \tfrac{1}{3}x\sum_2^\infty x^{n-1}/(n-1),\\ &= -\tfrac{1}{3}x\log(1-x);\end{aligned}$$

$$\begin{aligned}\tfrac{8}{3}\sum_2^\infty x^n/(n+2) &= \tfrac{8}{3}x^{-2}\sum_2^\infty x^{n+2}/(n+2),\\ &= -\tfrac{8}{3}x^{-2}\{x/1+x^2/2+x^3/3+\log(1-x)\}.\end{aligned}$$

Hence the whole sum is

$$x^2/(1-x)^2 - \tfrac{8}{3}x^{-1} - \tfrac{4}{3} - \tfrac{8}{3}x - \tfrac{1}{3}(x+8x^{-2})\log(1-x).$$

Example 2. Evaluate $\sum_2^\infty 1/(n-1)(n+2)$.

By the same process as before, we find

$$\sum_2^\infty x^n/(n-1)(n+2) = \tfrac{1}{3}x^{-1} + \tfrac{1}{6} + \tfrac{1}{3}x + \tfrac{1}{3}(x^{-2}-x)\log(1-x).$$

Now, since $\underset{x=1}{\mathrm{L}}\,(1-x)^{x^{-2}-x}=1$ (chap. xxv., § 17), $\underset{x=1}{\mathrm{L}}\,(x^{-2}-x)\log(1-x)=0$.

Therefore $\overset{\infty}{\underset{2}{\Sigma}}1/(n-1)(n+2)=\frac{1}{3}+\frac{1}{8}+\frac{1}{15}=\frac{11}{18}$.

This result might be obtained in quite another way.

It happens that $\Sigma 1/(n-1)(n+2)$ can be summed to n terms. In fact, we have

$$1/(n-1)(n+2)\equiv\tfrac{1}{3}\{1/(n-1)-1/(n+2)\}.$$

Hence, since the series is now finite and commutation of terms therefore permissible,

$$\begin{aligned}3\overset{n}{\underset{2}{\Sigma}}1/(n-1)(n+2)=&\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}\\&-\frac{1}{4}-\ldots-\frac{1}{n-4}-\frac{1}{n-3}-\frac{1}{n-2}-\frac{1}{n-1}\\&-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2},\\=&\frac{1}{1}+\frac{1}{2}+\frac{1}{3}-\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}.\end{aligned}$$

Hence, taking the limit for $n=\infty$, we have

$$\overset{\infty}{\underset{2}{\Sigma}}=\frac{1}{3}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)=\frac{11}{18}.$$

Example 3. To sum the series

$$\left(\frac{1}{1}+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+\ldots \text{ ad } \infty.$$

(Lionnet, *Nouv. Ann.*, ser. ii., t. 18.)

Let the $(n+1)$th term be u_n, then, since $u_\infty=0$, association is permitted (see chapter xxvi., § 7), and we may write

$$\begin{aligned}u_n\equiv&\frac{1}{4n+1}+\frac{1}{4n+3}-\frac{1}{2n+2},\\\equiv&\frac{1}{4n+1}-\frac{1}{4n+2}+\frac{1}{4n+3}-\frac{1}{4n+4}+\frac{1}{4n+2}-\frac{1}{4n+4},\\\equiv&\left(\frac{1}{4n+1}-\frac{1}{4n+2}+\frac{1}{4n+3}-\frac{1}{4n+4}\right)+\frac{1}{2}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right),\\\equiv&v_n+w_n, \text{ say.}\end{aligned}$$

Now, as may be easily verified, v_n and w_n are rational functions of n, in which the denominator is higher in degree than the numerator by two units at least. Hence (chap. xxvi., § 6) Σv_n and Σw_n are absolutely convergent series. Therefore (chap. xxvi., § 13)

$$\begin{aligned}\overset{\infty}{\underset{0}{\Sigma}}u_n=&\overset{\infty}{\underset{0}{\Sigma}}(v_n+w_n),\\=&\overset{\infty}{\underset{0}{\Sigma}}v_n+\overset{\infty}{\underset{0}{\Sigma}}w_n.\end{aligned}$$

Hence, again dissociating v_n and w_n (as is evidently permissible) we have

$$\sum_0^\infty u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$$
$$+ \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots\right)$$
$$= \log_e 2 + \tfrac{1}{2}\log_e 2, \quad \text{by § 9 above,}$$
$$= \tfrac{3}{2}\log_e 2.$$

This example is an interesting specimen of the somewhat delicate operation of evaluating a semi-convergent series. The process may be described as consisting in the conversion of the semi-convergent into one or more absolutely convergent series, whose terms can be commutated with safety. It should be observed that the terms in the given series are merely those of the series $1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots$ written in a different order. We have thus a striking instance of the truth of Abel's remark that the sum of a semi-convergent series may be altered by commutating its terms.

APPLICATIONS TO INEQUALITY AND LIMIT THEOREMS.

§ 15.] The Exponential and Logarithmic Series may be applied with effect in establishing theorems regarding inequality. Thus, for example, the reader will find it a good exercise to deduce from the logarithmic expansion the theorem, already proved in chapter XXV., that, if x be positive, then

$$x - 1 > \log x > 1 - 1/x \qquad (1).$$

It will also be found that the use of the three fundamental series—Binomial, Exponential, and Logarithmic—greatly facilitates the evaluation of limits. Both these remarks will be best brought home to the reader by means of examples.

Example 1. Show that

$$\log\frac{n}{m-1} > \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} > \log\frac{n+1}{m}.$$

If we put $1 - 1/x = 1/m$, that is, $x = m/(m-1)$, in the second part of (1) above, and then replace m by $m+1$, $m+2$, . . ., n successively, we get

$$\log m - \log(m-1) > 1/m,$$
$$\log(m+1) - \log m > 1/(m+1),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\log n - \log(n-1) > 1/n.$$

Hence, by addition,

$$\log n - \log(m-1) > 1/m + 1/(m+1) + \ldots + 1/n \qquad (2).$$

Next, if we put $x-1=1/m$ in the first part of (1), and proceed as before, we get

$$\log(m+1)-\log m<1/m,$$
$$\log(m+2)-\log(m+1)<1/(m+1),$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\log(n+1)-\log n<1/n.$$

Hence

$$\log(n+1)-\log m<1/m+1/(m+1)+\ldots+1/n \qquad (3).$$

From (2) and (3),

$$\log\{n/(m-1)\}>1/m+1/(m+1)+\ldots+1/n>\log\{(n+1)/m\}.$$

Example 2. If p and q be constant integers, show that

$$\mathrm{L}_{m=\infty}\{1/m+1/(m+1)+\ldots+1/(pm+q)\}=\log p.$$

(Catalan, *Traité Élémentaire des Séries*, p. 58.)

Put $n=pm+q$ of last example, and we find that

$$\log\{(pm+q)/(m-1)\}>1/m+1/(m+1)+\ldots+1/(pm+q)>\log\{(pm+q+1)/m\}.$$

Now $$\mathrm{L}_{m=\infty}\log\{(pm+q)/(m-1)\}=\log p,$$

and $$\mathrm{L}_{m=\infty}\log\{(pm+q+1)/m\}=\log p.$$

Hence the theorem.

Example 3. Evaluate $\mathrm{L}(e^x-1)^2/\{x-\log(1+x)\}$ when $x=0$.

Since $$(e^x-1)^2=(x+\tfrac{1}{2}x^2+\ldots)^2=x^2(1+\tfrac{1}{2}x+\ldots)^2;$$
$$x-\log(1+x)=\tfrac{1}{2}x^2-\tfrac{1}{3}x^3+\ldots=\tfrac{1}{2}x^2(1-\tfrac{2}{3}x+\ldots).$$

Therefore

$$(e^x-1)^2/\{x-\log(1+x)\}=2(1+\tfrac{1}{2}x+\ldots)^2/(1-\tfrac{2}{3}x+\ldots).$$

Since the series with the brackets are both convergent, it follows at once that $$\mathrm{L}(e^x-1)^2/\{x-\log(1+x)\}=2.$$

Exercises XIII.

(1.) If

$$\mathrm{P}=1/31+1/3.31^3+1/5.31^5+\ldots,$$
$$\mathrm{Q}=1/49+1/3.49^3+1/5.49^5+\ldots,$$
$$\mathrm{R}=1/161+1/3.161^3+1/5.161^5+\ldots,$$

then

$$\log 2=2(7\mathrm{P}+5\mathrm{Q}+3\mathrm{R}),$$
$$\log 3=2(11\mathrm{P}+8\mathrm{Q}+5\mathrm{R}),$$
$$\log 5=2(16\mathrm{P}+12\mathrm{Q}+7\mathrm{R}).$$

(See Glaisher, Art. "Logarithms," *Ency. Brit.*, 9th ed.)

(2.) If $a=-\log(1-1/10)$, $b=-\log(1-4/100)$, $c=\log(1+1/80)$, $d=-\log(1-2/100)$, $e=\log(1+8/1000)$, then $\log 2=7a-2b+3c$, $\log 3=11a-3b+5c$, $\log 5=16a-4b+7c$, $\log 7=\tfrac{1}{2}(39a-10b+17c-d)=19a-4b+8c+e$.

(Prof. J. C. Adams, *Proc.* R.S.L.; 1878.)

(3.) Calculate the logarithms of 2, 3, 5, 7 to ten places, by means of the formulæ of Example 1, or of Example 2.

(4.) Find the smallest integral value of x for which $(1{\cdot}01)^x>10x$.

Sum the series :—

(5.) $2^1/1!x^3 - 3x^1/1! + 2^2/2!3x^3 - 3x^2/2! + \ldots$

(6.) $1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \ldots$

(7.) $x^1/1.2 - x^2/2.3 + x^3/3.4 - \ldots (-)^{n-1}x^n/n(n+1)\ldots$

(8.) $x^2/3 + x^4/15 + \ldots + x^{2n}/(4n^2-1) + \ldots$

(9.) $x/1^2 + x^2/(1^2+2^2) + x^3/(1^2+2^2+3^2) + \ldots + x^n/(1^2+2^2+\ldots+n^2) + \ldots$; also $1/1^2 + 1/(1^2+2^2) + 1/(1^2+2^2+3^2) + \ldots + 1/(1^2+2^2+\ldots+n^2) + \ldots$

(10.) $4/1.2.3 + 6/2.3.4 + 8/3.4.5 + \ldots$

(11.) If $x>100$, then, to seven places of decimals at least, $\log(x+8) = 2\log(x+7) - \log(x+5) - \log(x+3) + 2\log x - \log(x-3) - \log(x-5) + 2\log(x-7) - \log(x-8)$.

(12.) Expand $\log(1+x+x^2)$ in ascending powers of x.

(13.) From $\log(x^3+1) \equiv \log(x+1) + \log(x^2-x+1)$, show that, if m be a positive integer, then

$$1 - \frac{6m-2}{2!} + \frac{(6m-3)(6m-4)}{3!} - \frac{(6m-4)(6m-5)(6m-6)}{4!} + \ldots = 0.$$

(Math. Trip., 1882.)

(14.) $\{\log_e(1+x)\}^2 = 2x^2/2 - 2(1/1+1/2)x^3/3 + \ldots (-)^n 2\{1/1+1/2+\ldots 1/(n-1)\}x^n/n \ldots$ Does this formula hold when $x=1$?

(15.) $\log(1+x)^{\log(1-x)} = -Q_1x^2/1 - Q_3x^4/2 - \ldots - Q_{2n-1}x^{2n}/n - \ldots$; where $Q_{2n-1} = 1/1 - 1/2 + 1/3 - \ldots + 1/(2n-1)$.

(16.) If $x<1$, show that

$x + \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{8}x^{16} \ldots = \log\{1/(1-x)\} - \frac{1}{3}P_3 - \frac{1}{5}P_5 + \frac{1}{6}P_6 - \frac{1}{7}P_7 - \frac{1}{9}P_9 + \frac{1}{10}P_{10} \ldots$; where $P_n = x^n + x^{2n} + x^{4n} + x^{8n} + x^{16n} + \ldots$, and the general term is $(-)^n P_n/n$, unless n is a power of 2, in which case there is no term.

(Trin. Coll., Camb., 1878.)

(17.) If $e^x \times e^{x^3/3} \times e^{x^5/5} \ldots = A_0 + A_1x + \ldots$, then $A_{2r} = A_{2r+1} = 1.3.5\ldots(2r-1)/2.4.6\ldots 2r$.

(18.) If $x + a_3x^3 + a_5x^5 + \ldots + y + a_3y^3 + a_5y^5 + \ldots = \{(x+y)/(1-xy)\}^1 + a_3\{(x+y)/(1-xy)\}^3 + a_5\{(x+y)/(1-xy)\}^5 + \ldots$, for all values of x and y which render the various series convergent, find a_3, a_5, ...

Show that

(19.) $\log(4/e) = 1/1.2 - 1/2.3 + 1/3.4 - 1/4.5 + \ldots$

(20.) $\log 2 = 4(1/1.2.3 + 1/5.6.7 + 1/9.10.11 + 1/13.14.15 + \ldots)$ (Euler.)

(21.) $(1 - 1/2 - 1/4) + (1/3 - 1/6 - 1/8) + (1/5 - 1/10 - 1/12) + \ldots = \frac{1}{2}\log 2$.

(See Lionnet, *Nouv. Ann.*, ser. ii., t. 18.)

(22.) $\sigma_1/1! - n\sigma_2/2! + n(n-1)\sigma_3/3! - \ldots$ to $\overline{n+1}$ terms $= 1/(n+1)^2$, where $\sigma_r = 1/1 + 1/2 + 1/3 + \ldots + 1/r$. (Math. Trip., 1888.)

(23.) $e \sim (1+1/m)^m$ lies between $e/(2m+1)$ and $e/(2m+2)$, whatever m may be. (*Nouv. Ann.*, ser. ii., t. 11.)

(24.) $\mathrm{L}\{x/(x-1) - 1/\log x\} = \frac{1}{2}$, when $x=1$. (Euler, *Inst. Calc. Diff.*)

(25.) $\mathrm{L}\{e^x - 1 - \log(1+x)\}/x^2 = 1$, when $x=0$. (Euler, *l.c.*)

(26.) $\mathrm{L}\{x^x - x\}/(1 - x + \log x) = -2$, when $x=1$. (Euler, *l.c.*)

(27.) $\mathrm{L}(1+1/n)^{1/n}(1+2/n)^{1/n}\ldots(1+n/n)^{1/n}=4/e$, when $n=\infty$.

(28.) $\mathrm{L}\{(2n-1)!/n^{2n-1}\}^{1/n}=4/e^2$, when $n=\infty$.

(29.) $e^x>1+x$, for all real values of x.

(30.) $x-1>\log x>1-1/x$, for all positive values of x; to be deduced from the logarithmic expansion.

(31.) $e^n>(1+n)^n/n!$, n being any integer.

(32.) If n be an integer $>e$, then $n^{n+1}>(n+1)^n$.

(33.) If A, B, a, b be all positive, then $(a-b)/(\mathrm{A}-\mathrm{B})+(\mathrm{A}a-\mathrm{B}b)\log(\mathrm{B}/\mathrm{A})(\mathrm{A}-\mathrm{B})^2$ is negative. (Tait.)

(34.) If $x>y>a$, then $\{(x+a)/(x-a)\}^x<\{(y+a)/(y-a)\}^y$.

(35.) $\mathrm{L}\{1/(n+1)+1/(n+2)+\ldots+1/2n\}=\log 2$, when $n=\infty$. (Catalan.)

(36.) $\log\{(n+\frac{1}{2})/(n-\frac{1}{2})\}>1/m+1/(m+1)+\ldots+1/n>\log\{(n+1)/m\}$. (Bourguet, *Nouv. Ann.*, ser. ii., t. 18.)

(37.) $\log 3=5/1.2.3+14/4.5.6+\ldots+(9n-4)/(3n-2)(3n-1)3n+\ldots$

(38.) If $\sum_1^\infty(-)^{n-1}\phi(n)/(n+a)(n+b)\ldots(n+k)$, where a, b, $\ldots$, k are all positive integers and $\phi(n)$ is an integral function of n, be absolutely convergent, its sum is

$$\mathrm{S}=\sum_{a,\,b,\,\ldots,\,k}\phi(-a)\{1/a-1/(a-1)\ldots(-)^{a-1}1/1\}/(b-a)(c-a)\ldots(k-a);$$

and, if it be semi-convergent, its sum is

$$\mathrm{S}+\log 2\sum_{a,\,b,\,\ldots,\,k}(-)^a\phi(-a)/(b-a)(c-a)\ldots(k-a).$$

(39.) Show that the residue in the expansion of $\log\{1/(1-x)\}$ lies between

$$x^{n+1}\{1+(n+1)x/(n+2)\}/(n+1)$$

and

$$x^{n+1}\{1+(n+1)x/(1-x)(n+2)\}/(n+1).$$

(40.) In a table of Briggian Logarithms in which the numbers are entered to 5 significant figures, and the mantissæ of the logarithms to 7 figures. Calculate the tabular difference of the logarithms when the number is near 30000; and find through what extent of the table it will remain constant.

CHAPTER XXIX.

Summation of the Fundamental Power Series for Complex Values of the Variable.

GENERALISATION OF THE ELEMENTARY TRANSCENDENTAL FUNCTIONS.

§ 1.] One of the objects of the present chapter is to generalise certain expansion theorems established in the two chapters which precede. In doing this, we are led to extend the definitions of certain functions such as a^x, $\log_a x$, $\cos x$, &c., already introduced, but hitherto defined only for real values of the variable x; and to introduce certain new functions analogous to the circular functions.

Seeing that the circular functions play an important part in what follows, it will be convenient here to recapitulate their leading properties. This is the more necessary, because it is not uncommon in English elementary courses so to define and discuss these functions that their general functional character is lost or greatly obscured.

§ 2.] *Definition and Properties of the Direct Circular Functions.* Taking, as in chap. xii., Fig. 1, a system of rectangular axes, we can represent any real algebraical quantity θ, by causing a radius vector OP of length r to rotate from OX through an angle containing θ radians, counter-clockwise if θ be a positive, clockwise if it be a negative quantity. If (x, y) be the algebraical values of the co-ordinates of P, any point on the radius vector of θ, then x/r, y/r, y/x, x/y, r/x, r/y are obviously all functions of θ, and of θ alone. The functions thus geometrically defined are called $\cos\theta$, $\sin\theta$,

$\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ respectively, and are spoken of collectively as the circular functions.

All the circular functions of one and the same argument, θ, are algebraically expressible in terms of one another, for their definition leads immediately to the equations

$$\left.\begin{array}{l} \tan\theta = \sin\theta/\cos\theta, \quad \cot\theta = \cos\theta/\sin\theta\,; \\ \sec\theta = 1/\cos\theta, \quad \operatorname{cosec}\theta = 1/\sin\theta\,; \\ \cos^2\theta + \sin^2\theta = 1, \quad \sec^2\theta - \tan^2\theta = 1\,; \end{array}\right\} \qquad (1);$$

from which it is easy to deduce an expression for any one of the six, $\cos\theta$, $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$, in terms of any other.

When $F(\theta)$ is such a function of θ that $F(-\theta) = F(\theta)$, it is said to be an even function of θ; and, when it is such that $F(-\theta) = -F(\theta)$, it is said to be an odd function of θ. For example, $1 + \theta^2$ is an even, and $\theta - \frac{1}{3}\theta^3$ is an odd function of θ.

It is easily seen from the definition of the circular functions that $\cos\theta$ and $\sec\theta$ are even, and $\sin\theta$, $\tan\theta$, $\cot\theta$, and $\operatorname{cosec}\theta$ odd functions of θ.

When $F(\theta)$ is such that for all values of θ, $F(\theta + n\lambda) = F(\theta)$, where λ is constant, and n any integer positive or negative, then $F(\theta)$ is said to be a periodic function of θ having the period λ.

It is obvious that the graph of such a function would consist of a number of parallel strips identical with one another, like the sections of a wall paper; so that, if we knew a portion of the graph corresponding to all values of θ between a and $a + \lambda$, we could get all the rest by simply placing side by side with this an infinite number of repetitions of the same.

Since the addition of $\pm 2\pi$ to θ corresponds to the addition or subtraction of a whole revolution to or from the rotation of the radius vector, it is obvious that all the circular functions are periodic and have the period 2π. This is the smallest period, that is, *the* period *par excellence*, in the case of $\cos\theta$, $\sin\theta$, $\sec\theta$, $\operatorname{cosec}\theta$. It is easily seen, by studying the defining diagram, that $\tan\theta$ and $\cot\theta$ have the smaller period π. Thus we have

$$\left.\begin{array}{ll} \cos(\theta + 2n\pi) = \cos\theta, & \sin(\theta + 2n\pi) = \sin\theta, \\ \sec(\theta + 2n\pi) = \sec\theta, & \operatorname{cosec}(\theta + 2n\pi) = \operatorname{cosec}\theta, \\ \tan(\theta + n\pi) = \tan\theta, & \cot(\theta + n\pi) = \cot\theta. \end{array}\right\} \quad (2).$$

Besides these relations for whole periods, we have also the following for half and quarter periods :—

$$\left.\begin{array}{ll} \cos(\pi \pm \theta) = -\cos\theta, & \sin(\pi \pm \theta) = \mp\sin\theta; \\ \cos(\frac{1}{2}\pi \pm \theta) = \mp\sin\theta, & \sin(\frac{1}{2}\pi \pm \theta) = +\cos\theta; \\ \tan(\frac{1}{2}\pi \pm \theta) = \mp\cot\theta, & \cot(\frac{1}{2}\pi \pm \theta) = \mp\tan\theta; \end{array}\right\} \quad (3),$$

&c.,

all easily deducible from the definition.

We have the following table of zero, infinite, and turning values :—

θ	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π	&c.
$\cos\theta$	+ 1	0	− 1	0	+ 1	
$\sin\theta$	0	+ 1	0	− 1	0	
$\tan\theta$	0	∞	0	∞	0	&c.
$\cot\theta$	∞	0	∞	0	∞	
$\sec\theta$	+ 1	∞	− 1	∞	+ 1	
$\operatorname{cosec}\theta$	∞	+ 1	∞	− 1	∞	

(4),

which might of course be continued forwards and backwards by adding and subtracting whole periods.

Hence $\cos\theta$ has an infinite number of zero values corresponding to $\theta = \frac{1}{2}(2n + 1)\pi$, where n is any positive or negative integer; no infinite values; an infinite number of maxima and of minima values corresponding to $\theta = 2n\pi$ and $\theta = (2n + 1)\pi$ respectively; and is susceptible of all real algebraical values lying between − 1 and + 1.

Sin θ is of like character.

But $\tan\theta$ is of quite a different character. It has an infinite number of zero values corresponding to $\theta = n\pi$; an infinite number of infinite values corresponding to $\theta = \frac{1}{2}(2n + 1)\pi$; no turning values; and is susceptible of all real algebraical values between $-\infty$ and $+\infty$.

Cot θ is of like character.

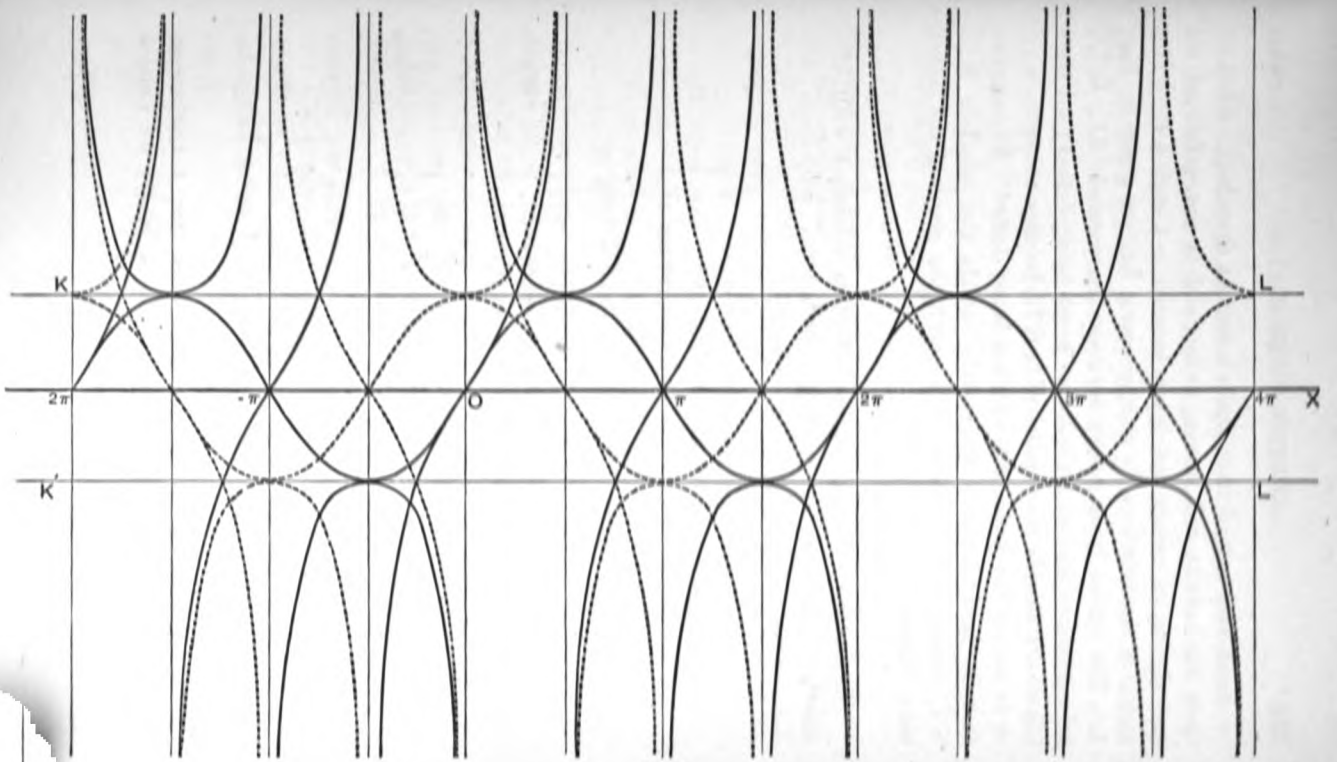

FIG. 1.

Sec θ and cosec θ have again a distinct character. Each of them has infinite and turning values, and is susceptible of all real algebraical values not lying between -1 and $+1$. The graphs of the functions $y = \sin x$, $y = \cos x$, &c., are given in Fig. 1. The curves lying wholly between the parallels KL, K'L', belong to $\cos x$ and $\sin x$, the cosine graph being dotted; all that lies wholly outside the parallels KL, K'L', belongs either to $\sec x$ or to $\operatorname{cosec} x$, the graph of the former being dotted. The curves that lie partly between and partly outside the parallels KL, K'L', belong either to $\tan x$ or to $\cot x$, the graph of the latter being dotted.

Again, from the geometrical definition combined with elementary considerations regarding orthogonal projection are deduced the following *Addition Formulæ*:—

$$\left.\begin{aligned} \cos(\theta \pm \phi) &= \cos\theta\cos\phi \mp \sin\theta\sin\phi; \\ \sin(\theta \pm \phi) &= \sin\theta\cos\phi \pm \cos\theta\sin\phi; \\ \tan(\theta \pm \phi) &= (\tan\theta \pm \tan\phi)/(1 \mp \tan\theta\tan\phi). \end{aligned}\right\} \quad (5).$$

As consequences of these, we have the following:—

$$\left.\begin{aligned} \cos\theta + \cos\phi &= 2\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi); \\ \cos\phi - \cos\theta &= 2\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi); \\ \sin\theta \pm \sin\phi &= 2\sin\tfrac{1}{2}(\theta\pm\phi)\cos\tfrac{1}{2}(\theta\mp\phi). \end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned} \cos\theta\cos\phi &= \tfrac{1}{2}\cos(\theta+\phi) + \tfrac{1}{2}\cos(\theta-\phi); \\ \sin\theta\sin\phi &= \tfrac{1}{2}\cos(\theta-\phi) - \tfrac{1}{2}\cos(\theta+\phi); \\ \sin\theta\cos\phi &= \tfrac{1}{2}\sin(\theta+\phi) + \tfrac{1}{2}\sin(\theta-\phi). \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\ &= (1 - \tan^2\theta)/(1 + \tan^2\theta). \\ \sin 2\theta &= 2\sin\theta\cos\theta = 2\tan\theta/(1+\tan^2\theta). \\ \tan 2\theta &= 2\tan\theta/(1-\tan^2\theta). \end{aligned}\right\} \quad (8).$$

§ 3.] *Inverse Circular Functions.* When, for a continuum (continuous stretch) of values of y, denoted by (y), we have a relation

$$x = \mathrm{F}(y) \quad (1),$$

which enables us to calculate a single value of x for each value of y, and the resulting values of x form a continuum (x), then the graph of $F(y)$ is continuous; and we can use it either to find x when y is given or y when x is given. We thus see that (1) not only determines x as a continuous function of y, but also y as a continuous function of x. The two functions are said to be inverse to each other; and it is usual to denote the latter function by $F^{-1}(x)$. So that the equation

$$y = F^{-1}(x) \qquad (2)$$

is identically equivalent to (1).

It must be noticed, however, that, although $F^{-1}(x)$ is continuous, it will not in general be single-valued, unless the values in the continuum (x) do not recur. This condition, as the student is already aware, is not fulfilled even in some of the simplest cases. Thus, for example, if $x = y^2$, for $-\infty < y < +\infty$, the continuum (x) is given by $0 \not> x < +\infty$; and each value of x occurs twice over. We have, in fact, $y = \pm x^{\frac{1}{2}}$; that is, the inverse function is two-valued.

It is also important to notice that, even when the direct function, $F(y)$, is completely defined for all real values of y, the inverse function, $F^{-1}(x)$, may not be completely defined for all values of x. $F^{-1}(x)$ is, in fact, defined by (1) solely for the values in the continuum (x). Take, for example, the relation $x = y^2$, for $-\infty < y < +\infty$. The continuum (x) is given by $0 \not> x < +\infty$; hence y is defined, by the above relation, as a function of x for values of x between 0 and $+\infty$ and for no others.

The application of the above ideas to the circular functions leads to some important remarks. It is obvious from the geometrical definition of $\sin y$ that the equation

$$x = \sin y \qquad (3)$$

completely defines x as a single-valued continuous function of y, for $-\infty < y < +\infty$. Hence, we may write

$$y = \sin^{-1} x \qquad (4),$$

where the inverse function, $\sin^{-1}x$,* is continuous, but neither single-valued, nor completely defined for all real values of x.

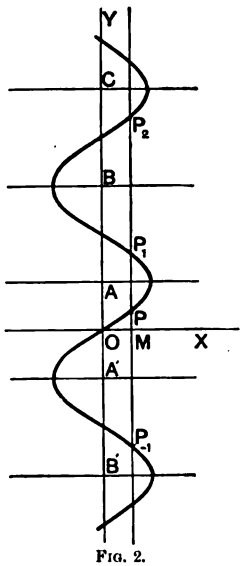

FIG. 2.

Since, by the properties of $\sin y$, x lies between -1 and $+1$ for all real values of y, $\sin^{-1}x$ is, in fact, defined by (3) only for values of x lying between -1 and $+1$. For other values of x the meaning of $\sin^{-1}x$ is at present arbitrary.

By looking graphically at the problem "to determine y for any value of x lying between -1 and $+1$," we see at once that $\sin^{-1}x$ is multiple-valued to an infinite extent.

If, however, we confine ourselves to values of $\sin^{-1}x$ lying between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$, we see at once from the graph (Fig. 2) that for any value of x lying between -1 and $+1$ there is one, and only one, value of $\sin^{-1}x$. If we draw parallels to the axis of x through the points A, B, C, . . ., A′, B′, . . ., whose ordinates are $+\frac{1}{2}\pi$, $+\frac{3}{2}\pi$, $+\frac{5}{2}\pi$, . . ., $-\frac{1}{2}\pi$, $-\frac{3}{2}\pi$, . . ., then between every pair of consecutive parallels we find, for a given value of x ($-1<x<+1$), one, and only one, value of $y=\sin^{-1}x$.

The values of y corresponding to points between the parallels A′ and A constitute what we may call the *Principal Branch* of the function. Similarly, the part of the graph between A and B represents the 1st positive branch; the part between B and C the 2nd positive branch; the part between A′ and B′ the 1st negative branch; and so on.

If, as is usual, we understand the symbol $\sin^{-1}x$ to give the value of y corresponding to x, for the principal branch only, and use y_n or ${}_n\sin^{-1}x$ for the nth branch, then it is easy to see that

$$y_n = {}_n\sin^{-1}x = n\pi + (-1)^n \sin^{-1}x \qquad (5),$$

* This may be read "angle whose sine is x" or "arc-sine x." In Continental works the latter name is contracted into arc-sin x; and this is used instead of $\sin^{-1}x$.

where n is a positive or negative integer according as the branch in question is positive or negative.

It is obviously to some extent arbitrary what portion of the graph shall be marked off as corresponding to the principal branch of the function; in other words, what part of the function shall be called the principal branch. But it is clearly necessary, if we are to avoid ambiguity—and this is the sole object of the present procedure—that no value of y should recur within the part selected; and, to secure completeness, all the different values of y should, if possible, be represented. Attending to these considerations, and drawing the corresponding figures, the reader will easily understand the reasons for the following conventions regarding $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\operatorname{cosec}^{-1}x$, wherein y and the inverse functional symbols $\cos^{-1}x$, &c., relate to the principal branch only, and y_n to the nth branch, positive or negative.

$$\left.\begin{aligned} y &= \cos^{-1}x, \quad y \text{ between } 0 \text{ and } +\pi; \\ y_n &= (n + \tfrac{1}{2} + (-)^{n-1}\tfrac{1}{2})\pi + (-)^n \cos^{-1}x. \end{aligned}\right\} \quad (6)$$

$$\left.\begin{aligned} y &= \tan^{-1}x, \quad y \text{ between } -\tfrac{1}{2}\pi \text{ and } +\tfrac{1}{2}\pi; \\ y_n &= n\pi + \tan^{-1}x. \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} y &= \cot^{-1}x, \quad y \text{ between } 0 \text{ and } \pi; \\ y_n &= n\pi + \cot^{-1}x. \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} y &= \sec^{-1}x, \quad y \text{ between } 0 \text{ and } \pi; \\ y_n &= (n + \tfrac{1}{2} + (-)^{n-1}\tfrac{1}{2})\pi + (-)^n \sec^{-1}x. \end{aligned}\right\} \quad (9)$$

$$\left.\begin{aligned} y &= \operatorname{cosec}^{-1}x, \quad y \text{ between } -\tfrac{1}{2}\pi \text{ and } +\tfrac{1}{2}\pi; \\ y_n &= n\pi + (-)^n \operatorname{cosec}^{-1}x. \end{aligned}\right\} \quad (10)$$

Since every function must, in practice, be unambiguously defined, it is necessary, in any particular case, to specify what branch of an inverse function is in question. If nothing is specified, it is understood that the principal branch alone is in question.

It is obvious that all the formulæ relating to direct circular functions could be translated into the notation of inverse circular functions. In this translation, however, close attention must be paid to the points just discussed. Thus

If x be positive, the formula $\cos\theta = \pm\sqrt{(1-\sin^2\theta)}$ becomes

$$\sin^{-1}x = \cos^{-1}\sqrt{(1-x^2)};$$

but, if x be negative, it becomes

$$\sin^{-1}x = -\cos^{-1}\sqrt{(1-x^2)}.$$

If $0<x<1/\sqrt{2}$, $0<y<1/\sqrt{2}$, we deduce from the addition formulæ for the direct functions

$$\sin^{-1}x+\sin^{-1}y = \cos^{-1}[\sqrt{\{(1-x^2)(1-y^2)\}}-xy];$$

if $0<x<1$, $0<y<1$,

$$\tan^{-1}x+\tan^{-1}y = \tan^{-1}[(x+y)/(1-xy)].$$

If x and y be both positive, but such that $xy>1$, then

$$\tan^{-1}x+\tan^{-1}y = \pi+\tan^{-1}[(x+y)/(1-xy)];*$$

and, in general, it is easy to show that

$$\begin{aligned} {}_m\tan^{-1}x+{}_n\tan^{-1}y &= (m+n+\rho)\pi+\tan^{-1}\{(x+y)/(1-xy)\}, \\ &= {}_{m+n+\rho}\tan^{-1}\{(x+y)/(1-xy)\} \quad (11), \end{aligned}$$

where $\rho=1$, 0, or -1, according as $\tan^{-1}x+\tan^{-1}y$ is greater than $\frac{1}{2}\pi$, lies between $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, or is less than $-\frac{1}{2}\pi$.

ON THE INVERSION OF $w=z^n$.

§ 4.] When the argument, and, consequently, in general, the value of the function are not restricted to be real, the discussion of the inverse function becomes more complicated, but the fundamental notions are the same.

For the present it will be sufficient to confine ourselves to the case of a binomial algebraical equation. Let us first consider the case

$$w=z^n \quad (1),$$

where n is a positive integer, z is a complex number, say $z=x+yi$, and, consequently, w also in general a complex number, say $w=u+vi$.

To attain absolute clearness in our discussion it will be

* In English Text-Books equations of this kind are often loosely stated; and the result has been some confusion in the higher branches of mathematics, such as the integral calculus, where these inverse functions play an important part.

necessary to pursue a little farther the graphical method of chap. xv., § 17.

It follows from what has there been laid down, and from the fact that any integral function of x and y is continuous for all finite values of x and y, that, if we form two Argand Diagrams, one for $x + yi$ (the z-plane), and one for $u + vi$ (the w-plane), then, whenever the graphic point of z* describes a continuous curve, the graphic point of w also describes a continuous curve. In this sense, therefore, the equation (1) defines w as a continuous function of z for all values, real or complex, of the latter. For simplicity in what follows we shall suppose the curve described by z to be the whole or part of a circle described about the origin of the z-plane. We shall also represent z by the standard form $r(\cos\theta + i\sin\theta)$, and w by the standard form $s(\cos\phi + i\sin\phi)$; but we shall, contrary to the practice followed in chap. xii., allow the amplitudes θ and ϕ to assume negative values. Thus, for example, if we wish to give z all values corresponding to a given modulus r, without repetition of the same value, we shall, in general, cause θ to vary continuously from $-\pi$ to $+\pi$, and not from 0 to 2π, as heretofore. In either way we get a complete single revolution of the graphic radius; and it happens that the plan now adopted is more convenient for our present purpose.

It is obvious that by varying the amplitude in this way, and then giving all different values to r from 0 to $+\infty$, we shall get every possible complex value of z, once over; and thus effect a complete exploration of any one-valued function of z.

Substituting in (1) the standard forms for w and z, and taking, for simplicity, $n = 3$, we have

$$\begin{aligned} s(\cos\phi + i\sin\phi) &= r^3(\cos\theta + i\sin\theta)^3 \\ &= r^3(\cos 3\theta + i\sin 3\theta) \qquad (2) \end{aligned}$$

by Demoivre's Theorem. Hence we deduce

$$s = r^3, \quad \phi = 3\theta + 2n\pi;$$

* For shortness, in future, instead "of graphic point of z" we shall say "z" simply.

or, if (as will be sufficient for our purpose) we confine ourselves to a single complete revolution of the graphic radius of z,

$$s = r^3, \quad \phi = 3\theta \qquad (3).$$

If, therefore, we give to r any particular value, s has the fixed value r^3; that is to say, w describes a circle about the origin of the w-plane (Fig. 4). Also, if we suppose z to describe its circle (Fig. 3) with uniform velocity, since $\phi = 3\theta$, w will describe the corresponding circle with a uniform velocity three times as great. To one complete revolution of z will therefore

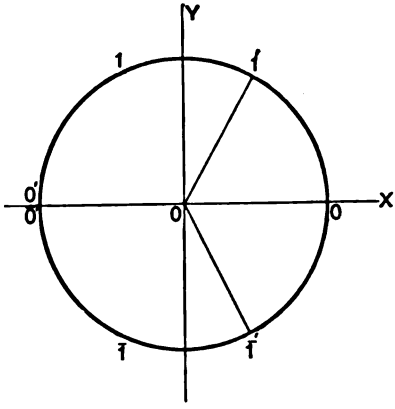

FIG. 3.

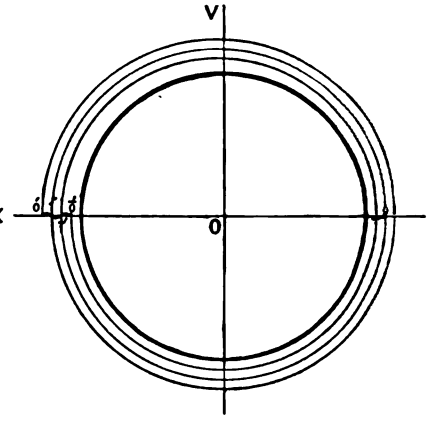

FIG. 4.

correspond three complete revolutions of w. In other words, the values in the (w)-continuum which correspond to those in the (z)-continuum *are each repeated three times over.**

The actual course of w is the circle of radius r^3 taken three times over. We may represent this multiple course of w by drawing round its actual circular course the spiral $\bar{0}'$, $\bar{1}$, $\bar{1}'$, 0, 1', 1, 0', which re-enters into itself at $\bar{0}'$ and 0'. The actual course may then be imagined to be what this spiral becomes when it is shrunk tight upon the circle.

* To indicate this peculiarity of w we shall occasionally use the term "Repeating Function." A repeating function need not, however, be periodic as $w = z^3$ is.

If we now letter the corresponding points on the z-circle with the same symbols we have the circle $\bar{0}'\bar{1}\bar{1}'$ in the w-plane, corresponding to the circular arc $\bar{0}'\bar{1}\bar{1}'$ in the z-plane, and so on, in this sense that, when z describes the arc $\bar{0}'\bar{1}\bar{1}'$, then w describes the complete circle $\bar{0}'\bar{1}\bar{1}'$, and so on.

It follows from this graphical discussion that *the equation* $w = z^3$, *which defines* w *as a one-valued continuous function of* z *for all values of* z, *defines* z *as a three-valued continuous function of* w *for all values of* w.

In other words, since, in accordance with a notation already defined, (1) may be written

$$z = \sqrt[3]{w} \qquad (1'),$$

we have shown that *the cube root of* w *is a three-valued continuous function of* w *for all values of* w.

It is obvious that there is nothing in the above reasoning peculiar to the case $n = 3$, except the fact that we have a triple spiral in the w-plane, and a trisected circumference in the z-plane. Hence, if we consider the equation

$$w = z^n \qquad (4),$$

and its equivalent inverse form

$$z = \sqrt[n]{w} \qquad (4'),$$

all the alteration necessary is to replace the triple by an n-ple spiral, returning into itself on the negative or positive part of the u-axis, according as n is odd or even; and the trisected circumference by a circumference divided into n equal parts.

Thus we see that *the equation* (4), *which defines* w *as a continuous one-valued function of* z *for all values of* z, *defines* z (*that is, the* n*th root of* w) *as a continuous* n*-valued function of* w *for all values of* w.

§ 5.] *Riemann's Surface.* It may be useful for those who are to pursue their mathematical studies beyond the elements, to illustrate, by means of the simple case $w = z^3$, a beautiful method for representing the continuous variation of a repeating function which was devised by the German mathematician Riemann, who ranks, along with Cauchy, as a founder of that branch of modern algebra whose fundamental conceptions we are now explaining.

Instead of supposing all the spires of the w-path in Fig. 4 to lie in one plane, we may conceive each complete spire to lie in a separate plane superposed on the w-plane. Instead of the single w-plane, we have thus three separate planes, $\bar{P}_1$, P_0, P_1, superposed upon each other. To secure continuity between the planes, each of them is supposed to be slit along the u-axis from 0 to $-\infty$; and the three joined together, so that the upper edge of the slit in P_0 is joined to the lower edge of the slit in $\bar{P}_1$; the lower edge of the slit in P_0 to the upper edge of the slit in P_1; the lower edge of the slit in P_1 to the upper edge of the slit in $\bar{P}_1$, this last junction taking place across the two intervening, now continuous, leaves. We have thus clothed the whole of the w-plane with a three-leaved continuous flat helicoidal * surface, any continuous path on which must, if it circulates about the origin at all, do so three times before it can return into itself. This surface is called a *Riemann's Surface.* The origin, about which the surface winds three times before returning into itself, is called a *Winding Point of the Third Order.* Upon this three-leaved surface w will describe a continuous single path corresponding to any continuous single path of z, provided we suppose that there is no continuity between the leaves except at the junctions above described.

§ 6.] If we confine θ to that part $\bar{1}'01'$ of its circle which is bisected by OX, and ϕ to the corresponding spire $\bar{1}'01'$ of its path, so that ϕ lies between $-\pi$ and $+\pi$, and θ between $-\pi/n$ and $+\pi/n$, then z becomes a one-valued function of w for all values of w. We call this the principal branch of the n-valued function $\sqrt[n]{w}$; and, as we have the distinct notation $w^{1/n}$ at our disposal, we may restrict it to denote this particular branch of the function z. In other words, if

$$w = s(\cos\phi + i\sin\phi), \qquad -\pi < \phi < +\pi,$$

we define $w^{1/n}$ by the equation

$$w^{1/n} = s^{1/n}(\cos.\,\phi/n + i\sin.\,\phi/n);$$

and we also restrict $(\cos\phi + i\sin\phi)^{1/n}$ to mean $\cos.\,\phi/n + i\sin.\,\phi/n$.

Just as in § 4, we take the next spire after $\bar{1}'01'$ in the positive direction (counter-clock) to represent the first positive branch of $\sqrt[n]{w}$; the next in the negative direction to represent the first negative branch of $\sqrt[n]{w}$; and so on, the last positive and the last negative being full spires, or only half spires, according as n is odd or even.

If, as is usual, we represent the actual analytical value of w

* Like a spiral staircase.

by the form $s(\cos\phi + i\sin\phi)$, where ϕ is always taken between $-\pi$ and $+\pi$, then it is easy to find expressions for the values of z, belonging to the $n-1$ positive and negative branches of $\sqrt[n]{w}$ and corresponding to any given value of w, in terms of the value belonging to the principal branch. We have, obviously, merely to add or subtract multiples of 2π to represent the successive positive and negative whole revolutions of the graphic radius of w. Thus, if z, z_t, z_{-t} relate to the principal, tth positive, and tth negative branches of $z = \sqrt[n]{w}$ respectively, we have

$$\left.\begin{aligned} z &= s^{1/n}\{\cos.\,\phi/n + i\sin.\,\phi/n\}\,; \\ z_t &= s^{1/n}\{\cos.\,(\phi+2t\pi)/n + i\sin.\,(\phi+2t\pi)/n\}\,; \\ z_{-t} &= s^{1/n}\{\cos.\,(\phi-2t\pi)/n + i\sin.\,(\phi-2t\pi)/n\}. \end{aligned}\right\} \qquad (5).$$

We have thus been led back by a purely graphical process to results equivalent to those already found in chap. xii., § 18.

Cor. 1. *Hence, if z denote the principal value of the nth root of w, and $\omega_n = \cos.\,2\pi/n + i\sin.\,2\pi/n$, then*

$$\left.\begin{aligned} z_t &= z\omega_n{}^t, & z_{-t} &= z\omega_n{}^{-t}\,; \\ \text{that is,}\quad z_t &= w^{1/n}\omega_n{}^t, & z_{-t} &= w^{1/n}\omega_n{}^{-t} \end{aligned}\right\} \qquad (6).$$

Cor. 2. *The principal value of the nth root of a positive real number r is the real positive nth root, that is, what has already been denoted by $r^{1/n}$* (see chap. x., § 2).

For, in this case, we have $w = r(\cos 0 + i\sin 0)$, that is, $\phi = 0$. Hence $w^{1/n} = r^{1/n}$.

Cor. 3. *There is continuity between the last values of any branch of $w^{1/n}$ and the first values of the next in succession, and between the last values of the last positive branch and the first values of the last negative branch; but elsewhere two values of $w^{1/n}$ belonging to different branches, and corresponding to the same value of w, differ by a finite amount.*

It should be noticed as a consequence of the above that the principal value of the nth root of a real negative number, such as -1, is not definite until its amplitude is assigned. For we may write $-1 = \cos\pi + i\sin\pi$ or $= \cos(-\pi) + i\sin(-\pi)$; and the principal value in the former case is $\cos.\,\pi/n + i\sin.\,\pi/n$, in the latter $\cos(-\pi/n) + i\sin(-\pi/n)$. This ambiguity does not exist for complex numbers differing from -1, even when they differ infinitely little, as will be at once seen by referring to Figs. 3 and 4.

§ 7.] It should be observed that if, instead of restricting ϕ in the expression $z = s^{1/n}\{\cos . \phi/n + i \sin . \phi/n\}$ to lie between $-\pi$ and $+\pi$, we cause it to vary continuously from $-n\pi$ to $+n\pi$, then $s^{1/n}\{\cos . \phi/n + i \sin . \phi/n\}$ varies continuously and passes once through every possible value of $\sqrt[n]{w}$ where mod w is given $= s$.

It follows also that, if w describe any continuous path starting from P and returning thereto, the value of $\sqrt[n]{w}$ will vary continuously; and will return to its original value, if w have circulated round the origin of the w-plane pn times, where p is 0 or any integer; and, in general, will return to its original value multiplied by $\omega_n{}^t$, where t is the algebraical value of $+\omega - \nu$, ω and ν being the number of times that w has circulated round the origin in the positive and negative directions respectively.

§ 8.] Let us now consider briefly the equation

$$w^p = z^q \qquad (1),$$

where p and q are positive integers. We shall suppose p and q to be prime to each other, because that is the only case with which we shall hereafter be concerned.*

Our symbols having the same meanings as before, we derive from (1)

$$s^p(\cos p\phi + i \sin p\phi) = r^q(\cos q\theta + i \sin q\theta) \qquad (2).$$

Hence, taking the simplest correspondence that will give a complete view of the variation of both sides of the equation last written, we have

$$s^p = r^q, \quad p\phi = q\theta \qquad (3).$$

If, then, we fix r, and therefore s, the paths of z and w will be circles about the origins of the z- and w-planes respectively; and, since p is prime to q, if z and w start from the positive part

* If p and q had the G.C.M. k, so that $p = kp'$, $q = kq'$, where p' and q' are mutually prime, then the equation (1) could be written $(w^{p'})^k = (z^{q'})^k$, which is equivalent to the k equations, $w^{p'} = z^{q'}$, $w^{p'} = \omega_k z^{q'}$, $w^{p'} = \omega_k{}^2 z^{q'}$, . . ., $w^{p'} = \omega_k{}^{k-1} z^{q'}$, where ω_k is a primitive kth root of $+1$. Each of these k equations falls under the case above discussed.

of the x- and u-axes simultaneously, they will not again be simultaneously at the starting place before z has made p, and w has made q revolutions.

To get a complete representation of the variation we must therefore cause θ to vary from $-p\pi$ to $+p\pi$, and ϕ from $-q\pi$ to $+q\pi$. The graphs of z and w will therefore be spirals having p and q spires respectively. To each whole spire of the q-spiral will correspond the p/qth part of the p-spiral. The case where $p = 3$ and $q = 4$ is illustrated by Figs. 5 and 6.

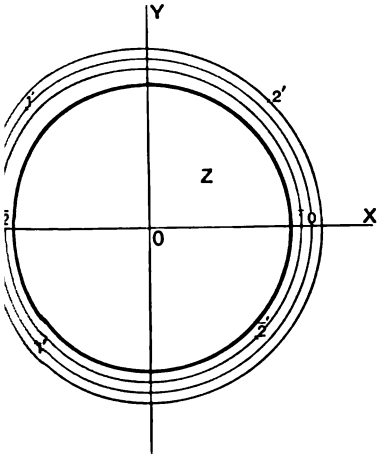

FIG. 5.

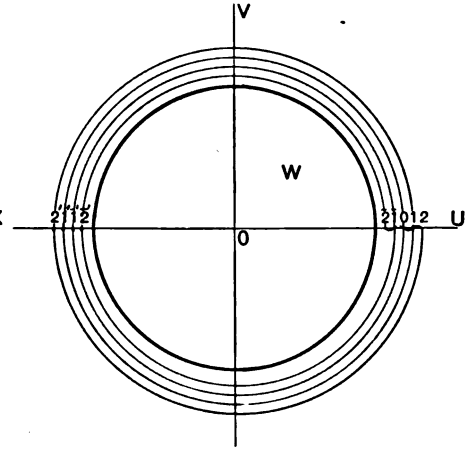

FIG. 6.

It follows, therefore, that *the equation* (1) *determines* w *as a continuous* p*-valued function of* z, *and* z *as a continuous* q*-valued function of* w. Taking the latter view, and writing (1) in the form

$$z = \sqrt[q]{w^p} \qquad (1'),$$

and (3) in the form

$$r = s^{p/q}, \quad \theta = p\phi/q \qquad (3'),$$

we see that, if we cause ϕ to vary continuously from $-q\pi$ to $+q\pi$, then $s^{p/q}(\cos\frac{p}{q}\phi + i\sin\frac{p}{q}\phi)$ will vary continuously through all the values which $\sqrt[q]{w^p}$ can assume so long as $\operatorname{mod} w = s$, and will return to the same value from which it started. In fact, we

see in general that, if w start from any point and return to the same point again after circulating ω times round the origin in the positive direction, and ν times in the negative direction, then $\sqrt[q]{w^p}$ returns to its original value multiplied by $\cos.2pt\pi/q + i\sin.2pt\pi/q$ where $t = +\omega - \nu$; that is, by $\omega_q{}^{pt}$, where ω_q denotes a primitive qth root of $+1$.

If, as usual, we divide up the circular graph of w into whole spires, counting forwards and backwards as before, and consider the separate branches of the function $\sqrt[q]{w^p}$ corresponding to these, then each of these branches is a single-valued function of θ.

The spire corresponding to $-\pi < \phi < +\pi$ is taken as the *principal spire*, and corresponding thereto we have the principal branch of the function $z = \sqrt[q]{w^p}$, namely,

$$z = s^{p/q}\left\{\cos\frac{p}{q}\phi + i\sin\frac{p}{q}\phi\right\}, \qquad -\pi < \phi < +\pi.$$

For the $(+t)$th and $(-t)$th branches respectively, we have

$$\begin{aligned} z_t &= s^{p/q}\{\cos.p(\phi + 2t\pi)/q + i\sin.p(\phi + 2t\pi)/q\}, \\ &= \omega_q{}^{pt}z\,; \\ z_{-t} &= s^{p/q}\{\cos.p(\phi - 2t\pi)/q + i\sin.p(\phi - 2t\pi)/q\}, \\ &= \omega_q{}^{-pt}z. \end{aligned}$$

As before, we may use $w^{p/q}$ to stand for the principal branch of $\sqrt[q]{w^p}$, and we observe, as before, that the principal value of $\sqrt[q]{w^p}$ when w is a real positive quantity is the real positive value of the qth root, that is, what we have, in chap. x., denoted by $w^{p/q}$.

§ 9.] It must be observed that, when p is not prime to q, the expressions $s^{p/q}\{\cos.p(\phi\pm2t\pi)/q + i\sin.p(\phi\pm2t\pi)/q\}$ no longer furnish all the q values of $\sqrt[q]{w^p}$, but (as may be easily verified) only q/k of them, where k is the G.C.M. of p and q. The appropriate expression in this case would be $s^{p/q}\{\cos.(p\phi\pm2t\pi)/q + i\sin.(p\phi\pm2t\pi)/q\}$.

This last expression gives in all cases the q different values of $\sqrt[q]{w^p}$; but it has this great inconvenience, that, if we arrange the branches by taking successively $t=0$, $t=1$, $t=2$, . . ., the end value of each branch is equal, not to the initial value of the succeeding branch, but to the initial value of a branch several orders farther on. There will therefore be more than one crossing in the graphic spiral. The investigation from this point of view will

be a good exercise for the student. When p is prime to q, the two expressions for $\sqrt[q]{w^p}$ are equivalent; and we have preferred to use the one which leads to the simpler graphic spiral.

If we adopt Riemann's method for the graphical representation of the equation $w^p = z^q$, then we shall have to cover the z-plane with a p-leaved Riemann's surface, having at the origin a winding point of the pth order; and the w-plane with a q-leaved surface, having at the origin a winding point of the qth order.

EXERCISES XIV.

(1.) Solve the equation

$$\tan^{-1}\{(x+1)/(x-1)\} + \tan^{-1}\{(x+2)/(x-2)\} = \tfrac{1}{4}\pi,$$

and examine whether the solutions obtained really satisfy the equation when $\tan^{-1}$ denotes the principal branch of the inverse function.

(2.) If $27r^2 < 4q^3$, show that the roots of the equation $x^3 - qx - r = 0$ are $2(q/3)^{1/2}\cos\alpha$, $2(q/3)^{1/2}\cos(\frac{2}{3}\pi + \alpha)$, $2(q/3)^{1/2}\cos(\frac{2}{3}\pi - \alpha)$, where α is determined by the equation $\cos 3\alpha = \frac{1}{2}r(3/q)^{3/2}$.

Show that the solution of any cubic equation, whose roots are all real, can be effected in this way; and work out the roots of $x^3 - 5x + 3 = 0$ to six places of decimals. (See Lock's *Higher Trigonometry*, § 135, or Todhunter's *Trigonometry*, 7th ed., § 260.)

Trace the graphs of the following, x being a real argument:—

(3.) $y = \sin x + \sin 2x$.
(4.) $y = \sin x + \cos 2x$.
(5.) $y = \sin x \sin 2x$.
(6.) $y = \tan x + \tan 2x$.
(7.) $y = x \sin x$.
(8.) $y = \sin x/x$.
(9.) $y = \sin 3x/\cos x$.
(10.) $y = \sin^{-1} x^2$.
(11.) $y^2 = \sin^{-1} x$.
(12.) $\sin y = \tan x$.

Discuss graphically the following functional equations connecting the complex variables w and z. In particular, trace in each case the w-paths when the z-paths are circles about the origin of the z-plane, or parallels to the real and to the imaginary axis.

(13.) $w^2 = z^2$.
(14.) $w = 1/z$.
(15.) $w = 1/z^2$.
(16.) $w^2 = 1/z^3$.
(17.) $w^2 = (z-a)(z-b)$.
(18.) $w^2 = (z-a)^2(z-b)$.
(19.) $w^3 = (z-a)^2$.
(20.) $w^2 = (z-a)^3$.
(21.) $w = (az+b)/(cz+d)$.
(22.) $w^2 = 1/(z-a)(z-b)$.

§ 10.] We can now extend to their utmost generality some of the theorems regarding the summation of series already established in previous chapters.

It is important to remark that the peculiar difficulties of this

part of the subject do not arise where we have to deal merely with a finite summation; that is to say, the summation of a series to n terms. For any such summation involves merely a statement of the identity of two chains of operations, each containing a finite number of links, and any such identity rests directly on the fundamental laws of algebra, which apply alike to real and to complex quantities.

Even when the series is infinite, provided it be convergent, and its sum be a one-valued function, the difficulty is merely one that has already been fully settled in chap. xxvi.

The fresh difficulty arises when the sum depends upon a multiple-valued function. We have then to determine which branch of the function represents the series; for the series, by its nature, is always one-valued.

We commence with some cases where the last-mentioned point does not arise.

GEOMETRIC AND INTEGRO-GEOMETRIC SERIES.

§ 11.] The summation

$$1 + z + z^2 + \ldots + z^n = (1 - z^{n+1})/(1 - z) \qquad (1),$$

since it depends merely on a finite identity, holds for all values of z. We may therefore suppose that $z = x + yi = r(\cos\theta + i\sin\theta)$, and the equation (1) will still hold.

Also, since $\underset{n=\infty}{\mathrm{L}}\ z^{n+1} = \mathrm{L}r^{n+1}(\cos\overline{n+1}\theta + i\sin\overline{n+1}\theta) = 0$,

when $r < 1$, we have, provided $\mathrm{mod}\, z < 1$, the infinite summation

$$1 + z + z^2 + \ldots \text{ ad } \infty = 1/(1 - z) \qquad (2)$$

for complex as well as for real values of z.

In like manner, the finite summation of the integro-geometric series $\overset{n}{\Sigma}\phi(n)z^n$, which we have seen can always be effected for real values of z (see chap. xx., § 14), holds good for all values of z; and, since $\overset{\infty}{\Sigma}\phi(n)z^n$ is convergent provided $\mathrm{mod}\, z < 1$, the infinite summation deducible from the finite one will hold good for all complex values of z such that $\mathrm{mod}\, z < 1$.

By substituting in (1) or (2), and in the corresponding equations for $\Sigma\phi(n)z^n$, the value $r(\cos\theta + i\sin\theta)$ for z, and then equating the real and imaginary parts on both sides, we can deduce a large number of summations of series involving circular functions of multiples of θ.

Example 1. To sum the series

$$S_n = 1 + r\cos\theta + r^2\cos 2\theta + \ldots + r^n\cos n\theta,$$
$$T_n = r\sin\theta + r^2\sin 2\theta + \ldots + r^n\sin n\theta,$$
$$U_n = \cos\alpha + r\cos(\alpha+\theta) + r^2\cos(\alpha+2\theta) + \ldots + r^n\cos(\alpha+n\theta),$$
$$V_n = \sin\alpha + r\sin(\alpha+\theta) + r^2\sin(\alpha+2\theta) + \ldots + r^n\sin(\alpha+n\theta),$$

to n terms; and to ∞ when $r<1$.

Starting with equation (1), let us put $z = r(\cos\theta + i\sin\theta)$, and equate real and imaginary parts on both sides. We find

$$1 + r(\cos\theta + i\sin\theta) + r^2(\cos 2\theta + i\sin 2\theta) + \ldots + r^n(\cos n\theta + i\sin n\theta)$$
$$= \{1 - r^{n+1}(\cos(n+1)\theta + i\sin(n+1)\theta)\}\{1 - r(\cos\theta + i\sin\theta)\} \quad (3);$$

whence *

$$S_n = \{1 - r\cos\theta - r^{n+1}\cos(n+1)\theta + r^{n+2}\cos n\theta\}/\{1 - 2r\cos\theta + r^2\} \quad (4);$$
$$T_n = \{r\sin\theta - r^{n+1}\sin(n+1)\theta + r^{n+2}\sin n\theta\}/\{1 - 2r\cos\theta + r^2\} \quad (5).$$

Again, since
$$U_n = \cos\alpha S_n - \sin\alpha T_n,$$
$$V_n = \sin\alpha S_n + \cos\alpha T_n,$$

we deduce from (4) and (5) the following:—

$$U_n = \{\cos\alpha - r\cos(\alpha-\theta) - r^{n+1}\cos(\overline{n+1}\theta+\alpha) + r^{n+2}\cos(n\theta+\alpha)\}/$$
$$\{1 - 2r\cos\theta + r^2\} \quad (6),$$
$$V_n = \{\sin\alpha - r\sin(\alpha-\theta) - r^{n+1}\sin(\overline{n+1}\theta+\alpha) + r^{n+2}\sin(n\theta+\alpha)\}/$$
$$\{1 - 2r\cos\theta + r^2\} \quad (7).$$

From these results, by putting $r = +1$, or $r = -1$, we deduce several important particular cases. For example, (6) and (7) give

$$\cos\alpha + \cos(\alpha+\theta) + \cos(\alpha+2\theta) + \ldots + \cos(\alpha+n\theta)$$
$$= \cos\tfrac{1}{2}\{\alpha + (\alpha+n\theta)\}\sin\tfrac{1}{2}(n+1)\theta/\sin\tfrac{1}{2}\theta \quad (6');$$
$$\sin\alpha + \sin(\alpha+\theta) + \sin(\alpha+2\theta) + \ldots + \sin(\alpha+n\theta)$$
$$= \sin\tfrac{1}{2}\{\alpha + (\alpha+n\theta)\}\sin\tfrac{1}{2}(n+1)\theta/\sin\tfrac{1}{2}\theta \quad (7').$$

Finally, if $r<1$, we may make n infinite in (4), (5), (6), (7); and we thus find

$$S_\infty = (1 - r\cos\theta)/(1 - 2r\cos\theta + r^2) \quad (4'');$$
$$T_\infty = r\sin\theta/(1 - 2r\cos\theta + r^2) \quad (5'');$$
$$U_\infty = \{\cos\alpha - r\cos(\alpha-\theta)\}/\{1 - 2r\cos\theta + r^2\} \quad (6'');$$
$$V_\infty = \{\sin\alpha - r\sin(\alpha-\theta)\}/\{1 - 2r\cos\theta + r^2\} \quad (7'').$$

* For brevity, and in order to keep the attention of the reader as closely as possible to the essentials of the matter, we leave it to him, or to his teacher, to supply the details of the analysis.

Example 2. Sum to infinity the series

$$S = 1 - 2r\cos\theta + 3r^2\cos 2\theta - 4r^3\cos 3\theta + \ldots \qquad (r<1).$$

If $z = r(\cos\theta + i\sin\theta)$, then S is the real part of the sum of the series

$$T = 1 - 2z + 3z^2 - 4z^3 + \ldots .$$

Now, by chap. xx., § 14, Example 2,

$$T = 1/(1+z)^2.$$

Hence
$$S = R\{1/(1 + r\cos\theta + ri\sin\theta)^2\},*$$
$$= R\{(1 + r\cos\theta - ri\sin\theta)^2/(\overline{1 + r\cos\theta}^2 + r^2\sin^2\theta)^2\},$$
$$= (1 + 2r\cos\theta + r^2\cos 2\theta)/(1 + 2r\cos\theta + r^2)^2.$$

Example 3. Exemplify the fact that every algebraical identity leads to two trigonometrical identities in the particular case of the identity

$$-(b-c)(c-a)(a-b) = bc(b-c) + ca(c-a) + ab(a-b).$$

In the given identity put $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, and observe that

$$\cos\beta + i\sin\beta - \cos\gamma - i\sin\gamma = 2i\sin\tfrac{1}{2}(\beta-\gamma)\{\cos\tfrac{1}{2}(\beta+\gamma) + i\sin\tfrac{1}{2}(\beta+\gamma)\}.$$

We thus get

$$4\Pi\sin\tfrac{1}{2}(\beta-\gamma)\{\cos\tfrac{1}{2}(\beta+\gamma) + i\sin\tfrac{1}{2}(\beta+\gamma)\}$$
$$= \Sigma\sin\tfrac{1}{2}(\beta-\gamma)\{\cos\beta + i\sin\beta\}\{\cos\gamma + i\sin\gamma\}\{\cos\tfrac{1}{2}(\beta+\gamma) + i\sin\tfrac{1}{2}(\beta+\gamma)\},$$

whence

$$4\cos(\alpha+\beta+\gamma)\Pi\sin\tfrac{1}{2}(\beta-\gamma) = \Sigma\sin\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{3}{2}(\beta+\gamma);$$
$$4\sin(\alpha+\beta+\gamma)\Pi\sin\tfrac{1}{2}(\beta-\gamma) = \Sigma\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{3}{2}(\beta+\gamma).$$

FORMULÆ CONNECTED WITH DEMOIVRE'S THEOREM AND THE BINOMIAL THEOREM FOR AN INTEGRAL INDEX.

§ 12.] By chap. xii., § 15 (3), we have

$$\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$
$$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n).$$

If we expand the right-hand side, and use P_r to denote $\Sigma\cos\theta_1\cos\theta_2\ldots\cos\theta_r\sin\theta_{r+1}\ldots\sin\theta_n$, that is, the sum of all the partial products that can be formed by taking the cosines of r of the angles $\theta_1, \theta_2, \ldots, \theta_n$ and the sines of the rest, then we find that

$$\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$$
$$= P_n + iP_{n-1} - P_{n-2} - iP_{n-3} + P_{n-4} + iP_{n-5} - \ldots .$$

* We use $Rf(x+yi)$ and $If(x+yi)$ to denote the real and imaginary parts of $f(x+yi)$ respectively.

Hence

$$\cos(\theta_1+\theta_2+\ldots+\theta_n) = P_n - P_{n-2} + P_{n-4} - P_{n-6} + \ldots \quad (1);$$
$$\sin(\theta_1+\theta_2+\ldots+\theta_n) = P_{n-1} - P_{n-3} + P_{n-5} - P_{n-7} + \ldots \quad (2).$$

From these, or, more directly, from

$$\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1\cos\theta_2 \ldots \cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)\ldots(1+i\tan\theta_n),$$

we derive

$$\tan(\theta_1+\theta_2+\ldots+\theta_n) = (T_1-T_3+T_5-\ldots)/(1-T_2+T_4-\ldots) \quad (3),$$

where $T_r = \Sigma\tan\theta_1\tan\theta_2\ldots\tan\theta_r$.

The formulæ (1), (2), (3) are generalisations of the familiar addition formulæ for the cosine, sine, and tangent.

From the usual form of Demoivre's Theorem, namely,

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n,$$

we derive, by expansion of the right-hand side,

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\,{}_nC_1\cos^{n-1}\theta\sin\theta - {}_nC_2\cos^{n-2}\theta\sin^2\theta - i\,{}_nC_3\cos^{n-3}\theta\sin^3\theta + {}_nC_4\cos^{n-4}\theta\sin^4\theta + \ldots .$$

Hence

$$\cos n\theta = \cos^n\theta - {}_nC_2\cos^{n-2}\theta\sin^2\theta + {}_nC_4\cos^{n-4}\theta\sin^4\theta - \ldots \quad (4);^*$$
$$\sin n\theta = {}_nC_1\cos^{n-1}\theta\sin\theta - {}_nC_3\cos^{n-3}\theta\sin^3\theta + {}_nC_5\cos^{n-5}\theta\sin^5\theta - \ldots \quad (5);$$
$$\tan n\theta = \frac{{}_nC_1\tan\theta - {}_nC_3\tan^3\theta + {}_nC_5\tan^5\theta - \ldots}{1 - {}_nC_2\tan^2\theta + {}_nC_4\tan^4\theta - \ldots} \quad (6).$$

These are generalisations of the formulæ (8) of § 2.

The formulæ (4) and (5) above at once suggest that $\cos n\theta$ can always be expanded in a series of descending powers of $\cos\theta$; that, when n is even, $\cos n\theta$ can be expanded in a series of even powers of $\sin\theta$ or of $\cos\theta$; $\sin n\theta/\sin\theta$ in a series of odd powers of $\cos\theta$; and $\sin n\theta/\cos\theta$ in a series of odd powers of $\sin\theta$: and, when n is odd, $\cos n\theta$ in a series of odd powers of $\cos\theta$; $\cos n\theta/\cos\theta$ in a series of even powers of $\sin\theta$; $\sin n\theta$ in a series of odd powers of $\sin\theta$; $\sin n\theta/\sin\theta$ in a series of even powers of $\cos\theta$.

* The formulæ (4), (5), (6), (8) were first given by John Bernoulli in 1701 (see *Op.*, t. i., p. 387).

Knowing, *a priori*, that these series exist, we could in various ways determine their coefficients; or we could obtain certain of them from (1) and (2) by direct transformation; and then deduce the rest by writing $\frac{1}{2}\pi - \theta$ in place of θ. (See Todhunter's *Trigonometry*, §§ 286-288.)

We may, however, deduce the expansions in question from the results of chap. xxvii., § 7. If in the equations (9), (10), (9'), (9''), (10'), (10'') there given we put $\alpha = \cos\theta + i\sin\theta$, $\beta = \cos\theta - i\sin\theta$, and therefore $p = 2\cos\theta$, $q = 1$, we deduce

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1!}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \ldots$$

$$(-)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}(2\cos\theta)^{n-2r} + \ldots \quad (7);^*$$

$$\sin n\theta/\sin\theta = (2\cos\theta)^{n-1} - \frac{n-2}{1!}(2\cos\theta)^{n-3} + \frac{(n-3)(n-4)}{2!}$$

$$(2\cos\theta)^{n-5} - \ldots (-)^r\frac{(n-r-1)(n-r-2)\ldots(n-2r)}{r!}$$

$$(2\cos\theta)^{n-2r-1} + \ldots \quad (8);$$

$$\cos n\theta = (-)^{n/2}\left\{1 - \frac{n^2}{2!}\cos^2\theta + \frac{n^2(n^2-2^2)}{4!}\cos^4\theta - \ldots\right.$$

$$\left.(-)^s\frac{n^2(n^2-2^2)\ldots(n^2-\overline{2s-2}^2)}{(2s)!}\cos^{2s}\theta + \ldots\right\}(n \text{ even}) \quad (9);$$

$$\cos n\theta = (-)^{(n-1)/2}\left\{\frac{n}{1!}\cos\theta - \frac{n(n^2-1^2)}{3!}\cos^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\right.$$

$$\left.\cos^5\theta - \ldots (-)^s\frac{n(n^2-1^2)(n^2-3^2)\ldots(n^2-\overline{2s-1}^2)}{(2s+1)!}\cos^{2s+1}\theta + \ldots\right\}$$

$$(n \text{ odd}) \quad (10);$$

$$\sin n\theta/\sin\theta = (-)^{n/2-1}\left\{\frac{n}{1!}\cos\theta - \frac{n(n^2-2^2)}{3!}\cos^3\theta + \ldots\right.$$

$$\left.(-)^s\frac{n(n^2-2^2)\ldots(n^2-\overline{2s}^2)}{(2s+1)!}\cos^{2s+1}\theta + \ldots\right\}(n \text{ even}) \quad (11);$$

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* The series (7), (9'), (10') were first given by James Bernoulli in 1702 (see *Op.*, t. ii., p. 926). He deduced them from the formula

$$2\sin^2 n\theta = \frac{n^2}{2!}(2\sin\theta)^2 - \frac{n^2(n^2-1^2)}{4!}(2\sin\theta)^4 + \frac{n^2(n^2-1^2)(n^2-3^2)}{6!}(2\sin\theta)^6 - \ldots,$$

which he established by an induction based on the previous results of Vieta regarding the multisection of an angle.

$$\sin n\theta/\sin\theta = (-)^{(n-1)/2}\left\{1 - \frac{n^2-1}{2!}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{4!}\cos^4\theta - \ldots\right.$$

$$\left.(-)^s\frac{(n^2-1^2)(n^2-3^2)\ldots(n^2-\overline{2s-1}^2)}{(2s)!}\cos^{2s}\theta + \ldots\right\}(n \text{ odd}) \quad (12).$$

If in the above six formulæ we put $\frac{1}{2}\pi - \theta$ in place of θ, we derive six more in which all the series contain sines instead of cosines. In this way we get, *inter alia*, the following :—

$$\cos n\theta = 1 - \frac{n^2}{2!}\sin^2\theta + \frac{n^2(n^2-2^2)}{4!}\sin^4\theta - \ldots \quad (n \text{ even}) \quad (9');$$

$$\sin n\theta = \frac{n}{1!}\sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \ldots$$

$$(n \text{ odd}) \quad (10');$$

$$\sin n\theta/\cos\theta = \frac{n}{1!}\sin\theta - \frac{n(n^2-2^2)}{3!}\sin^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{5!}\sin^5\theta - \ldots$$

$$(n \text{ even}) \quad (11');$$

$$\cos n\theta/\cos\theta = 1 - \frac{n^2-1^2}{2!}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{4!}\sin^4\theta - \ldots$$

$$(n \text{ odd}) \quad (12').$$

The formulæ of this paragraph are generalisations of the familiar expressions for cos 2θ, sin 2θ, cos 3θ, and sin 3θ, in terms of cos θ and sin θ.

§ 13.] The converse problem to express cos$^n\theta$, sin$^n\theta$, and, generally, sin$^m\theta$ cos$^n\theta$ in a series of sines or cosines of multiples of θ, can also be readily solved by means of Demoivre's Theorem.

If, for shortness, we denote cos $\theta + i \sin\theta$ by x, then we have, by Demoivre's Theorem, the following results :—

$$\left.\begin{aligned} &x = \cos\theta + i\sin\theta, \quad 1/x = \cos\theta - i\sin\theta\,; \\ &x^n = \cos n\theta + i\sin n\theta, \quad 1/x^n = \cos n\theta - i\sin n\theta\,: \\ &\cos\theta = \frac{1}{2}(x + 1/x), \quad \sin\theta = \frac{1}{2i}(x - 1/x)\,; \\ &\cos n\theta = \frac{1}{2}(x^n + 1/x^n), \quad \sin n\theta = \frac{1}{2i}(x^n - 1/x^n). \end{aligned}\right\} \quad (1).$$

Hence

$$\cos^{2m}\theta = \frac{1}{2^{2m}}(x + 1/x)^{2m},$$

$$= \frac{1}{2^{2m}}\{(x^{2m} + 1/x^{2m}) + {}_{2m}C_1(x^{2m-2} + 1/x^{2m-2}) + {}_{2m}C_2(x^{2m-4} + 1/x^{2m-4}) + \ldots + {}_{2m}C_m\},$$

$$= \frac{1}{2^{2m-1}}\{\cos 2m\theta + {}_{2m}C_1\cos(2m-2)\theta + {}_{2m}C_2\cos(2m-4)\theta + \ldots + \tfrac{1}{2}\,{}_{2m}C_m\} \quad (2).$$

Similarly,

$$\cos^{2m+1}\theta = \frac{1}{2^{2m}}\{\cos(2m+1)\theta + {}_{2m+1}C_1\cos(2m-1)\theta + {}_{2m+1}C_2\cos(2m-3)\theta + \ldots + {}_{2m+1}C_m\cos\theta\} \quad (3);$$

$$\sin^{2m}\theta = \frac{(-1)^m}{2^{2m-1}}\{\cos 2m\theta - {}_{2m}C_1\cos(2m-2)\theta + {}_{2m}C_2\cos(2m-4)\theta + \ldots (-)^m\tfrac{1}{2}\,{}_{2m}C_m\} \quad (4);$$

$$\sin^{2m+1}\theta = \frac{(-1)^m}{2^{2m}}\{\sin(2m+1)\theta - {}_{2m+1}C_1\sin(2m-1)\theta + {}_{2m+1}C_2\sin(2m-3)\theta + \ldots (-)^m{}_{2m+1}C_m\sin\theta\} \quad (5).$$

These formulæ are generalisations of the ordinary trigonometrical formulæ $\sin^2\theta = -\frac{1}{2}(\cos 2\theta - 1)$, $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$, &c.

In any particular case, especially when products, such as $\sin^m\theta\cos^n\theta$, have to be expanded, the use of detached coefficients after the manner of the following example will be found to conduce both to rapidity and to accuracy.

Example 1. To expand $\sin^5\theta\cos^3\theta$ in a series of sines of multiples of θ.

$$\sin^5\theta\cos^3\theta = \frac{1}{2^8 i^5}(x - 1/x)^5(x + 1/x)^3.$$

Starting with the coefficients of the highest power which happens to be remembered, say the 4th, we proceed thus—

Coefficients of Multiplier.	Coefficients of Product.
	1 − 4 + 6 − 4 + 1
1 − 1	1 − 5 + 10 − 10 + 5 − 1
1 + 1	1 − 4 + 5 + 0 − 5 + 4 − 1
1 + 1	1 − 3 + 1 + 5 − 5 − 1 + 3 − 1
1 + 1	1 − 2 − 2 + 6 + 0 − 6 + 2 + 2 − 1

The coefficients in the last line are those in the expansion of $(x - 1/x)^5(x + 1/x)^3$. Hence, arranging together the terms at the beginning and end, and replacing

$\frac{1}{2i}(x^8 - 1/x^8)$ by $\sin 8\theta$, $\frac{1}{2i}(x^6 - 1/x^6)$ by $\sin 6\theta$, and so on, we find

$$\sin^5\theta\cos^3\theta = \frac{1}{2^7}\{\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta + \tfrac{1}{2}.0\},$$
$$= \frac{1}{128}\{\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta\}.$$

The student will see that $\sin^m\theta\cos^n\theta$ can be expanded in a series of sines or of cosines of multiples of θ, according as m is odd or even. The highest multiple occurring will be $(m+n)\theta$.

Example 2. If $\theta = 2\pi/n$, and α any angle whatever, and

$$_mU_n = \cos^m\alpha + \cos^m(\alpha+\theta) + \ldots + \cos^m(\alpha+\overline{n-1}\theta),$$
$$_mV_n = \sin^m\alpha + \sin^m(\alpha+\theta) + \ldots + \sin^m(\alpha+\overline{n-1}\theta),$$

where m is any positive integer which is not of the form $r + sn/2$, then

$$_{2m}U_n = {}_{2m}V_n = n.1.3\ldots(2m-1)/2.4\ldots 2m;$$
$$_{2m+1}U_n = {}_{2m+1}V_n = 0.$$

This will be found to follow from a combination of the formulæ of the present paragraph with the summation formulæ of § 11.

EXERCISES XV.

Sum the following series to n terms, and also, where admissible, to infinity :—

(1.) $\cos\alpha - \cos(\alpha+\theta) + \cos(\alpha+2\theta) - \ldots$

(2.) $\sin\alpha - \sin(\alpha+\theta) + \sin(\alpha+2\theta) - \ldots$

(3.) $\Sigma\sin^3 n\theta$. (4.) $n\cos\theta + (n-1)\cos 2\theta + (n-2)\cos 3\theta + \ldots$

(5.) $\Sigma\sin n\theta\cos(n+1)\theta$. (6.) $\Sigma\sin n\theta\sin 2n\theta\sin 3n\theta$.

(7.) $\sin\alpha - \cos\alpha\sin(\alpha+\theta) + \cos^2\alpha\sin(\alpha+2\theta) - \ldots$

(8.) $1 + \cos\theta/\cos\theta + \cos 2\theta/\cos^2\theta + \cos 3\theta/\cos^3\theta + \ldots$ to n terms, where $\theta = n\pi$.

(9.) $1 - 2r\cos\theta + 3r^2\cos 2\theta - 4r^3\cos 3\theta + \ldots$

(10.) $\sin\theta + 3\sin 2\theta + 5\sin 3\theta + 7\sin 4\theta + \ldots$

(11.) $\Sigma n^2\cos(n\theta+\alpha)$. (12.) $\Sigma n(n+1)\sin(2n+1)\theta$.

(13.) $\sin 2n\theta - {}_{2n}C_1\sin(2n-2)\theta + {}_{2n}C_2\sin(2n-4)\theta - \ldots$ (n a positive integer).

(14.) $\sin(2n+1)\theta + {}_{2n+1}C_1\sin(2n-1)\theta + {}_{2n+1}C_2\sin(2n-3)\theta + \ldots$ (n a positive integer).

(15.) $\Sigma m(m+1)\ldots(m+n-1)r^n\cos(\alpha+n\theta)/n!$ to infinity, m being a positive integer.

(16.) Does the function

$$(\sin^2\theta + \sin^2 2\theta + \ldots + \sin^2 n\theta)/(\cos^2\theta + \cos^2 2\theta + \ldots + \cos^2 n\theta)$$

approach a definite limit when $n = \infty$?

(17.) Expand $1/(1 - 2\cos\theta.x + x^2)$ in a series of ascending powers of x.

(18.) Expand $1/(1-2\cos\theta.x+x^2)^2$ in a series of ascending powers of x.

(19.) Expand $(1+2x)/(1-x^3)$ in a series of ascending powers of x; and show that

$$1-3n+\frac{(3n-1)(3n-2)}{2!}-\frac{(3n-2)(3n-3)(3n-4)}{3!}+\ldots=(-1)^n.$$

(20.) Show that $1/(1+x+x^2)=1-x+x^3-x^4+x^6-x^7+x^9-x^{10}+\ldots$; and that, if the sum of the even terms of this expansion be $\phi(x)$, and the sum of the odd terms $\psi(x)$, then $\{\phi(x)\}^2-\{\psi(x)\}^2=\phi(x^2)+\psi(x^2)$.

Prove the following identities by means of Demoivre's Theorem, or otherwise. Σ and Π refer to the letters α, β, γ:—

(21.) $\Sigma\sin\alpha/(1+\Sigma\cos\alpha)=-\Pi\tan\frac{1}{2}\alpha$, where $\alpha+\beta+\gamma=0$.

(22.) $\Sigma\sin(\theta-\beta)\sin(\theta-\gamma)/\sin(\alpha-\beta)\sin(\alpha-\gamma)=1$.

(23.) $\Sigma\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)\cos\alpha/\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\alpha-\gamma)=\cos(\alpha+\beta+\gamma)$.

(24.) $\cos\sigma\cos(\sigma-2\alpha)\cos(\sigma-2\beta)\cos(\sigma-2\gamma)+\sin\sigma\sin(\sigma-2\alpha)\sin(\sigma-2\beta)\sin(\sigma-2\gamma)=\cos2\alpha\cos2\beta\cos2\gamma$, where $\sigma=\alpha+\beta+\gamma$.

Expand in series of cosines or sines of multiples of θ:—

(25.) $\cos^{10}\theta$. (26.) $\sin^7\theta$. (27.) $\sin^8\theta$.

(28.) $\cos^6\theta\sin^3\theta$. (29.) $\cos^6\theta\sin^4\theta$.

Expand in series of powers of sines or cosines:—

(30.) $\cos10\theta$. (31.) $\sin7\theta$.

(32.) $\sin3\theta\cos6\theta$. (33.) $\cos m\theta\cos n\theta$.

EXPANSION OF COS θ AND SIN θ IN POWERS OF θ.

§ 14.] We propose next to show that, for all finite real values of θ,

$$\cos\theta=1-\theta^2/2!+\theta^4/4!-\theta^6/6!+\ldots \text{ ad } \infty \qquad (1);$$

$$\sin\theta=\theta-\theta^3/3!+\theta^5/5!-\theta^7/7!+\ldots \text{ ad } \infty \qquad (2).$$

These expansions * are of fundamental importance in the part of algebraical analysis with which we are now concerned. They may be derived by the method of limits either from the formulæ of § 12, or from two or more of the equivalent formulæ of § 13. We shall here choose the former course. It will appear, however, afterwards that this is by no means the only way in which these important expansions might be introduced into algebra.

* First given by Newton in his tract *Analysis per æquationes numero terminorum infinitas*, which was shown to Barrow in 1669. The leading idea of the above demonstration was given by Euler (*Introd. in Anal. Inf.*, t. i., § 122), but his demonstration was not rigorous in its details.

From (4) and (5) of § 12, writing, as is obviously permissible, θ/m in place of θ, and taking $n = m$, we deduce, after a little rearrangement,

$$\cos\theta = \cos^m\frac{\theta}{m}\left\{1 - \frac{1-1/m}{2!}\theta^2\left(\tan\frac{\theta}{m}\Big/\frac{\theta}{m}\right)^2 + \frac{(1-1/m)(1-2/m)(1-3/m)}{4!}\theta^4\left(\tan\frac{\theta}{m}\Big/\frac{\theta}{m}\right)^4 - \ldots\right\} \quad (3),$$

$$= \cos^m\frac{\theta}{m}\{1 - u_2 + u_4 - \ldots\}, \text{ say,} \quad (3');$$

and

$$\sin\theta = \cos^m\frac{\theta}{m}\left\{\theta\left(\tan\frac{\theta}{m}\Big/\frac{\theta}{m}\right) - \frac{(1-1/m)(1-2/m)}{3!}\theta^3\left(\tan\frac{\theta}{m}\Big/\frac{\theta}{m}\right)^3 + \ldots\right\} \quad (4),$$

$$= \cos^m\frac{\theta}{m}\{u_1 - u_3 + \ldots\}, \text{ say,} \quad (4').$$

Here, from the nature of the original formula, m must be a positive integer; but nothing hinders our giving it as large a value as we please, and we propose in fact ultimately to increase it without limit. On the other hand, we take θ to be a fixed finite real quantity, positive or negative.

The series (3), as it stands, terminates; and its terms alternate in sign.

We have

$$\operatorname{mod}\frac{u_{2n+2}}{u_{2n}} = \frac{(1-2n/m)(1-\overline{2n+1}/m)}{(2n+1)(2n+2)}\theta^2\left(\tan\frac{\theta}{m}\Big/\frac{\theta}{m}\right)^2.$$

Hence, so long as n is finite,

$$\operatorname*{L}_{m=\infty}\operatorname{mod}\frac{u_{2n+2}}{u_{2n}} = \frac{\theta^2}{(2n+1)(2n+2)}.$$

If, therefore, we take $2n + 1 > \theta$,* we can always, by taking m large enough, secure that, on and after the term u_{2n}, the numerical value of the convergency-ratio of the series (3) shall be less than unity.

* Strictly speaking, it is sufficient if $\theta < \sqrt{\{(2n+1)(2n+2)\}}$.

From this it follows that, if $2n + 1 > \theta$, and m be only taken large enough, $\cos\theta$ will be intermediate in value between

$$\cos^m\frac{\theta}{m}\{1 - u_2 + u_4 - \ldots (-)^n u_{2n}\} \qquad (5),$$

and

$$\cos^m\frac{\theta}{m}\{1 - u_2 + u_4 - \ldots (-)^n u_{2n} + (-)^{n+1}u_{2n+2}\} \quad (6).$$

Therefore $\cos\theta$ will always lie between the limits of (5) and (6) for $m = \infty$.

Now (see chap. xxv., § 23)

$$\mathrm{L}\cos^m(\theta/m) = 1, \quad \mathrm{L}\, u_2 = \theta^2/2\,!, \quad \mathrm{L}\, u_4 = \theta^4/4\,!, \ \ldots$$
$$\mathrm{L}\, u_{2n} = \theta^{2n}/(2n)\,!, \quad \mathrm{L}\, u_{2n+2} = \theta^{2n+2}/(2n+2)\,!.$$

Hence $\cos\theta$ lies between

$$1 - \theta^2/2\,! + \theta^4/4\,! - \ldots (-)^n\theta^{2n}/(2n)\,!$$

and

$$1 - \theta^2/2\,! + \theta^4/4\,! - \ldots (-)^n\theta^{2n}/(2n)\,! + (-)^{n+1}\theta^{2n+2}/(2n+2)\,!.$$

In other words, *provided* $2n + 1 > \theta$,

$$\cos\theta = 1 - \theta^2/2\,! + \theta^4/4\,! - \ldots (-)^n\theta^{2n}/(2n)\,! + (-)^{n+1}\mathrm{R}_{2n} \quad (7),$$

where $$\mathrm{R}_{2n} < \theta^{2n+2}/(2n+2)\,!.$$

Here $2n$ may be made as large as we please, therefore since $\mathrm{L}_{n=\infty}\, \theta^{2n+2}/(2n+2)\,! = 0$ (chap. xxv., § 15, Example 2), we may write

$$\cos\theta = 1 - \theta^2/2\,! + \theta^4/4\,! - \ldots \text{ad } \infty \qquad (7').$$

By an identical process of reasoning, we may show that, *provided* $2n + 2 > \theta$,* *then*

$$\sin\theta = \theta - \theta^3/3\,! + \ldots (-)^n\theta^{2n+1}/(2n+1)\,! + (-)^{n+1}\mathrm{R}_{2n+1} \quad (8),$$

where $$\mathrm{R}_{2n+1} < \theta^{2n+3}/(2n+3)\,!,$$

and therefore

$$\sin\theta = \theta - \theta^3/3\,! + \theta^5/5\,! - \ldots \text{ad } \infty \qquad (8').$$

It has already been shown, in chap. xxvi., that the series (7') and (8') are convergent for all real finite values of θ; they are

* More closely, if $\theta < \sqrt{\{(2n+2)(2n+3)\}}$.

therefore legitimately equivalent to the one-valued functions $\cos\theta$ and $\sin\theta$ for all real values of θ, that is, for all values of the argument for which these functions are as yet defined. From this it follows that the two series must be periodic functions of θ having the period 2π. This conclusion may at first sight startle the reader; but he can readily verify it by arithmetical calculation through a couple of periods at least.

When θ is not very large, say $\not> \frac{1}{4}\pi$, which is the utmost value of the argument we need use for the purposes of calculation,* the series converge with great rapidity, five or six terms being amply sufficient to secure accuracy to the 7th decimal place.

We shall not interrupt our exposition to dwell on the many uses of these fundamental expansions. A few examples will be sufficient, for the present, on that head.

Example 1. To calculate to seven places the cosine and sine of the radian.

We have

$$\cos 1 = 1 - 1/2! + 1/4! - 1/6! + 1/8! - 1/10! + R_{10},$$

$$R_{10} < 1/12!,$$

$$= 1 - \cdot 500,000,0 + \cdot 041,666,7 - \cdot 001,388,9 + \cdot 000,024,8 - \cdot 000,000,3 + R_{10},$$

$$R_{10} < \cdot 000,000,003.$$

$$= \cdot 540,302,3.$$

Similarly,

$$\sin 1 = 1 - 1/3! + 1/5! - 1/7! + 1/9! - R_9,$$

$$R_9 < 1/11! < \cdot 000,000,03,$$

$$= \cdot 841,471,0.$$

The error in each case does not exceed a unit in the 7th place.

Example 2. If $\theta < 3$, then $\theta > \sin\theta > \theta - \frac{1}{6}\theta^3$; $1 - \frac{1}{2}\theta^2 < \cos\theta < 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$. These inequalities follow at once from (7) and (8) above. They are extensions of those previously deduced, in chap. xxv., § 21, from geometrical considerations.

Example 3. Expand $\cos(\alpha + \theta)$ in powers of θ.

Result. $\cos(\alpha + \theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta,$

$$= \cos\alpha - \sin\alpha\,\theta - \cos\alpha\,\theta^2/2! + \sin\alpha\,\theta^3/3! + \cos\alpha\,\theta^4/4! - \ldots$$

* Seeing that the cosine or sine of every angle between $\frac{1}{4}\pi$ and $\frac{1}{2}\pi$ is the sine or cosine of an angle between 0 and $\frac{1}{4}\pi$.

Example 4. Find the limit of

$$\theta(1-\cos\theta)/(\tan\theta-\theta) \text{ when } \theta=0.$$

$$\begin{aligned}
\mathrm{L}\theta(1-\cos\theta)/(\tan\theta-\theta) &= \mathrm{L}\sec\theta\ \mathrm{L}\theta(1-\cos\theta)/(\sin\theta-\theta\cos\theta),\\
&= 1\times \mathrm{L}\theta(\theta^2/2-\theta^4/4!+\ldots)/(\theta-\theta^3/3!+\ldots-\theta+\theta^3/2-\ldots),\\
&= \mathrm{L}(\theta^3/2-\theta^5/4!+\ldots)/(\theta^3/3+\ldots),\\
&= \mathrm{L}(1/2+\mathrm{P}\theta^2+\ldots)/(1/3+\mathrm{Q}\theta^2+\ldots),\\
&= 3/2.
\end{aligned}$$

EXERCISES XVI.

(1.) Expand $\sin(\alpha+\theta)\sin(\beta+\theta)$ in powers of θ.

(2.) Calculate $\sin 45^\circ\ 32'\ 30''$ to five places of decimals.

(3.) Given $\tan\theta/\theta = 1001/1000$, calculate θ.

(4.) Expand $\cos^2\theta$, $\sin^2\theta$, and $\sin^3\theta\cos\theta$ in powers of θ; and find the general term in each case.

(5.) Show that $\cos^m\theta$ (m a positive integer) can be expanded in a convergent series of even powers of θ; and that the coefficient of θ^{2n} in this expansion is

$$(-)^n\{m^{2n}+{}_m\mathrm{C}_1(m-2)^{2n}+{}_m\mathrm{C}_2(m-4)^{2n}+\ldots\}/2^{m-1}(2n)!.$$

(6.) Show that, if m and n be positive integers, and $1<n<m$, then

$$m^n-{}_m\mathrm{C}_1(m-2)^n+{}_m\mathrm{C}_2(m-4)^n-\ldots=0.$$

Examine how this result is modified when $n=1$, or $n=m$.

Evaluate the following limits :—

(7.) $(\sin^2 m\theta-\sin^2 n\theta)/(\cos p\theta-\cos q\theta)$, $\theta=0$.

(8.) $\{\sin p(\alpha+\theta)-\sin p\alpha\}/\theta$, $\theta=0$.

(9.) $\{\sin^n p(\alpha+\theta)-\sin^n p\alpha\}/\theta$, $\theta=0$.

(10.) $\{\sin^n p(\alpha+\theta)\cos(\alpha+\theta)-\sin^n p\alpha\cos\alpha\}/\theta$, $\theta=0$.

(11.) $(a^\theta\sin a\theta-b^\theta\sin b\theta)/(b^\theta\tan a\theta-a^\theta\tan b\theta)$, $\theta=0$.

(12.) $1/2x^2-\pi/2x\tan\pi x-1/(1-x^2)$, $x=1$ (Euler).

(13.) $\{\sin x/x\}^{1/x^n}$, $x=0$.

(14.) $\{(x/a)\sin(a/x)\}^{x^m}$, $x=\infty$, $(m=>2)$.

(15.) Show, by employing the process used in chap. xxvii., § 2, that the series for $\sin n\theta/\cos\theta$ in powers of $\sin\theta$ can be derived from the series for $\cos n\theta$ in powers of $\sin\theta$; and so on.

(16.) Show, by using the process of chap. xxvii., § 2, twice over, that, if

$$\cos n\theta = 1+\mathrm{A}_1\sin^2\theta+\mathrm{A}_2\sin^4\theta+\ldots+\mathrm{A}_r\sin^{2r}\theta+\ldots,$$

then

$$\begin{aligned}
-n^2\cos n\theta = 2\mathrm{A}_1+(3\cdot 4\mathrm{A}_2-2^2\mathrm{A}_1)\sin^2\theta+\ldots\\
+\{(2r+1)(2r+2)\mathrm{A}_{r+1}-(2r)^2\mathrm{A}_r\}\sin^{2r}\theta+\ldots.
\end{aligned}$$

Hence determine the coefficients A_1, A_2, &c.; and, by combining Exercise 15 with Exercise 16, deduce all the series (7)...(12') of § 12.

(17.) Show (from § 13) that $\cos^n\theta$ and $\sin^n\theta$ can each be expanded in a convergent series of powers of θ; and find an expression for the coefficient of the general term in each case.

In particular, show that

$$\sin^3 x/3! = x^3/3! - (1+3^2)x^5/5!+(1+3^2+3^4)x^7/7!-(1+3^2+3^4+3^6)x^9/9!+\ldots.$$

BINOMIAL THEOREM FOR ANY COMMENSURABLE INDEX.

§ 15.] If, as in chap. xxvii., § 3, we write

$$f(m) = 1 + \Sigma_m C_n z^n \qquad (10),$$

where m is any commensurable number as before, but z is now a complex variable, then, so long as $\text{mod}\, z < 1$, $\Sigma_m C_n z^n$ will (chap. xxvi., § 3) be an absolutely convergent series; and $f(m)$ will be a one-valued continuous function both of m and of z. Hence the reasoning of chap. xxvii., § 3, which established the addition theorem $f(m_1)f(m_2) = f(m_1 + m_2)$ will still hold good; and all the immediate consequences of this theorem—for example, the equations (4), (5), (6), (7), (8), (9) in the paragraph referred to—will hold for the more general case now under consideration.

In particular, if p and q be any positive integers (which, for simplicity, we suppose prime to each other), then

$$\begin{aligned} \{f(p/q)\}^q &= \{f(1)\}^p, \\ &= (1+z)^p \end{aligned} \qquad (11).$$

It follows that $f(p/q)$ represents part of the q-valued function $\sqrt[q]{(1+z)^p}$; and it remains to determine what part.

Let $z = r(\cos\theta + i\sin\theta)$, then, since we have merely to explore the variation of the one-valued function $f(p/q)$, it will be sufficient to cause θ to vary between $-\pi$ and $+\pi$.

Also, let

$$\left.\begin{aligned} w = 1 + z &= 1 + x + yi, \\ &= 1 + r\cos\theta + ir\sin\theta, \\ &= \rho(\cos\phi + i\sin\phi), \end{aligned}\right\} \qquad (a),$$

so that

$$\left.\begin{aligned} \rho = \{(1+x)^2 + y^2\}^{1/2} &= (1 + 2r\cos\theta + r^2)^{1/2}; \\ \tan\phi = y/(1+x) &= r\sin\theta/(1 + r\cos\theta), \end{aligned}\right\} \qquad (b).$$

If we draw the Argand diagram for $w = 1 + x + yi$, we see that when r is given w describes a circle of radius r, whose centre is the point (1, 0). Since $r < 1$, this circle falls short of the origin. Hence ϕ, the inclination to the x-axis of the vector drawn from the origin to the point w, is never greater than

$\tan^{-1}\{r/(1-r^2)^{1/2}\}$, and never less than $-\tan^{-1}\{r/(1-r^2)^{1/2}\}$. Hence ϕ lies in all cases between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Therefore, since $f(p/q)$ is continuous, only one branch of the function $\sqrt[q]{(1+z)^p}$ is in question. Now, if we denote the principal branch by $(1+z)^{p/q}$, so that

$$(1+z)^{p/q} = \rho^{p/q}(\cos p\phi/q + i \sin p\phi/q),$$

we have, by § 8,

$$\sqrt[q]{(1+z)^p} = (1+z)^{p/q}\omega_q{}^{pt} \qquad (12),$$

where $t = 0, \pm 1, \pm 2, \ldots$, according to the branch of the function which is in question. Hence we have

$$f(p/q) = (1+z)^{p/q}\omega_q{}^{pt},$$

where t has to be determined.

Now, when $z = 0$, we have $f(p/q) = 1$, hence we must have

$$1 = \omega_q{}^{pt}.$$

Hence $t = 0$, and we have

$$f(p/q) = (1+z)^{p/q} = \rho^{p/q}(\cos.p\phi/q + i \sin.p\phi/q),$$

where $$-\tfrac{1}{2}\pi < \phi < \tfrac{1}{2}\pi.$$

Next consider any negative commensurable quantity, say $-p/q$. Then (by chap. xxvii., § 3 (9)),

$$f(-p/q) = f(0)/f(p/q),$$
$$= 1/f(p/q).$$

If, therefore, we define $(1+z)^{-p/q}$ to mean the reciprocal of the principal value of $(1+z)^{p/q}$, we have

$$f(-p/q) = (1+z)^{-p/q} = 1/(1+z)^{p/q}$$
$$\rho^{-p/q}\{\cos(-p\phi/q) + i\sin(-p\phi/q)\} \quad (13).$$

To sum up: *We have now established the following expansion for the principal value of* $(1+z)^m$, *in all cases where* m *is any commensurable number, and mod* $z < 1$:—

$$(1+z)^m = 1 + \Sigma_m C_n z^n \qquad (14).$$

The theorem may also be written in the following forms:—

$$1 + \Sigma_m C_n (x+yi)^n = \{(1+x)^2 + y^2\}^{m/2}[\cos.m \tan^{-1}\{y/(1+x)\}$$
$$+ i \sin.m \tan^{-1}\{y/(1+x)\}] \quad (15);$$

$$1 + \Sigma_m C_n r^n(\cos n\theta + i \sin n\theta)$$
$$= (1 + 2r\cos\theta + r^2)^{m/2}(\cos m\phi + i \sin m\phi),$$

where $-\frac{1}{2}\pi < \phi = \tan^{-1}\{r\sin\theta/(1 + r\cos\theta)\} < +\frac{1}{2}\pi$ (16).

§ 16.] The results of last paragraph were first definitely established by Cauchy.* In a classical memoir on the present subject,† Abel demonstrated the still more general theorem

$$1 + \Sigma_{m+ki} C_n (x + yi)^n$$
$$= [(1 + x)^2 + y^2]^{m/2}[\cos\{m\tan^{-1}\{y/(1 + x)\} + \tfrac{1}{2}k\log\{(1 + x)^2 + y^2\}\}$$
$$+ i\sin\{m\tan^{-1}\{y/(1 + x)\} + \tfrac{1}{2}k\log\{(1 + x)^2 + y^2\}\}]$$
$$\mathrm{Exp}[-k\tan^{-1}\{y/(1 + x)\}].$$

Into the proof of this theorem we shall not enter, as the theorem itself is not necessary for our present purpose.

§ 17.] The demonstration of § 15 fails when $\mathrm{mod}\, z = 1$. Here, however, the theorem of Abel, given in chap. xxvi., § 20, comes to our aid. From it we see that the summation of, say, (16) will hold, provided the series on the left hand remain convergent when $r = 1$.

Now the series $1 + \Sigma_m C_n(\cos n\theta + i \sin n\theta)$ will be convergent if, and will not be convergent unless, each of the series

$$S = 1 + \Sigma_m C_n \cos n\theta,$$
$$T = \Sigma_m C_n \sin n\theta$$

be convergent.

In the first place, we remark that, if $m < -1$, $L_m C_n = \pm\infty$ when $n = \infty$, so that neither of the series S, T can be convergent.

If $m = -1$, then ${}_mC_n = (-1)^n$, $S = 1 + \Sigma(-1)^n \cos n\theta$, $T = \Sigma(-1)^n \sin n\theta$, neither of which is convergent (see chap. xxvi., § 9).

If $-1 < m < 0$, then $L_m C_n = 0$; and the coefficients ultimately alternate in sign. Hence, by chap. xxvi., § 9, both the series S and T are convergent, provided $\theta \neq \pm\pi$. When θ has one or other of these excepted values, then $S = 1 + \Sigma(-1)^n {}_mC_n$, which is divergent when m lies between -1 and 0 (see chap. xxvi., § 6, Example 3).

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* See his *Analyse Algébrique*.

† *Œuvres Complètes* (ed. by Sylow & Lie), t. i., p. 238.

If $m > 0$, then, as we have already proved (see chap. xxvi., § 6, Example 4), $\Sigma_m C_n$ is absolutely convergent, and, *a fortiori*, $1 + \Sigma_m C_n \cos n\theta$ and $\Sigma_m C_n \sin n\theta$ are both absolutely convergent.

It follows, therefore, that *the equation*

$$(1 + z)^m = 1 + \Sigma_m C_n z^n$$

will hold when $\mod z = 1$, *in all cases where* $m > 0$; *and also when* m *lies between* -1 *and* 0, *provided that in this last case the imaginary part of* z *do not vanish, that is, provided the amplitude of* z *is not* $\pm \pi$.

In other cases where $\mod z = 1$, the theorem is not in question, owing to the non-convergency of $\Sigma_m C_n z^n$.

In all cases where $\mod z > 1$, the series $\Sigma_m C_n z^n$ is divergent, and the validity of the theorem is of course out of the question.

EXPONENTIAL AND LOGARITHMIC SERIES—GENERALISATION OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

§ 18.] The series

$$1 + z + z^2/2! + z^3/3! + \ldots$$

is absolutely convergent for all complex values of z having a finite modulus (see chap. xxvi., § 10). Hence it defines a single-valued continuous function of z for all values of z. We may call this function the Exponential of z, or shortly $\mathrm{Exp}\, z$;* so that $\mathrm{Exp}\, z$ is defined by the equation

$$\mathrm{Exp}\, z = 1 + z + z^2/2! + z^3/3! + \ldots \qquad (1).$$

The reasoning of chap. xxviii., § 5, presupposes nothing but the absolute convergence of the Exponential Series, and is therefore applicable when the variable is complex. We have therefore *the following addition theorem for the function* $\mathrm{Exp}\, z$:—

* When it is necessary to distinguish between the general function of a complex variable z and the ordinary exponential function of a real variable x, we shall use Exp (with a capital letter) for the former, and either e^x or $\exp x$ for the latter. After the student fully understands the theory, he may of course drop this distinction. It seems to be forgotten by some writers that the e in $e^{i\theta}$ is a mere *nominis umbra*—a contraction for the name of a function, and not 2·71828 . . . Oblivion of this fact has led to some strange pieces of mathematical logic.

$$\operatorname{Exp} z_1 \operatorname{Exp} z_2 \ldots \operatorname{Exp} z_m = \operatorname{Exp}(z_1 + z_2 + \ldots + z_m) \quad (2),$$

where $z_1, z_2, \ldots, z_m$ are any values of z whatever.

In particular, we have, if m be any positive integer,

$$(\operatorname{Exp} z)^m = \operatorname{Exp}(mz) \quad (3).$$

Also

$$\operatorname{Exp} z \operatorname{Exp}(-z) = \operatorname{Exp} 0,$$
$$= 1;$$

and therefore

$$\operatorname{Exp}(-z) = 1/\operatorname{Exp} z \quad (4).$$

We have, further,

$$\operatorname{Exp} 1 = 1 + 1 + 1/2! + 1/3! + \ldots,$$
$$= e \quad (5);$$

and, if x be any real commensurable number,

$$\operatorname{Exp} x = 1 + x + x^2/2! + x^3/3! + \ldots,$$
$$= e^x \quad (6),$$

by chap. xxviii., where e^x denotes, of course, the principal value of any root involved if x be not integral.

It appears, therefore, that $\operatorname{Exp} x$ coincides in meaning with e^x, so far as e^x is yet defined.

We may therefore, for real values of x and for the corresponding values of y, take the graph of $y = \operatorname{Exp} x$ to be identical with the graph of $y = e^x$, already discussed in chap. xxi. Hence the equation

$$y = \operatorname{Exp} x \quad (7)$$

defines x as a continuous one-valued function of y, for all positive real values of y greater than 0. We might, in fact, write (7) in the form

$$x = \operatorname{Exp}^{-1} y \quad (8);$$

and it is obvious that *$Exp^{-1}y$ may, for real values of y greater than 0, be taken to be identical with $\log y$ as previously defined.*

If we consider the purely imaginary arguments $+iy$ and $-iy$, we have, by the definition of $\operatorname{Exp} z$,

$$\begin{aligned}\operatorname{Exp}(+iy) &= 1 + iy - y^2/2! - iy^3/3! + y^4/4! + iy^5/5! - \ldots, \\ &= (1 - y^2/2! + y^4/4! - \ldots) \\ &\qquad + i(y - y^3/3! + y^5/5! - \ldots), \\ &= \cos y + i \sin y \qquad (9);\end{aligned}$$

$$\begin{aligned}\operatorname{Exp}(-iy) &= (1 - y^2/2! + y^4/4! - \ldots) \\ &\qquad - i(y - y^3/3! + y^5/5! - \ldots), \\ &= \cos y - i \sin y \qquad (9'),\end{aligned}$$

by § 14.

Finally, by the addition theorem,

$$\begin{aligned}\operatorname{Exp}(x + yi) &= \operatorname{Exp}(x) \operatorname{Exp}(yi), \\ &= e^x(\cos y + i \sin y) \qquad (10).\end{aligned}$$

The General Exponential Function is therefore always expressible by means of the Elementary Transcendental Functions e^x, $\cos y$, $\sin y$, already defined.

Inasmuch as the function $\operatorname{Exp} z$ possesses all the characteristics which e^z has when z is real, and is identical with e^z in all cases where e^z is already defined, it is usual to employ the notation e^z for $\operatorname{Exp} z$ in all cases. This simply amounts to defining e^z in all cases by means of the equation

$$e^z = 1 + z + z^2/2! + z^3/3! + \ldots,$$

which, as we now see, will lead to no contradiction.

§ 19.] *Graphic Discussion of the General Exponential Function —Definition of the General Logarithmic Function.* Let w be defined as a function of z by the equation

$$w = \operatorname{Exp} z \qquad (1);$$

and let $z = x + yi$, and $w = u + vi = s(\cos\phi + i \sin\phi)$. Then, since $\operatorname{Exp}(x + yi) = e^x(\cos y + i \sin y)$, we have

$$s(\cos\phi + i \sin\phi) = e^x(\cos y + i \sin y) \qquad (2).$$

Hence

$$s = e^x, \quad \phi = y \qquad (3),$$

where we take the simplest relation between the amplitudes that will suit our purpose.

Suppose now that in the z-plane (Fig. 7) we draw a straight line $\bar{2}'\bar{1}'1'2'$ parallel to the y-axis, and at a distance x from it. Then,

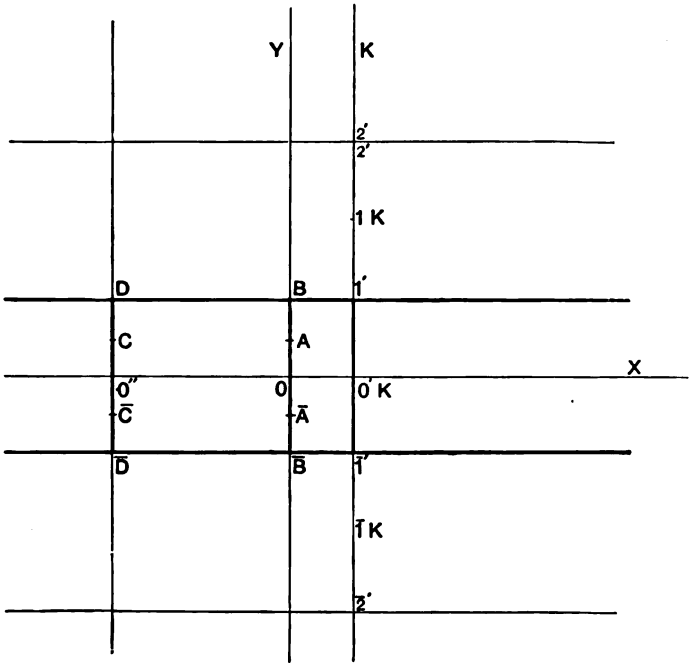


FIG. 7.

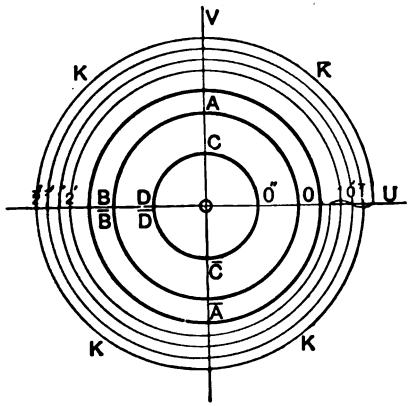


FIG. 8.

if we cause z to describe this line, x will remain constant, and therefore e^x will remain constant; that is to say, the point w will describe a circle (K) (Fig. 8) whose radius is e^x about the origin in the w-plane. If we draw parallels to the x-axis in the z-plane, at distances $0'1' = \pi$, $0'2' = 3\pi$, . . ., above, and $0'\bar{1}' = \pi$, $0'\bar{2}' = 3\pi$, . . ., below, then, as y varies from $-\pi$ to $+\pi$, z travels from $\bar{1}'$ to $1'$; as y varies from $+\pi$ to $+3\pi$, z travels from $1'$ to $2'$, and so on; and each of these pieces of the straight line corresponds to the circumference of the circle K taken once over. To make the correspondence clearer, we may, as heretofore, replace the repeated circle K by a spiral supposed ultimately to coincide with it. Then to the infinite number of pieces, each equal to 2π, on the line K corresponds an infinite number of spires of the spiral K.

In like manner, to every parallel to the y-axis in the z-plane corresponds a spiral circle in the w-plane concentric with the circle K. To the axis of y itself corresponds the spiral circle $\bar{B}\bar{A}OAB$; to the parallel $\bar{D}O''D$ to the left of the y-axis the spiral circle $\bar{D}O''D$; and so on.

To the whole strip between the infinite parallels DB and $\bar{D}\bar{B}$ corresponds the whole of the w-plane taken once over; namely, to the right half of the infinite strip corresponds the part of the w-plane outside the circle $\bar{B}\bar{A}OAB$; to the left half of the strip the part of the w-plane inside the circle $\bar{B}\bar{A}OAB$.

To each such parallel strip of the z-plane corresponds the whole of the w-plane taken once over.

Hence the values of w are repeated infinitely often, and we see *that the equation* (1) *defines w as a continuous periodic function of z having the period $2\pi i$.*

Conversely, the above graphic discussion shows that the equation (1) *defines z as a continuous ∞-ple valued function of w.*

Taking the latter view, we might write the equation in the form

$$z = \mathrm{Exp}^{-1} w \qquad (1').$$

Instead of $\mathrm{Exp}^{-1}w$ we shall, for the most part, employ the more usual notation $\mathrm{Log}\, w$, using, however, for the present at least, a capital letter to distinguish from the one-valued function $\log y$, which arises from the inversion of $y = e^x$, when x and y are both restricted to be real.

In accordance with the view we are now taking, we may write (3) in the form

$$x = \log s, \quad y = \phi.$$

Hence $$z = \mathrm{Log}\, w$$

gives $$x + yi = \mathrm{Log}\, \{s(\cos\phi + i \sin\phi)\},$$

where $$x = \log s, \text{ and } y = \phi.$$

In other words, we have

$$\mathrm{Log}\, w = \log (\mathrm{mod}\, w) + i \,\mathrm{amp}\, (w) \qquad (2');$$

and, if we cause ϕ (that is, amp (w)) to vary continuously through all values between $-\infty$ and $+\infty$, then the left-hand side of the equation (2′) will vary continuously through all values which $\mathrm{Log}\, w$ can assume for a given value of mod w.

If we confine ϕ to lie between $-\pi$ and $+\pi$, then $\mathrm{Log}\, w$ becomes one-valued; and we have

$$\mathrm{Log}\, w = \log s + i\phi \qquad (4),$$

where $s = \mathrm{mod}\, w = \sqrt{(u^2 + v^2)}$, and $\cos\phi = u / \sqrt{(u^2 + v^2)}$, $\sin\phi = v / \sqrt{(u^2 + v^2)}$, $-\pi \geqslant \phi \geqslant +\pi$.

This is called *the principal branch of Log w*; and we may denote it by z.

It is obvious from the graphic discussion that, *if* z_t *or* ${}_t\mathrm{Log}\, w$ *denote the value of Log w in its t-th branch, z being the value in the principal branch corresponding to the same value of w (that is, a value of w whose amplitude differs by an integral multiple of* 2π*), then*

$$\begin{aligned} {}_t\mathrm{Log}\, w = z_t &= z + 2t\pi i, \\ &= \log s + i(\phi + 2t\pi) \qquad (5), \end{aligned}$$

where ϕ *is the amplitude (confined between the limits* $-\pi$ *and* $+\pi$*) of w, and t is any integer positive or negative.*

If w be a real positive quantity, $= u$ say, then $s = \mathrm{mod}\, w = u$, $\phi = \mathrm{amp}\, w = 0$; and we have, for the principal value of $\mathrm{Log}\, u$,

$$\mathrm{Log}\, u = \log u.$$

Hence, for real positive values of the argument, log u is the principal value of Log u. The other values are of course given by ${}_t\mathrm{Log}\, u = \log u + 2t\pi i$, t being the order of the branch.

We have also the following particular principal values:—

$$\mathrm{Log}\,(+i) = \tfrac{1}{2}\pi i,$$
$$\mathrm{Log}\,(-i) = -\tfrac{1}{2}\pi i,$$
$$\mathrm{Log}\,(-1) = \pm \pi i:$$

the principal value in the last case is not determinate until we know the amplitude; and the same applies to all purely real negative arguments.

§ 20.] *Definition of $\mathrm{Exp}_a z$.* The meaning of a^z, or, as it is sometimes written, $\mathrm{Exp}_a z$, has not as yet been defined for values of z which are not real and commensurable.

We now define it to mean $\mathrm{Exp}\,(z.{}_t\mathrm{Log}\, a)$, where ${}_t\mathrm{Log}\, a$ is the t-th branch of the inverse function $\mathrm{Log}\, a$, and t may have any positive or negative integral value including 0.

Thus defined, a^z is in general multiple-valued to an infinite extent. In fact, since ${}_t\mathrm{Log}\, a = \log s + i(\phi + 2t\pi)$, where $s = \mathrm{mod}\, a$, and $\phi = \mathrm{amp}\, a\ (-\pi < \phi < +\pi)$, we have, if $z = x + yi$,

$$\begin{aligned} a^{x+yi} &= \mathrm{Exp}\,[(x+yi)\{\log s + i(\phi + 2t\pi)\}], \\ &= \mathrm{Exp}\,[\{x\log s - (\phi + 2t\pi)y\} + i\{y\log s + (\phi + 2t\pi)x\}], \\ &= \exp\{x\log s - (\phi + 2t\pi)y\}.[\cos\{y\log s + (\phi + 2t\pi)x\} \\ &\qquad + i\sin\{y\log s + (\phi + 2t\pi)x\}] \quad (1). \end{aligned}$$

If we put $t = 0$, that is, take the principal branch of $\mathrm{Log}\, a$, in the defining equation, then we get what may be called the principal branch of a^{x+yi}, namely,

$$\begin{aligned} a^{x+yi} &= \mathrm{Exp}\,(z\,\mathrm{Log}\, a), \\ &= \exp\{x\log s - \phi y\}.[\cos\{y\log s + \phi x\} + i\sin\{y\log s + \phi x\}] \quad (2). \end{aligned}$$

The value given in (1) would then be called the t-th branch, and might for distinction be denoted by ${}_t a^{x+yi}$ or by ${}_t\mathrm{Exp}_a(x+yi)$.

It is important to notice that the *above definition of a^z agrees with that already given for real commensurable values of z provided we take the corresponding branches.* In fact, when $y = 0$, (1) gives

$$a^x = \exp(x\log s).[\cos(\phi + 2t\pi)x + i\sin(\phi + 2t\pi)x];$$

that is, if $x = p/q$,

$$[s(\cos\phi + i\sin\phi)]^{p/q} = s^{p/q}[\cos.(\phi + 2t\pi)p/q + i\sin.(\phi + 2t\pi)p/q] \quad (3);$$

the right-hand side of which is the t-th branch of the left as ordinarily defined.

Cor. *It follows from the above that when x is an incommensurable number the function a^x has an infinite number of values even when both a and x are real.*

The principal value of a^x, however, when both a and x are real and a is positive, is $\exp(x \log a)$, which differs infinitely little from the principal value of $a^{x'}$, if x' be a commensurable quantity differing infinitely little from x.

§ 21.] *The Addition Theorem for Log z.*

By the result of § 19 we have

$${}_m\mathrm{Log}\, w_1 + {}_n\mathrm{Log}\, w_2 = \log(\mathrm{mod}\, w_1) + \log(\mathrm{mod}\, w_2) + i\,\mathrm{amp}\, w_1 + i\,\mathrm{amp}\, w_2 + 2(m+n)\pi i.$$

Now (chap. xii., § 15) $\mathrm{mod}\, w_1 \,\mathrm{mod}\, w_2 = \mathrm{mod}\,(w_1 w_2)$, and, if $\mathrm{amp}\,(w_1 w_2)$ were not restricted in any way, we should have $\mathrm{amp}\, w_1 + \mathrm{amp}\, w_2 = \mathrm{amp}\,(w_1 w_2)$. Since, however, $\mathrm{amp}\,(w_1 w_2)$ is restricted in the definition of $\mathrm{Log}\,(w_1 w_2)$ to lie between $-\pi$ and π, we have

$$\mathrm{amp}\, w_1 + \mathrm{amp}\, w_2 = \mathrm{amp}\,(w_1 w_2) + 2\rho\pi,$$

where $\rho = +1$, 0, or -1 according as $\mathrm{amp}\, w_1 + \mathrm{amp}\, w_2 > +\pi$ lies between $+\pi$ and $-\pi$, or $< -\pi$. Hence we have

$${}_m\mathrm{Log}\, w_1 + {}_n\mathrm{Log}\, w_2 = {}_{m+n+\rho}\mathrm{Log}\,(w_1 w_2) \quad (1),$$

where ρ is as defined.

In like manner, it may be shown that

$${}_m\mathrm{Log}\, w_1 - {}_n\mathrm{Log}\, w_2 = {}_{m-n+\rho}\mathrm{Log}\,(w_1 w_2) \quad (2),$$

where $\rho = +1$, 0, or -1 according as $\mathrm{amp}\, w_1 - \mathrm{amp}\, w_2 > +\pi$, between $+\pi$ and $-\pi$, or $< -\pi$.

Taking the definition of a^{x+yi} given in § 20, and making use of equation (1) of that paragraph, we have

$${}_k\mathrm{Log}\,{}_ta^{x+yi} = \log(\mathrm{mod}\,{}_ta^{x+yi}) + (\mathrm{amp}\,{}_ta^{x+yi} + 2k\pi)i,$$
$$= x\log s - (\phi + 2t\pi)y + \{y\log s + (\phi + 2t\pi)x\}i + 2(k+l)\pi i,$$

where l is an integer, positive or negative, chosen so that

$$-\pi < y\log s + (\phi + 2t\pi)x + 2l\pi < +\pi.$$

Hence

$${}_k\mathrm{Log}\,{}_ta^{x+yi} = (x+yi)\{\log s + (\phi + 2t\pi)i\} + 2(k+l)\pi i,$$
$$= (x+yi)\,{}_t\mathrm{Log}\,a + 2(k+l)\pi i \qquad (3).$$

The equations (1), (2), (3) are generalisations of formulæ for $\log x$ with which the reader is already familiar.

If we confine each of the multiple-valued functions ${}_t\mathrm{Log}$ and ${}_t\mathrm{Exp}_a$ to its principal branch, we have

$$\mathrm{Log}\,a^{x+yi} = (x+yi)\,\mathrm{Log}\,a + 2l\pi i \qquad (3'),$$

where l is so chosen that

$$-\pi < y\log s + \phi x + 2l\pi < +\pi.$$

§ 22.] *Expansion of ${}_t\mathrm{Log}\,(1+z)$ in powers of z.*

Consider first the principal branch of the function $\mathrm{Log}\,(1+z)$. By the definition and discussion of § 20, we see that, when x is any real quantity, the principal branch of $(1+z)^x$ has for its value $\mathrm{Exp}\{x\,\mathrm{Log}\,(1+z)\}$. Hence we have

$$(1+z)^x = 1 + \{x\,\mathrm{Log}\,(1+z)\} + \{x\,\mathrm{Log}\,(1+z)\}^2/2\,! + \ldots;$$

and, since the series $1 + \Sigma_x \mathrm{C}_n z^n$ represents the principal branch of $(1+z)^x$, we have

$$1 + \Sigma_x \mathrm{C}_n z^n = 1 + \{x\,\mathrm{Log}\,(1+z)\} + \ldots\ .$$

Now all the conditions involved in the reasoning of chap. xxviii., § 9, will be fulfilled here, provided the complex variable z be so restricted that $\mathrm{mod}\,z < 1$.

Hence, if $\mathrm{mod}\,z < 1$, we must have, as before,

$$\mathrm{Log}\,(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \ldots \qquad (1).$$

In other words, so long as mod $z<1$, the series $z - z^2/2 + z^3/3 - \ldots$ represents the principal branch of $Exp^{-1}(1+z)$.

Cor. *Since ${}_t\mathrm{Log}\,(1+z) = \mathrm{Log}\,(1+z) + 2t\pi i$, we have*

$${}_t\mathrm{Log}\,(1+z) = 2t\pi i + z - z^2/2 + z^3/3 - z^4/4 + \ldots \qquad (2),$$

which gives us an expansion for the t-th branch of $\mathrm{Exp}^{-1}(1+z)$ within the region of the z-plane for which $\mathrm{mod}\, z < 1$.

It follows readily, from the principles of chap. xxvi., § 9, that when $\mathrm{mod}\, z = 1$ the series $z - z^2/2 + z^3/3 - \ldots$ is convergent, provided $\mathrm{amp}\, z \neq \pm\pi$ (other odd multiples of π are not in question here). Hence, by the theorem of Abel so often quoted already, the expansion-formulæ (1) and (2) will still hold when $\mathrm{mod}\, z = 1$, provided $\mathrm{amp}\, z \neq \pm\pi$.

GENERALISATION OF THE CIRCULAR FUNCTIONS—INTRODUCTION OF THE HYPERBOLIC FUNCTIONS.

§ 23.] *General definition of Cos z, Sin z, Tan z, Cot z, Sec z, Cosec z.* Since the series $1 - z^2/2! + z^4/4! - \ldots$, $z - z^3/3! + z^5/5! - \ldots$ are convergent for all values of z having a finite modulus, however large, they are each single-valued continuous functions of z throughout the z-plane. Let us call the functions thus defined $\mathrm{Cos}\, z$ and $\mathrm{Sin}\, z$, using capital initial letters, for the present, to distinguish from the geometrically defined real functions $\cos x$ and $\sin x$. We thus have

$$\mathrm{Cos}\, z = 1 - z^2/2! + z^4/4! - \ldots \qquad (1),$$
$$\mathrm{Sin}\, z = z - z^3/3! + z^5/5! - \ldots \qquad (2).$$

We also define $\mathrm{Tan}\, z$, $\mathrm{Cot}\, z$, $\mathrm{Sec}\, z$, $\mathrm{Cosec}\, z$ by the following equations:—

$$\left.\begin{aligned} &\mathrm{Tan}\, z = \mathrm{Sin}\, z/\mathrm{Cos}\, z; \quad \mathrm{Cot}\, z = \mathrm{Cos}\, z/\mathrm{Sin}\, z; \\ &\mathrm{Sec}\, z = 1/\mathrm{Cos}\, z; \quad \mathrm{Cosec}\, z = 1/\mathrm{Sin}\, z. \end{aligned}\right\} \qquad (3).$$

In the first place, we observe that when z is real, $= x$ say, we have, by § 14,

$$\mathrm{Cos}\, x = 1 - x^2/2! + x^4/4! - \ldots = \cos x,$$
$$\mathrm{Sin}\, x = x - x^3/3! + x^5/5! - \ldots = \sin x;$$

so that, when the argument is real, the more general functions Cos., Sin., Tan., Cot., Sec., Cosec. coincide with the functions cos., sin., tan., cot., sec., cosec. already geometrically defined for real values of the argument.

Since

$$1 - z^2/2! + z^4/4! - \ldots = \frac{1}{2}\{\mathrm{Exp}\,(iz) + \mathrm{Exp}\,(-iz)\},$$

$$z - z^3/3! + z^5/5! - \ldots = \frac{1}{2i}\{\mathrm{Exp}\,(iz) - \mathrm{Exp}\,(-iz)\},$$

it follows from (1) and (2) *that we have for all values of* z

$$\left.\begin{aligned} \mathrm{Cos}\,z &= \frac{1}{2}\{\mathrm{Exp}\,(iz) + \mathrm{Exp}\,(-iz)\}, \\ \mathrm{Sin}\,z &= \frac{1}{2i}\{\mathrm{Exp}\,(iz) - \mathrm{Exp}\,(-iz)\}; \end{aligned}\right\} \qquad (4)^*$$

with corresponding expressions for Tan z, Cot z, Sec z, and Cosec z.

By (4) we have

$$\begin{aligned} \mathrm{Cos}^2 z + \mathrm{Sin}^2 z &= \tfrac{1}{4}[\{\mathrm{Exp}\,(iz)\}^2 + \{\mathrm{Exp}\,(-iz)\}^2 + 2\,\mathrm{Exp}\,(iz)\,\mathrm{Exp}\,(-iz) \\ &\quad - \{\mathrm{Exp}\,(iz)\}^2 + \mathrm{Exp}\,\{(-iz)\}^2 - 2\,\mathrm{Exp}\,(iz)\,\mathrm{Exp}\,(-iz)]. \end{aligned}$$

Hence, bearing in mind that we have, by the exponential addition theorem,

$$\mathrm{Exp}\,(iz)\,\mathrm{Exp}\,(-iz) = \mathrm{Exp}\,(iz - iz) = \mathrm{Exp}\,0 = 1,$$

we see that

$$\mathrm{Cos}^2 z + \mathrm{Sin}^2 z = 1 \qquad (5),$$

from which we deduce at once, for the generalised functions, all the algebraical relations which were formerly established for the circular functions properly so called.

We also see, from (4), that $\mathrm{Cos}\,(-z) = \mathrm{Cos}\,z$ and $\mathrm{Sin}\,(-z) = -\mathrm{Sin}\,z$; that is to say, Cos z is an even, and Sin z an odd function of z.

Since, by (4), we have

$$\begin{aligned} \mathrm{Cos}\,z + i\,\mathrm{Sin}\,z &= \mathrm{Exp}\,(iz), \\ \mathrm{Cos}\,z - i\,\mathrm{Sin}\,z &= \mathrm{Exp}\,(-iz), \end{aligned}$$

* These formulæ were first given by Euler. See *Int. in Anal. Inf.*, t. i., § 138. He gave, however, no sufficient justification for their usage, resting merely on a bold analogy, as Bernoulli and Demoivre had done before him.

it follows from the exponential addition theorem, namely,

$$\text{Exp}\,(iz_1 + iz_2) = \text{Exp}\,(iz_1)\ \text{Exp}\,(iz_2),$$

that

$$\text{Cos}\,(z_1 + z_2) + i\,\text{Sin}\,(z_1 + z_2) = (\text{Cos}\,z_1 + i\,\text{Sin}\,z_1)\,(\text{Cos}\,z_2 + i\,\text{Sin}\,z_2)$$
$$= (\text{Cos}\,z_1\,\text{Cos}\,z_2 - \text{Sin}\,z_1\,\text{Sin}\,z_2) + i\,(\text{Sin}\,z_1\,\text{Cos}\,z_2 + \text{Cos}\,z_1\,\text{Sin}\,z_2).*$$

Hence, changing the signs of z_1 and z_2, and remembering that Cos is even and Sin odd, we have

$$\text{Cos}\,(z_1 + z_2) - i\,\text{Sin}\,(z_1 + z_2) = (\text{Cos}\,z_1\,\text{Cos}\,z_2 - \text{Sin}\,z_1\,\text{Sin}\,z_2)$$
$$- i(\text{Sin}\,z_1\,\text{Cos}\,z_2 + \text{Cos}\,z_1\,\text{Sin}\,z_2).$$

Therefore, by addition and subtraction, we deduce

$$\left.\begin{aligned}\text{Cos}\,(z_1 + z_2) &= \text{Cos}\,z_1\,\text{Cos}\,z_2 - \text{Sin}\,z_1\,\text{Sin}\,z_2\,;\\ \text{Sin}\,(z_1 + z_2) &= \text{Sin}\,z_1\,\text{Cos}\,z_2 + \text{Cos}\,z_1\,\text{Sin}\,z_2.\end{aligned}\right\} \qquad (6).$$

In other words, the addition theorem for Cos. and Sin. in general is identical with that for cos. and sin.

By (6) we have

$$\text{Cos}\,(z + 2n\pi) = \text{Cos}\,z\,\text{Cos}\,2n\pi - \text{Sin}\,z\,\text{Sin}\,2n\pi,$$

that is, if n be any positive or negative integer, so that $\text{Cos}\,2n\pi = \cos 2n\pi = 1$, and $\text{Sin}\,2n\pi = \sin 2n\pi = 0$, then

$$\text{Cos}\,(z + 2n\pi) = \text{Cos}\,z.$$

In like manner, $\text{Sin}\,(z + 2n\pi) = \text{Sin}\,z$; $\text{Tan}\,(z + n\pi) = \text{Tan}\,z$; &c. That is to say, *the Generalised Circular Functions have the same real periods as the Circular Functions proper.*

Just in the same way, we can establish all the relations for half and quarter periods given in equations (3) of § 2. Thus, for example,

$$\text{Cos}\,(\pi + z) = \text{Cos}\,\pi\,\text{Cos}\,z - \text{Sin}\,\pi\,\text{Sin}\,z,$$
$$= \cos\pi\,\text{Cos}\,z - \sin\pi\,\text{Sin}\,z,$$
$$= -\,\text{Cos}\,z.$$

Also all the equations (5), (6), (7) *of* § 2 *will hold for the generalised functions*; for they are merely deductions from the addition theorem.

* We cannot here equate the coefficients of i, &c., on both sides, because $\text{Sin}\,(z_1 + z_2)$, &c., are no longer necessarily real.

§ 24.] We proceed next to discuss briefly the variation of the generalised circular functions.

Consider first the case where the argument is wholly imaginary, say $z = iy$. In this case we have

$$\begin{aligned} \operatorname{Cos}(iy) &= \frac{1}{2}\{\operatorname{Exp}(iiy) + \operatorname{Exp}(-iiy)\}, \\ &= \frac{1}{2}(e^{-y} + e^{y}) \qquad (1); \\ \operatorname{Sin}(iy) &= \frac{1}{2i}(e^{-y} - e^{y}), \\ &= \frac{i}{2}(e^{y} - e^{-y}) \qquad (2). \end{aligned}$$

We are thus naturally led to introduce and discuss two new functions, namely, $\frac{1}{2}(e^{y} + e^{-y})$ and $\frac{1}{2}(e^{y} - e^{-y})$, which are called the Hyberbolic Cosine and the Hyperbolic Sine. These functions are usually denoted by $\cosh y$ and $\sinh y$; so that, for real values of y, $\cosh y$ and $\sinh y$ are defined by the equations

$$\cosh y = \tfrac{1}{2}(e^{y} + e^{-y}), \quad \sinh y = \tfrac{1}{2}(e^{y} - e^{-y}) \qquad (3).$$

In general, when y is complex, we define the more general functions $\operatorname{Cosh} z$ and $\operatorname{Sinh} z$ by the equations

$$\begin{aligned} \operatorname{Cosh} z &= \tfrac{1}{2}\{\operatorname{Exp}(z) + \operatorname{Exp}(-z)\}, \\ \operatorname{Sinh} z &= \tfrac{1}{2}\{\operatorname{Exp}(z) - \operatorname{Exp}(-z)\} \qquad (3'). \end{aligned}$$

We also introduce $\tanh y$, $\coth y$, $\operatorname{sech} y$, and $\operatorname{cosech} y$ by the definitions

$$\begin{aligned} &\tanh y = \sinh y/\cosh y, \quad \coth y = \cosh y/\sinh y; \\ &\operatorname{sech} y = 1/\cosh y, \quad \operatorname{cosech} y = 1/\sinh y; \end{aligned}$$

and the more general functions $\operatorname{Tanh} z$, $\operatorname{Coth} z$, &c., in precisely the same way.

From the equations (1) and (2) we have

$$\left.\begin{aligned} &\operatorname{Cos}(iy) = \cosh y, \quad \operatorname{Sin}(iy) = i\sinh y; \\ &\operatorname{Tan}(iy) = i\tanh y, \quad \operatorname{Cot}(iy) = -i\coth y; \\ &\operatorname{Sec}(iy) = \operatorname{sech} y, \quad \operatorname{Cosec}(iy) = -i\operatorname{cosech} y; \end{aligned}\right\} \qquad (4),$$

and, of course, in general, $\operatorname{Cos} iz = \operatorname{Cosh} z$, &c.

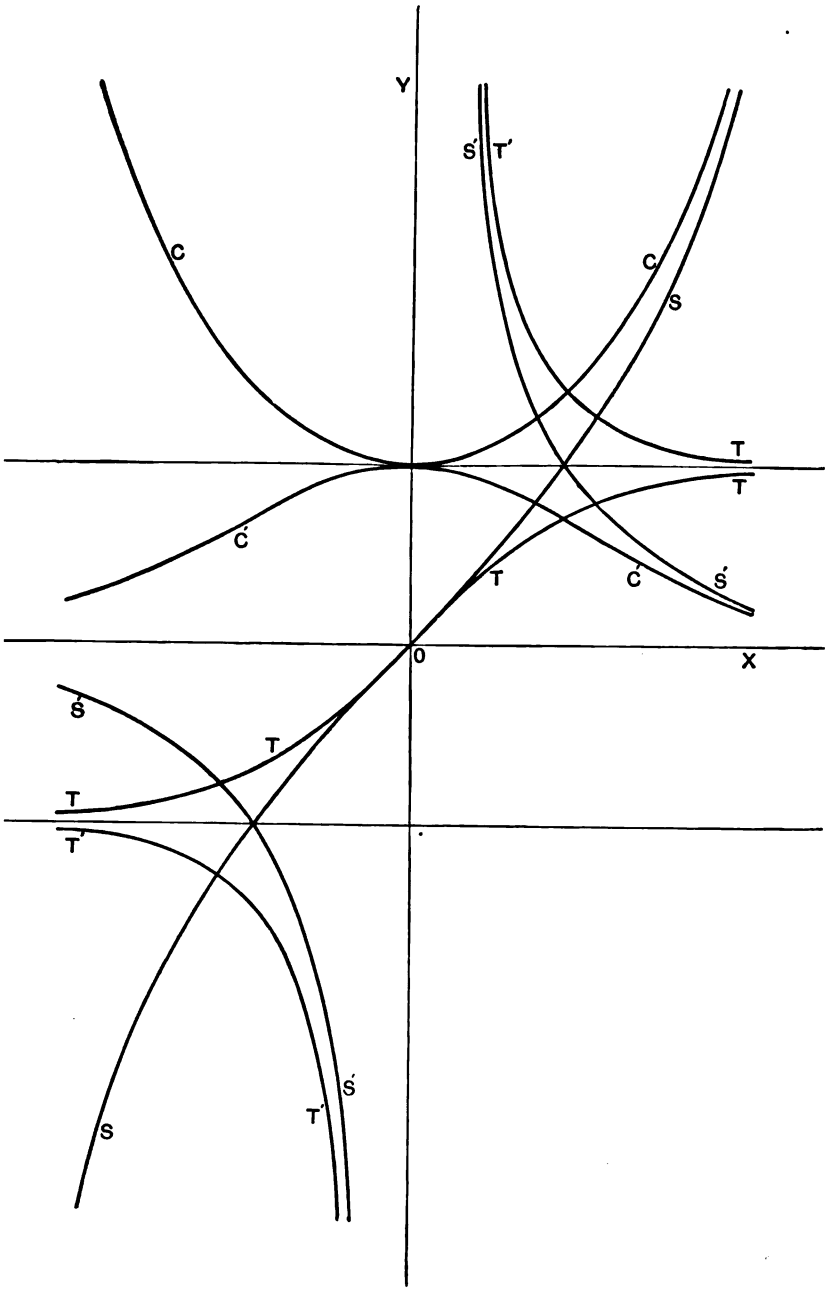


FIG. 9.

The discussion of the variation of the circular functions for purely imaginary arguments reduces, therefore, to the discussion of the hyperbolic functions for purely real arguments.

§ 25.] *Variation of the Hyperbolic Functions for real arguments.* The graphs of $y = \cosh x$, $y = \sinh x$, &c., are given in Fig. 9 as follows :—

$\cosh x$, CC; $\sinh x$, SOS;
$\coth x$, T′T′T′T′; $\tanh x$, TTOTT;
$\operatorname{sech} x$, C′C′; $\operatorname{cosech} x$, S′S′S′S′.

By studying these curves the reader will at once see the truth of the following remarks regarding the direct and inverse hyperbolic functions of a real argument.

(1) $\cosh x$ is an even function of x, having two positive infinite values corresponding to $x = \pm\infty$, no zero value, and a minimum value 1 corresponding to $x = 0$.

$\cosh^{-1} y$ is a two-valued function of y, defined for the continuum $1 \not> y \not> \infty$, having a zero value corresponding to $y = 1$, and infinite values corresponding to $y = \infty$, but no turning value.

(2) $\sinh x$ is an odd function of x, having a zero value when $x = 0$, and positive and negative infinite values when $x = +\infty$ and $x = -\infty$ respectively.

$\sinh^{-1} y$ is one-valued, and defined for all values of y; it has a zero value for $y = 0$, and positive and negative infinite values when $y = +\infty$ and $= -\infty$ respectively.

(3) $\tanh x$ is an odd function, has a zero value for $x = 0$, positive maximum + 1, and negative minimum − 1, corresponding to $x = +\infty$ and $x = -\infty$ respectively.

$\tanh^{-1} y$ is a one-valued odd function, defined for $-1 \not> y \not> +1$; has zero value for $y = 0$, positive and negative infinite values corresponding to $y = +1$ and $y = -1$.

(4) $\coth x$ is an odd function, having no zero value, but an infinite value for $x = 0$, and minimum + 1, and maximum − 1, for $x = +\infty$ and $y = -\infty$ respectively.

$\coth^{-1} y$ is a one-valued odd function, defined, except for the continuum $-1 \not> y \not> +1$, having positive and negative infinite values corresponding to $y = +1$ and $y = -1$ respectively, and a zero value for $y = \infty$.

(5) $\operatorname{sech} x$ is an even function, having a maximum + 1 for $x = 0$, and a zero value for $x = \pm\infty$.

$\operatorname{sech}^{-1} y$ is a two-valued function, defined for $0 \not> y \not> 1$, having a zero value for $y = 1$, and infinite values for $y = 0$.

(6) $\operatorname{cosech} x$ is an odd function, having zero values for $x = \pm\infty$, and an infinite value for $x = 0$.

$\operatorname{cosech}^{-1} y$ is one-valued and defined for all values of y, having zero values for $y = \pm\infty$, and infinite values for $y = 0$.

§ 26.] *Logarithmic expressions for $\cosh^{-1} y$, $\sinh^{-1} y$, &c.*

If $x = \cosh^{-1} y$, we have

$$y = \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \qquad (1).$$

Therefore

$$\pm\sqrt{(y^2 - 1)} = \tfrac{1}{2}(e^x - e^{-x}) \qquad (2).$$

From (1) and (2),

$$e^x = y \pm \sqrt{(y^2 - 1)}.$$

Hence

$$x = \log\{y \pm \sqrt{(y^2 - 1)}\};$$

that is,

$$\cosh^{-1} y = \log\{y \pm \sqrt{(y^2 - 1)}\} \qquad (3),$$

the upper sign corresponding to the positive or principal branch of $\cosh^{-1} y$, the lower sign to the negative branch.

In like manner we can show that

$$\sinh^{-1} y = \log\{y + \sqrt{(y^2 + 1)}\} \qquad (4);$$

$$\tanh^{-1} y = \tfrac{1}{2}\log\{(1 + y)/(1 - y)\} \qquad (5);$$

$$\coth^{-1} y = \tfrac{1}{2}\log\{(y + 1)/(y - 1)\} \qquad (6);$$

$$\operatorname{sech}^{-1} y = \log[\{1 \pm \sqrt{(1 - y^2)}\}/y] \qquad (7);$$

$$\operatorname{cosech}^{-1} y = \log[\{1 + \sqrt{(1 + y^2)}\}/y] \qquad (8).$$

§ 27.] *Properties of the General Hyperbolic Functions analogous to those of the Circular Functions.*

We have already seen that the properties of the circular functions, both for real and for complex values of the argument, might be deduced from the equations of Euler, namely,

$$\left.\begin{aligned} \operatorname{Cos} z &= \tfrac{1}{2}\{\operatorname{Exp}(+iz) + \operatorname{Exp}(-iz)\}; \\ \operatorname{Sin} z &= \tfrac{1}{2i}\{\operatorname{Exp}(+iz) - \operatorname{Exp}(-iz)\} \end{aligned}\right\} \qquad (A).$$

In like manner, the properties of the general hyperbolic functions spring from the defining equations

$$\left.\begin{aligned} \operatorname{Cosh} z &= \tfrac{1}{2}\{\operatorname{Exp}(+z) + \operatorname{Exp}(-z)\}; \\ \operatorname{Sinh} z &= \tfrac{1}{2}\{\operatorname{Exp}(+z) - \operatorname{Exp}(-z)\} \end{aligned}\right\} \quad \text{(B).}$$

We should therefore expect a close analogy between the functional relations in the two cases. In what follows we state those properties of the hyperbolic functions which are analogous to the properties of the circular functions tabulated in § 2. The demonstrations are for the most part omitted; they all depend on the use of the equations (B), combined with the properties of the general exponential function, already fully discussed.

The demonstrations might also be made to depend on the relations connecting the general circular functions with the general hyperbolic functions given in § 24,* namely,

$$\left.\begin{aligned} \operatorname{Cosh} z = \operatorname{Cos} iz, \quad & i \operatorname{Sinh} z = \operatorname{Sin} iz; \\ + i\operatorname{Tanh} z = \operatorname{Tan} iz, \quad & - i \operatorname{Coth} z = \operatorname{Cot} iz; \\ \operatorname{Sech} z = \operatorname{Sec} iz, \quad & - i \operatorname{Cosech} z = \operatorname{Cosec} iz; \end{aligned}\right\} \quad \text{(C).}$$

Algebraic Relations.

$$\operatorname{Cosh}^2 z - \operatorname{Sinh}^2 z = 1, \quad \operatorname{Sech}^2 z + \operatorname{Tanh}^2 z = 1 \qquad (1),$$

&c.

Periodicity.—All the hyperbolic functions have the period $2\pi i$; and $\operatorname{Tanh} z$ and $\operatorname{Coth} z$ have the smaller period πi.

Thus

$$\left.\begin{aligned} &\operatorname{Cosh}(z + 2n\pi i) = \operatorname{Cosh} z; \ \&c. \\ &\operatorname{Tanh}(z + n\pi i) = \operatorname{Tanh} z; \ \&c. \end{aligned}\right\} \quad (2).$$

Also,

$$\left.\begin{aligned} &\operatorname{Cosh}(\pi i \pm z) = -\operatorname{Cosh} z, \quad \operatorname{Sinh}(\pi i \pm z) = \mp \operatorname{Sinh} z; \\ &\operatorname{Cosh}(\tfrac{1}{2}\pi i \pm z) = \pm i \operatorname{Sinh} z, \quad \operatorname{Sinh}(\tfrac{1}{2}\pi i \pm z) = i \operatorname{Cosh} z; \\ &\operatorname{Tanh}(\tfrac{1}{2}\pi i \pm z) = \pm \operatorname{Coth} z, \quad \operatorname{Coth}(\tfrac{1}{2}\pi i \pm z) = \pm \operatorname{Tanh} z; \end{aligned}\right\} \quad (3).$$

Addition Formulæ.

$$\left.\begin{aligned} &\operatorname{Cosh}(z_1 \pm z_2) = \operatorname{Cosh} z_1 \operatorname{Cosh} z_2 \pm \operatorname{Sinh} z_1 \operatorname{Sinh} z_2; \\ &\operatorname{Sinh}(z_1 \pm z_2) = \operatorname{Sinh} z_1 \operatorname{Cosh} z_2 \pm \operatorname{Cosh} z_1 \operatorname{Sinh} z_2; \\ &\operatorname{Tanh}(z_1 \pm z_2) = (\operatorname{Tanh} z_1 \pm \operatorname{Tanh} z_2)/(1 \pm \operatorname{Tanh} z_1 \operatorname{Tanh} z_2). \end{aligned}\right\} \quad (5).$$

* This connection furnishes the simplest *memoria technica* for the hyperbolic formulæ.

$$\left.\begin{aligned}
&\operatorname{Cosh} z_1 + \operatorname{Cosh} z_2 = 2 \operatorname{Cosh} \tfrac{1}{2}(z_1 + z_2) \operatorname{Cosh} \tfrac{1}{2}(z_1 - z_2);\\
&\operatorname{Cosh} z_1 - \operatorname{Cosh} z_2 = 2 \operatorname{Sinh} \tfrac{1}{2}(z_1 + z_2) \operatorname{Sinh} \tfrac{1}{2}(z_1 - z_2);\\
&\operatorname{Sinh} z_1 \pm \operatorname{Sinh} z_2 = 2 \operatorname{Sinh} \tfrac{1}{2}(z_1 \pm z_2) \operatorname{Cosh} \tfrac{1}{2}(z_1 \mp z_2).
\end{aligned}\right\} (6).$$

$$\left.\begin{aligned}
&\operatorname{Cosh} z_1 \operatorname{Cosh} z_2 = \tfrac{1}{2} \operatorname{Cosh} (z_1 + z_2) + \tfrac{1}{2} \operatorname{Cosh} (z_1 - z_2);\\
&\operatorname{Sinh} z_1 \operatorname{Sinh} z_2 = \tfrac{1}{2} \operatorname{Cosh} (z_1 + z_2) - \tfrac{1}{2} \operatorname{Cosh} (z_1 - z_2);\\
&\operatorname{Sinh} z_1 \operatorname{Cosh} z_2 = \tfrac{1}{2} \operatorname{Sinh} (z_1 + z_2) + \tfrac{1}{2} \operatorname{Sinh} (z_1 - z_2).
\end{aligned}\right\} (7).$$

$$\left.\begin{aligned}
\operatorname{Cosh} 2z &= \operatorname{Cosh}^2 z + \operatorname{Sinh}^2 z = 2 \operatorname{Cosh}^2 z - 1,\\
&= 1 + 2 \operatorname{Sinh}^2 z = (1 + \operatorname{Tanh}^2 z)/(1 - \operatorname{Tanh}^2 z).\\
\operatorname{Sinh} 2z &= 2 \operatorname{Sinh} z \operatorname{Cosh} z = 2 \operatorname{Tanh} z/(1 - \operatorname{Tanh}^2 z).\\
\operatorname{Tanh} 2z &= 2 \operatorname{Tanh} z/(1 + \operatorname{Tanh}^2 z).
\end{aligned}\right\} (8).$$

Inverse Functions.—Regarding the inverse functions Cosh^{-1}, Sinh^{-1}, &c., it is sufficient to remark that we can always express them by means of the functions Cos^{-1}, Sin^{-1}, &c. Thus, for example, if we have $\operatorname{Cosh}^{-1} z = w$, say, then

$$z = \operatorname{Cosh} w = \operatorname{Cos} iw.$$

Hence $$iw = \operatorname{Cos}^{-1} z;$$

that is, $$w = -i \operatorname{Cos}^{-1} z.$$

So that $$\operatorname{Cosh}^{-1} z = -i \operatorname{Cos}^{-1} z;$$

and so on.

In the practical use of such formulæ, however, we must attend to the multiple-valuedness of Cosh^{-1} and Cos^{-1}. If, for example, in the above equation, the two branches are taken at random in the two inverse functions, then the equation will take the form

$$\operatorname{Cosh}^{-1} z = 2m\pi i \pm i \operatorname{Cos}^{-1} z,$$

where m is some positive or negative integer, whose value and the choice of sign in the ambiguity $\pm$ both depend on circumstances.

§ 28.] *Formulæ for the Hyperbolic Functions analogous to Demoivre's Theorem and its consequences.*

We have at once, from the definition of $\operatorname{Cosh} z$ and $\operatorname{Sinh} z$,

$$\begin{aligned}\operatorname{Cosh}(z_1+z_2+\ldots+z_n)\pm\operatorname{Sinh}(z_1+z_2+\ldots+z_n)&=\operatorname{Exp}\pm(z_1+z_2+\ldots+z_n),\\&=\operatorname{Exp}\pm z_1\operatorname{Exp}\pm z_2\ldots\operatorname{Exp}\pm z_n,\\&=(\operatorname{Cosh}z_1\pm\operatorname{Sinh}z_1)(\operatorname{Cosh}z_1\pm\operatorname{Sinh}z_2)\\&\qquad\ldots(\operatorname{Cosh}z_n\pm\operatorname{Sinh}z_n)\quad\text{(A)};\end{aligned}$$

and, in particular, if n be any positive integer,

$$\operatorname{Cosh}nz\pm\operatorname{Sinh}nz=(\operatorname{Cosh}z\pm\operatorname{Sinh}z)^n\qquad\text{(B)}.$$

These correspond to the Demoivre-formulæ, with which the reader is already familiar.*

We can deduce from (A) and (B) a series of formulæ for the hyperbolic functions analogous to those established in § 12 for the circular functions.

Thus, in particular, we have

$$\operatorname{Cosh}(z_1+z_2+\ldots+z_n)=\mathrm{P}_n+\mathrm{P}_{n-2}+\mathrm{P}_{n-4}+\ldots\quad(1'),$$

where $\mathrm{P}_r=\Sigma\operatorname{Cosh}z_1\operatorname{Cosh}z_2\ldots\operatorname{Cosh}z_r\operatorname{Sinh}z_{r+1}\ldots\operatorname{Sinh}z_n$.

$$\operatorname{Tanh}(z_1+z_2+\ldots+z_n)=(\mathrm{T}_1+\mathrm{T}_3+\mathrm{T}_5+\ldots)/(1+\mathrm{T}_2+\mathrm{T}_4+\ldots)\quad(3'),$$

where $\mathrm{T}_r=\Sigma\operatorname{Tanh}z_1\operatorname{Tanh}z_2\ldots\operatorname{Tanh}z_r$.

$$\operatorname{Cosh}nz=\operatorname{Cosh}^nz+{}_n\mathrm{C}_2\operatorname{Cosh}^{n-2}z\operatorname{Sinh}^2z+{}_n\mathrm{C}_4\operatorname{Cosh}^{n-4}z\operatorname{Sinh}^4z+\ldots\quad(4').$$

$$\operatorname{Sinh}nz={}_n\mathrm{C}_1\operatorname{Cosh}^{n-1}z\operatorname{Sinh}z+{}_n\mathrm{C}_3\operatorname{Cosh}^{n-3}z\operatorname{Sinh}^3z+{}_n\mathrm{C}_5\operatorname{Cosh}^{n-5}z\operatorname{Sinh}^5z+\ldots\quad(5').$$

$$\operatorname{Cosh}nz=(-)^{n/2}\left\{1-\frac{n^2}{2!}\cosh^2z+\frac{n^2(n^2-2^2)}{4!}\cosh^4z-\ldots\right.$$
$$\left.(-)^s\frac{n^2(n^2-2^2)\ldots(n^2-2s-2^2)}{(2s)!}\cosh^{2s}z+\ldots\right\}\quad(9'),$$
$$(n\text{ even});$$

* As a matter of history, Demoivre first found (B) in the form $y=\frac{1}{2}[\sqrt[n]{\{\sqrt{(1+v^2)}-v\}}-\sqrt[n]{\{\sqrt{(1+v^2)}-v\}}]$, where y is the ordinate of P in Fig. 10 below, and v the ordinate of Q, Q corresponding to a vector OQ such that the area AOQ is n times AOP, and OA is taken to be 1. He then deduced the corresponding formula for the circle by an imaginary transformation. (See *Miscellanea Analytica*, Lib. II., cap. i.)

$$\text{Sinh } nz/\sinh z = (-)^{(n-2)/2}\left\{\frac{n}{1!}\cosh z - \frac{n(n^2-2^2)}{3!}\cosh^3 z + \ldots\right.$$
$$\left.(-)^s\frac{n(n^2-2^2)\ldots(n^2-\overline{2s}^2)}{(2s+1)!}\cosh^{2s+1}z + \ldots\right\}(11'),$$
$$(n \text{ even});$$

and so on.

We may also deduce formulæ analogous to those of § 13, such as

$$\text{Sinh}^{2m+1}z = \frac{1}{2^m}\{\sinh(2m+1)z - {}_{2m+1}C_1\sinh(2m-1)z + \ldots$$
$$(-)^m{}_{2m+1}C_m\sinh z\}.$$

§ 29. *Fundamental Inequality and Limit Theorems for the Hyperbolic Functions of a real argument.*

If u be any positive real quantity, then

$$\tanh u < u < \sinh u < \cosh u \qquad (1).$$

By the definitions of § 24 we have

$$\sinh u = \tfrac{1}{2}\{\exp(u) - \exp(-u)\};$$
$$= u + u^3/3! + u^5/5! + \ldots \qquad (2);$$
$$\cosh u = 1 + u^2/2! + u^4/4! + \ldots \qquad (3);$$

whence it appears at once that $\sinh u > u$.

Again, $\cosh u = +\sqrt{(1+\sinh^2 u)}$, so that $\cosh u > \sinh u$.

Finally, since

$$\tanh u = \sinh u/\cosh u$$
$$= u(1 + u^2/3! + u^4/5! + \ldots)/(1 + u^2/2! + u^4/4! + \ldots),$$

and $\qquad u^2/3! < u^2/2!, \quad u^4/5! < u^4/4!, \quad \&c.,$

we see that $\tanh u < u$.

Cor. *When* $u = 0$, $\mathrm{L}\sinh u/u = 1$, *and* $\mathrm{L}\tanh u/u = 1$. This may either be deduced from (1) or established directly by means of the series (2) and (3).

If a be a quantity which is either finite and independent of n or else has a finite limit when $n = \infty$, then, when $n = \infty$,

$$\mathrm{L}\left(\cosh\frac{a}{n}\right)^n = 1, \quad \mathrm{L}\left(\sinh\frac{a}{n}\Big/\frac{a}{n}\right)^n = 1, \quad \mathrm{L}\left(\tanh\frac{a}{n}\Big/\frac{a}{n}\right)^n = 1.$$

We have

$$\left(\cosh\frac{a}{n}\right)^n = \left(\frac{e^{a/n}+e^{-a/n}}{2}\right)^n = e^a\left(\frac{1+e^{-2a/n}}{2}\right)^n.$$

Hence, if we put $1+e^{-2a/n} = 2-2z$, so that $z=0$ corresponds to $n=\infty$, then we have

$$\mathop{\mathrm{L}}_{n=\infty}\left(\cosh\frac{a}{n}\right)^n = e^a \mathop{\mathrm{L}}_{z=0}\{(1-z)^{-1/z}\}^{a2z/\log(1-2z)}.$$

Now, $\mathrm{L}(1-z)^{-1/z} = e$, and $\mathrm{L}2z/\log(1-2z) = -1$. Hence, by chap. xxv., § 13,

$$\mathrm{L}\left(\cosh\frac{a}{n}\right)^n = e^a e^{-a} = 1.$$

We leave the demonstration of the second limit as an exercise for the reader. The third is obviously deducible from the other two.

A very simple proof of these theorems may also be obtained by using the convergent series for cosh. a/n and sinh. a/n.

§ 30.] *Geometrical Analogies between the Circular and Hyperbolic Functions.*

If θ be continuously varied from $-\pi$ to $+\pi$, and we connect x and y with θ by the equations

$$x = a\cos\theta, \quad y = a\sin\theta \qquad (1),$$

then we have

$$x^2+y^2 = a^2(\cos^2\theta+\sin^2\theta) = a^2 \qquad (2).$$

Hence, if (x, y) be the co-ordinates of a point P, as θ varies continuously from $-\pi$ to $+\pi$, P will describe continuously the circle A′AA″ (of radius a) in the direction indicated by the arrow-heads (Fig. 10).

Let P be the point corresponding to θ; and let Θ denote the area AOP, to be taken with the sign + or − according as θ is positive or negative. Then Θ is obviously a function of θ. We can determine the form of this function as follows:—

Divide θ into n equal parts, and let $\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_r, \ldots \mathrm{P}$ be the points corresponding to $\theta/n, 2\theta/n, \ldots, r\theta/n, \ldots n\theta/n$ respectively. Then we have, by the lemmas of Newton,

$$\text{Area AOP} = \mathop{\mathrm{L}}_{n=\infty}\sum_{r=1}^{r=n}\mathrm{P}_r\mathrm{OP}_{r+1}.$$

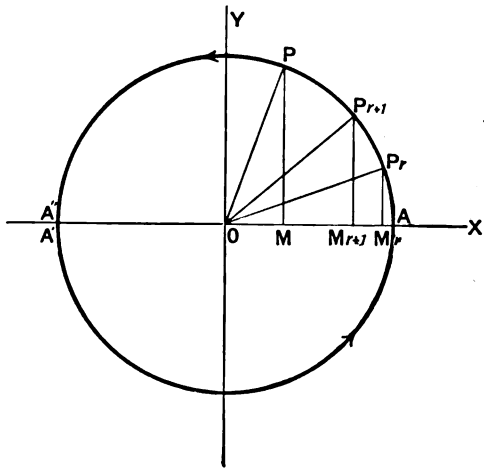


FIG. 10.

Now

$$\begin{aligned}
&P_rOP_{r+1}\\
&\quad = OM_{r+1}P_{r+1} + M_{r+1}P_{r+1}P_rM_r - OM_rP_r,\\
&\quad = \tfrac{1}{2}\{x_{r+1}y_{r+1} + (y_{r+1} + y_r)(x_r - x_{r+1}) - x_ry_r\},\\
&\quad = \tfrac{1}{2}(x_ry_{r+1} - x_{r+1}y_r),\\
&\quad = \tfrac{1}{2}a^2\{\cos.r\theta/n \sin.(r+1)\theta/n - \sin.r\theta/n \cos.(r+1)\theta/n\},\\
&\quad = \tfrac{1}{2}a^2 \sin.\theta/n.
\end{aligned}$$

Hence

$$\begin{aligned}
\Theta &= \tfrac{1}{2}a^2Ln \sin.\theta/n,\\
&= \tfrac{1}{2}a^2\theta L(\sin.\theta/n)/(\theta/n),\\
&= \tfrac{1}{2}a^2\theta. \qquad (3).
\end{aligned}$$

Hence, if $\theta = 2\Theta/a^2$, we have $\cos\theta = x/a$, $\sin\theta = y/a$, $\tan\theta = y/x$, $\cot\theta = x/y$, &c.

Next, let u be continuously varied from $-\infty$ to $+\infty$; and let

$$x = a\cosh u, \quad y = a\sinh u \qquad (1').$$

Then

$$x^2 - y^2 = a^2(\cosh^2 u - \sinh^2 u) = a^2 \qquad (2').$$

Hence, if (x, y) be the co-ordinates of P, as u* varies continuously from $-\infty$ to $+\infty$, P will describe continuously the right-hand branch A′AA″ of the rectangular hyperbola, whose

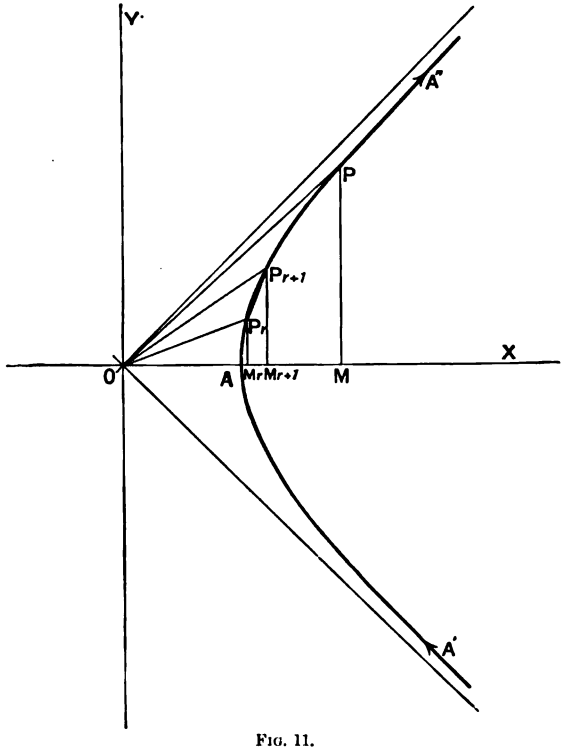


FIG. 11.

semi-axis-major is $OA = a$, in the direction indicated by the arrow-heads in Fig. 11.

If P be the point corresponding to u, P_r, P_{r+1} the points corresponding to ru/n and $(r+1)u/n$, and U the area AOP agreeing in sign with u, then, exactly as before,

* Adopting an astronomical term, we may call u the hyperbolic excentric anomaly of P. u plays in the theory of the hyperbola, in general, the same part as the eccentric angle in the theory of the ellipse.

$$U = \tfrac{1}{2} \operatorname*{L}_{n=\infty} \sum_{r=1}^{r=n} (x_r y_{r+1} - x_{r+1} y_r);$$

and

$$\begin{aligned} x_r y_{r+1} - x_{r+1} y_r &= a^2\{\cosh.ru/n \sinh.(r+1)u/n - \sinh.ru/n \cosh.(r+1)u/n\}, \\ &= a^2 \sinh.u/n. \end{aligned}$$

Therefore

$$\begin{aligned} U &= \tfrac{1}{2}a^2 Ln \sinh.u/n, \\ &= \tfrac{1}{2}a^2 u \mathrm{L}(\sinh.u/n)/(u/n), \\ &= \tfrac{1}{2}a^2 u, \qquad \text{by § 29,} \qquad (3'). \end{aligned}$$

Hence, if the area AOP = U, and $u = 2U/a^2$, then, x and y being the co-ordinates of P, we might give the following geometric definitions of $\cosh u$, $\sinh u$, &c. :—

$$\cosh u = x/a, \quad \sinh u = y/a,$$
$$\tanh u = y/x, \quad \coth u = x/y, \text{ &c.}$$

It will now be apparent that the hyperbolic functions are connected in the same way with one half of a rectangular hyperbola, as the circular functions are with the circle. It is from this relation that they get their name.

We know, from elementary geometrical considerations, that the area Θ is the product of $\frac{1}{2}a^2$ into the number of radians in the angle AOP. It therefore follows from (3) that the variable θ introduced above is simply the number of radians in the angle AOP. Our demonstration did not, however, rest upon this fact, but merely on the functional equation $\cos^2\theta + \sin^2\theta = 1$. This is an interesting point, because it shows us that we might have introduced the functions $\cos\theta$ and $\sin\theta$ by the definitions $\cos\theta = \frac{1}{2}\{\mathrm{Exp}(i\theta) + \mathrm{Exp}(-i\theta)\}$, $\sin\theta = \frac{1}{2i}\{\mathrm{Exp}(i\theta) - \mathrm{Exp}(-i\theta)\}$; and then, by means of the above reasoning, have deduced the property which is made the basis for their geometrical definition. When this point of view is taken, the theory of the circular and hyperbolic functions attains great analytical symmetry; for it becomes merely a branch of the general theory of the exponential function as defined in § 18.

When we attempt to get for u a connection with the arc AP, like that which subsists in the case of the circle, the parallel ceases to run on the same elementary line. To understand its nature in this respect we must resort to the theory of Elliptic Integrals.

§ 31.] *Expression of Real Hyperbolic Functions in terms of Real Circular Functions.*

Since the range of the variation of $\cosh u$ when u varies from $-\infty$ to $+\infty$ is the same as the range of $\sec\theta$ when θ varies from $-\pi$ to $+\pi$, it follows that, if we restrict θ and u to have the same sign, there is always one and only one value of u between $-\infty$ and $+\infty$ and of θ between $-\pi$ and $+\pi$ such that

$$\cosh u = \sec\theta \qquad (1).$$

If we determine θ in this way, we have

$$\begin{aligned}\sinh u &= \pm\sqrt{(\cosh^2 u - 1)},\\ &= \pm\sqrt{(\sec^2\theta - 1)};\end{aligned}$$

hence, bearing in mind the understanding as to sign, we have

$$\sinh u = \tan\theta \qquad (2).$$

From these we deduce

$$\begin{aligned}e^u &= \cosh u + \sinh u,\\ &= \sec\theta + \tan\theta;\\ u &= \log(\sec\theta + \tan\theta),\\ &= \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) \qquad (3).\end{aligned}$$

Also, as may be easily verified,

$$\tanh\tfrac{1}{2}u = \tan\tfrac{1}{2}\theta, \qquad (4).$$

When θ is connected with u by any of the four equivalent equations just given, it is called the *Gudermannian* * of u, and we write $\theta = \mathrm{gd}\, u$.

* This name was invented by Cayley in honour of the German mathematician Gudermann (1798-1852), to whom the introduction of the hyperbolic functions into modern analytical practice is largely due. The origin of the functions goes back to Mercator's discovery of the logarithmic quadrature of the hyperbola, and Demoivre's deduction therefrom (see p. 282). According to Houel, F. C. Mayer, a contemporary of Demoivre's, was the first to give shape to the analogy between the hyperbolic and the circular functions. The notation cosh. sinh. seems to be a contraction of coshyp. and sinhyp., proposed by Lambert, who worked out the hyperbolic trigonometry in considerable detail, and gave a short numerical table. Many of the hyperbolic formulæ were independently deduced by William Wallace (Professor of Mathematics in Edinburgh from 1819 to 1838) from the geometrical properties of the rectangular hyperbola, in a little known memoir entitled *New Series for the Quadrature of Conic Sections and the Computation of Logarithms* (*Trans. R.S.E.*, vol. vi., 1812). For further historical information, see Günther, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen* (Halle, 1881); also, *Beiträge zur Geschichte der Neueren Mathematik* (*Programmschrift*, Ansbach, 1881).

It is easy to give a geometrical form to the relation between θ and u. If, in Fig. 11, a circle be described about O with a as radius, and from M a tangent be drawn to touch this circle in Q (above or below OX according as u is positive or negative), then, since $MQ^2 = OM^2 - OQ^2 = x^2 - a^2 = y^2$, we have $a \cosh u = x = a \sec QOM$. Therefore $QOM = \theta$, and we have $y = MQ = a \tan \theta$. From this relation many interesting geometrical results arise which it would be out of place to pursue here. We may refer the reader who desires further information regarding this and other parts of the theory of the hyperbolic functions to the following authorities:—Greenhill, *Differential and Integral Calculus* (Macmillan, 1886), and also an important tract entitled *A Chapter in the Integral Calculus* (Hodgson, London, 1888); Laisant, "Essai sur les Fonctions hyperboliques," *Mém. de la Soc. Phys. et Nat. de Bordeaux*, 1875; Heis, *Die Hyperbolischen Functionen* (Halle, 1875). Tables of the functions have been calculated by Gudermann, *Theorie der Potential- oder Cyclisch-hyperbolischen Functionen* (Berlin, 1833); and by Gronau (Dantzig, 1863). See also Cayley, *Quarterly Journal of Mathematics*, vol. XX.

EXERCISES XVII.

(1.) Write down the values of the six hyperbolic functions corresponding to the arguments $\frac{1}{2}\pi i$, πi, $2\pi i$.

Draw the graphs of the following, x and y being real:—

(2.) $y = \sinh x/x$. (3.) $y = x \coth x$.

(4.) $y = \mathrm{gd}\, x$. (5.) $y = \sinh^{-1}\{1/(x-1)\}$.

(6.) Express $\mathrm{Sinh}^{-1}z$, $\mathrm{Tanh}^{-1}z$, $\mathrm{Sech}^{-1}z$, $\mathrm{Cosech}^{-1}z$ by means of $\mathrm{Sin}^{-1}z$, $\mathrm{Cos}^{-1}z$, &c.

(7.) Show that $\cosh^6 u - \sinh^6 u = 1 + 3 \sinh^2 u \cosh^2 u$.

(8.) Show that

$$4 \cosh^3 u - 3 \cosh u - \cosh 3u = 0;$$

$$4 \sinh^3 u + 3 \sinh u - \sinh 3u = 0.$$

(9.) Show that any cubic equation which has only one real root can be numerically solved by means of the equations of last exercise. In particular, show that the roots of $x^3 - qx - r = 0$ are $\sqrt{(q/3)} \cosh u$, $2\sqrt{(q/3)}\,(\cos \frac{1}{3}\pi \cosh u \pm i \sin \frac{1}{3}\pi \sinh u)$, u being determined by $\cosh 3u = 3r\sqrt{3}/2\sqrt{q^3}$.

(10.) Solve by the method of last exercise the equation $x^3 + 6x + 7 = 0$.

Express

(11.) $\tanh^{-1}x + \tanh^{-1}y$ in the form $\tanh^{-1}z$.

(12.) $\cosh^{-1}x + \cosh^{-1}y$ in the form $\cosh^{-1}z$.

(13.) $\sinh^{-1}x - \sinh^{-1}y$ in the form $\cosh^{-1}z$.

Expand in a series of hyperbolic sines or cosines of multiples of u:—

(14.) $\mathrm{Cosh}^{10}u$. (15.) $\sinh^7 u$. (16.) $\cosh^9 u \sinh^3 u$.

Expand in a series of powers of hyperbolic sines or cosines of u : —

(17.) Cosh $10u$. (18.) $\sinh 7u$.

(19.) $\cosh 6u \sinh 3u$. (20.) $\sinh mu \cosh nu$.

Establish the following identities :—

(21.) $\tanh \frac{1}{2}(u+v) - \tanh \frac{1}{2}(u-v) = 2 \sinh v/(\cosh u + \cosh v)$.

(22.) $\dfrac{\sinh(u-v)+\sinh u+\sinh(u+v)}{\cosh(u-v)+\cosh u+\cosh(u+v)} = \tanh u$.

(23.) $\tanh u + \tanh(\frac{1}{3}\pi i + u) + \tanh(\frac{2}{3}\pi i + u) = 3\tanh 3u$,
$\cosh 2u + \cosh 2v + \cosh 2w + \cosh 2(u+v+w) = 4\Pi \cosh(v+w)$.

(24.) Tan $\frac{1}{2}(u+iv) = (\sin u + i \sinh v)/(\cos u + \cosh v)$.

(25.) Express $\text{Cosh}^2(u+iv) + \text{Sinh}^2(u+iv)$ in terms of functions of u and v.

Eliminate u and v from the following equations :—

(26.) $x = a \cosh(u+\lambda)$, $y = b \sinh(u+\mu)$.

(27.) $y \cosh u - x \sinh u = a \cosh 2u$,
$y \sinh u + x \cosh u = a \sinh 2u$.

(28.) $x = \tanh u + \tanh v$, $y = \coth u + \coth v$, $u + v = c$.

(29.) Expand $\sinh(u+h)$ in powers of h.

(30.) Expand $\tanh^{-1}x$ in powers of x; and deduce the expansions of $\cosh^{-1}x$ and $\sinh^{-1}x$. Discuss the limits within which your expansions are valid.

(31.) Given $\sinh u/u = 1001/1000$, calculate u.

(32.) Show that the series $\sum_1^\infty \dfrac{x^{1/2n-1}-1}{x^{1/2^{n-1}}+1}$ is convergent, and that its sum is $(x^2+1)/(x^2-1) - 1/\log x$ (Wallace, *l.c.*).

(33.) Prove that the infinite product $\cosh\dfrac{u}{2^1}\cosh\dfrac{u}{2^2}\cosh\dfrac{u}{2^3}\ldots$ is convergent, and that its value is $\sinh u/u$.

(34.) Show that

$$\log x = \frac{x-x^{-1}}{2} \cdot \frac{2}{x^{1/2}+x^{-1/2}} \cdot \frac{2}{x^{1/4}+x^{-1/4}} \cdot \frac{2}{x^{1/8}+x^{-1/8}} \cdots \text{ ad } \infty.$$

(Wallace, *l.c.*).

(35.) If $\dfrac{1}{P_n} = \dfrac{x-x^{-1}}{2} \cdot \dfrac{2}{x^{1/2}+x^{-1/2}} \cdots \dfrac{2}{x^{1/2^n}+x^{-1/2^n}}$, show that P_n differs from $1/\log x$ (in defect) by less than

$$\{1 + \tfrac{1}{2}(x^{1/2^{n+1}} + x^{-1/2^{n+1}})\}/3.4^{n+1}P_n.$$

Evaluate the following limits :—

(36.) $(\sinh x - \sin x)/x^3$, $x = 0$.

(37.) $(\sinh^2 mx - \sinh^2 nx)/(\cosh px - \cosh qx)$, $x = 0$.

(38.) $(\tan^2 x - \tanh^2 x)/(\cos x - \cosh x)$, $x = 0$.

Show that, when $h=0$,

(39.) $\mathrm{L}\{\cosh a(x+h)-\cosh ax\}/h=a\sinh ax.$

(40.) $\mathrm{L}\{\sinh a(x+h)-\sinh ax\}/h=a\cosh ax.$

(41.) $\mathrm{L}\{\tanh a(x+h)-\tanh ax\}/h=a\,\mathrm{sech}^2 ax.$

(42.) $\mathrm{L}\{\coth a(x+h)-\coth ax\}/h=-a\,\mathrm{cosech}^2 ax.$

(43.) Show that

$$\frac{1}{2^n}\coth\frac{u}{2^n}=\coth u-\sum_1^n\frac{1}{2^n}\tanh\frac{u}{2^n},$$

$$\frac{1}{u}=\coth u-\sum_1^\infty\frac{1}{2^n}\tanh\frac{u}{2^n},$$

and state the corresponding formulæ for the circular functions (Wallace, *Trans. R.S.E.*, vol. vi.).

(44.) From the formulæ of last exercise, derive, by the process of chap. xxvii., § 2, the following:—

$$\frac{1}{2^{2n}}\coth^2\frac{u}{2^n}=\coth^2 u-\sum_1^n\frac{1}{2^{2n}}\tanh^2\frac{u}{2^n},$$

$$\frac{1}{u^2}=\coth^2 u-\sum_1^\infty\frac{1}{2^{2n}}\tanh^2\frac{u}{2^n}.$$

(Wallace, *l.c.*).

In the following, O is the centre of the hyperbola $x^2/a^2-y^2/b^2=1$; A one of its vertices; F the corresponding focus; P and P′ any two points on the curve, whose excentric anomalies are u and u', and whose co-ordinates are (x, y) (x', y'), so that $x=a\cosh u$, $y=b\sinh u$, &c.; and N is the projection of P on the axis a. Show that

(45.) Area $\mathrm{ANP}=\frac{1}{4}ab(\sinh 2u-2u)$.

(46.) Area of the right segment cut off by the double ordinate of P

$$=\frac{b}{a}x\sqrt{(x^2-a^2)}-ab\cosh^{-1}\frac{x}{a},$$

$$=\frac{b}{a}x\sqrt{(x^2-a^2)}-ab\log\frac{x+\sqrt{(x^2-a^2)}}{a}.$$

(47.) Area of the segment cut off by $\mathrm{PP}'=\frac{1}{2}ab\{\sinh(u'-u)-(u'-u)\}$.

Express this in terms of x, y, x', y'.

(48.) If R be the middle point of PP′, and OR meet the hyperbola in S, the co-ordinates of S are $\{a\cosh\frac{1}{2}(u+u'),\ b\sinh\frac{1}{2}(u+u')\}$.

(49.) OS bisects the hyperbolic area POP′.

(50.) If PP′ move parallel to itself, the locus of R is a straight line passing through O.

(51.) If PP′ cut off a segment of constant area, the locus of R is a hyperbola.

GRAPHICAL DISCUSSION OF THE GENERALISED CIRCULAR FUNCTIONS.

§ 32.] Let us now consider the general functional equation $w = \text{Cos}\, z$, or, as we may write it,

$$u + iv = \text{Cos}\,(x + yi) \qquad (1),$$

where u, v, x, y are all real.

Since $\text{Cos}\,(x + yi) = \text{Cos}\, x\ \text{Cos}\, yi - \text{Sin}\, x\ \text{Sin}\, yi = \cos x \cosh y - i \sin x \sinh y$, we have

$$u = \cos x \cosh y, \quad v = -\sin x \sinh y \qquad (2);$$

and therefore

$$u^2/\cos^2 x - v^2/\sin^2 x = 1 \qquad (3);$$
$$u^2/\cosh^2 y + v^2/\sinh^2 y = 1 \qquad (4).$$

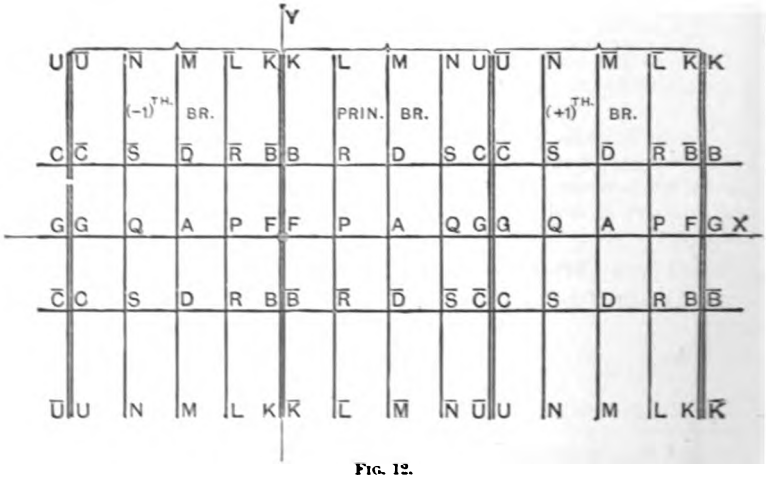

Y
U Ū N M L K K L M N U Ū N M L K K
(−1)TH. BR. PRIN. BR. (+1)TH. BR.
C C̄ S̄ Q̄ R̄ B̄ B R D S C C̄ S̄ D̄ R̄ B̄ B
G G Q A P F F P A Q G G Q A P F G X
C̄ C S D R B B̄ R̄ D̄ S̄ C̄ C S D R B B̄
Ū U N M L K K̄ L̄ M̄ N̄ Ū U N M L K K̄

FIG. 12.

In order to avoid repetition of the values u and v, arising from the periodicity of $\cos x$ and $\sin x$, we confine z, in the first instance, to lie between the axis of y and a parallel $\bar{U}\bar{C}GCU$ to this axis at a distance from it equal to π (Fig. 12).

If we draw a series of parallels to the y-axis within this strip, we see, from equation (3), that to each of these will belong half

of a hyperbola in the w-plane (Fig. 13), having its foci at the fixed points F and G, which are such that OF = OG = 1. Thus, for example, if in the z-plane FP = $\frac{1}{4}\pi$ and FQ = $\frac{3}{4}\pi$, then to the parallels LP$\overline{\text{L}}$, NQ$\overline{\text{N}}$ correspond the two halves LP$\overline{\text{L}}$, NQ$\overline{\text{N}}$ of a hyperbola whose transverse axis is PQ = $\sqrt{2}$.

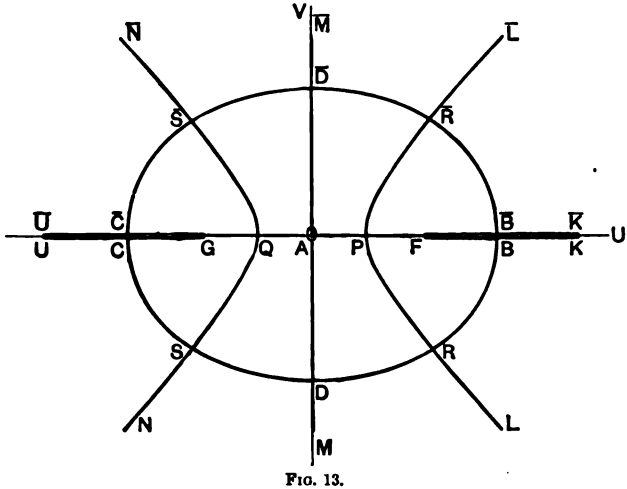
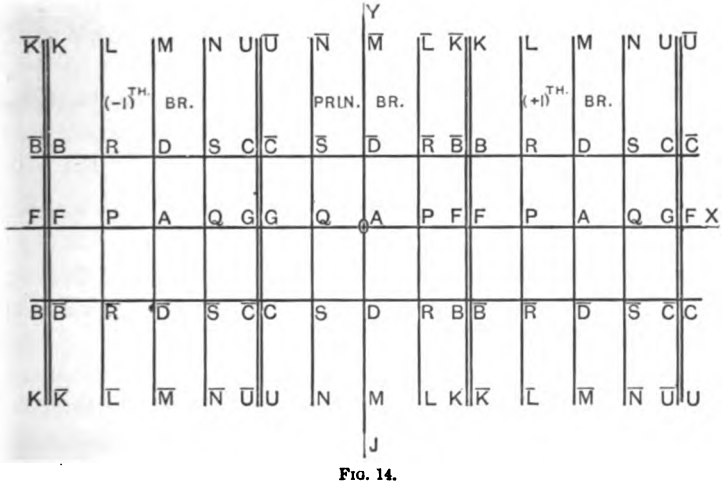


FIG. 13.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K K L M N U U N M L K K L M N U U
(−1)TH. BR. PRIN. BR. (+1)TH. BR.
B B R D S C C S D R B B R D S C C
F F P A Q G G Q A P F F P A Q G F X
B B R D S C C S D R B B R D S C C
K K L M N U U N M L K K L M N U U
J

FIG. 14.

To the parallel $MA\overline{M}$, which bisects the strip, corresponds the axis of v (which may be regarded as that hyperbola of the confocal system which has its transverse axis equal to 0); and to the parallels $KF\overline{K}$ and $UG\overline{U}$, which bound the strip, correspond the parts $KF\overline{K}$ and $UG\overline{U}$ of the u-axis, each regarded as a double line (flat hyperbola).

Again, if we draw parallels to the x-axis across the strip, to each of these will correspond one of the halves of an ellipse belonging to a confocal system having F and G for common foci. Thus to BRDSC and $\overline{B}\overline{R}\overline{D}\overline{S}\overline{C}$ equidistant from the x-axis correspond the two halves BRDSC and $\overline{B}\overline{R}\overline{D}\overline{S}\overline{C}$ of the same ellipse whose semi-axes are cosh y and sinh y. In particular, to FPAQG on the x-axis itself corresponds the double line (flat ellipse) FPAQG.

Thus, to the whole of the first parallel strip between $KO\overline{K}$ and $U\overline{U}$ corresponds uniquely the whole of the w-plane. Hence, if we confine ourselves to this strip, (1) defines w and z each as a continuous one-valued function of the other. To each succeeding or preceding strip corresponds the w-plane again taken once over, alternately one way or the opposite, as indicated by the lettering in Fig. 12. w is therefore a periodic function of z, having the real period 2π; and z is a multiple-valued function of w of infinite multiplicity, having two branches for each period of w.

The value of z corresponding to the first strip on the right of the y-axis is called the principal branch of $\mathrm{Cos}^{-1}w$, and the others are numbered as usual. We therefore have for the t-th branch

$$_t\mathrm{Cos}^{-1}w = z_t = (t + \tfrac{1}{2} + (-)^{t-1}\tfrac{1}{2})\pi + (-)^t \mathrm{Cos}^{-1}w \quad (5),$$

where $\mathrm{Cos}^{-1}w$ is the principal value as heretofore; and $\mathrm{Cos}^{-1}w = x + yi$, x and y being determined by (3) and (4), when u and v are given.

It should be noticed that for the same branch of z there is continuity from B to $\overline{B}$ not directly across the u-axis, but only by the route $BF\overline{B}$; whereas there is continuity from B to $\overline{B}$

directly, if we pass from one branch to the next. This may be represented to the eye by slitting the u-axis from F to $+\infty$ and from G to $-\infty$, as indicated in Fig. 13. If we were to construct a Riemann's surface for the w-plane, so as to secure unique correspondence between every w-point and its z-point, then the junctions of the leaves of this surface would be along these slits. The reader will find no difficulty in constructing the model.

Since to the line KFPAQGU (the whole of the u-axis) corresponds in the z-plane the three lines KF, FPAQG, GU taken in succession, we see that as w varies first from $+\infty$ to 1, then from 1 to -1, and finally from -1 to $-\infty$, $\mathrm{Cos}^{-1}w$ varies first from ∞i to 0, then from 0 to π, and finally from π to $\pi+\infty i$; so that an angle whose cosine is greater than 1 is either wholly or partly imaginary.

§ 33.] If $w = \mathrm{Sin}\, z$, say

$$u + iv = \mathrm{Sin}\,(x + yi) \qquad (1),$$

then, as in last paragraph,

$$u = \sin x \cosh y, \quad v = \cos x \sinh y \qquad (2);$$

$$u^2/\sin^2 x - v^2/\cos^2 x = 1 \qquad (3);$$

$$u^2/\cosh^2 y + v^2/\sinh^2 y = 1 \qquad (4).$$

The graphical representation is, as the student may easily verify, obtained by taking Fig. 13 for the w-plane and Fig. 14 for the z-plane.

We have also, for the t-th branch of the inverse function,

$$_t\mathrm{Sin}^{-1}w = z_t = t\pi + (-)^t \mathrm{Sin}^{-1}w,$$

where $\mathrm{Sin}^{-1}w = x + yi$, x and y being determined by equations (3) and (4), under the restrictions proper to the principal branch of the function.

§ 34.] If $w = \mathrm{Tan}\, z$, say

$$u + iv = \mathrm{Tan}(x + yi) \qquad (1),$$

then
$$(u + iv)\,\mathrm{Cos}\,(x + yi) = \mathrm{Sin}\,(x + yi),$$

that is,

$$(u\cos x\cosh y + v\sin x\sinh y) + i(-u\sin x\sinh y + v\cos x\cosh y) = \sin x\cosh y + i\cos x\sinh y.$$

Therefore

$$u\cos x\cosh y + v\sin x\sinh y = \sin x\cosh y,$$
$$-u\sin x\sinh y + v\cos x\cosh y = \cos x\sinh y.$$

From the last pair of equations it is easy, if we bear in mind the formulæ of § 27, to deduce the following:—

$$u = \sin 2x/(\cos 2x + \cosh 2y), \quad v = \sinh 2y/(\cos 2x + \cosh 2y) \qquad (2);$$
$$u^2 + v^2 + 2u\cot 2x - 1 = 0 \qquad (3);$$
$$u^2 + v^2 - 2v\coth 2y + 1 = 0 \qquad (4).$$

The graphical representation of these results is given by Figs. 15 and 16.

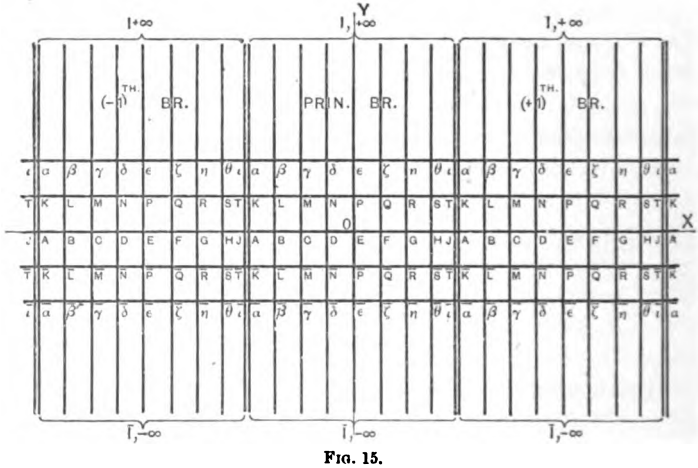

FIG. 15.

When x is kept constant, the equation to the path of w is given by (3), which evidently represents a series of circles passing through the points $(0, +1)$ and $(0, -1)$.

When y is constant, the equation to the path of w is (4), which represents a circle having its centre on the v-axis; and it is easy to verify that the square of the distance between the centres of the circles (3) and (4) is equal to the sum of the squares of their radii, from which it appears that they are orthotomic.

If we consider a parallel strip of the z-plane bounded by $x = -\frac{1}{2}\pi$, $x = +\frac{1}{2}\pi$, we find that to this corresponds the whole

w-plane taken once over. The corresponding values of z are said to belong to the principal branch of the function $\mathrm{Tan}^{-1}w$.

To the vertical parallels in the z-plane correspond the circles passing through I and $\bar{\mathrm{I}}$ in the w-plane, and to the horizontal parallels correspond the circles in the w-plane which cut the former orthogonally.

It should be noticed that I and $\bar{\mathrm{I}}$ in the w-plane correspond to $+\infty$ and $-\infty$ in the direction of the y-axis in the z-plane, and

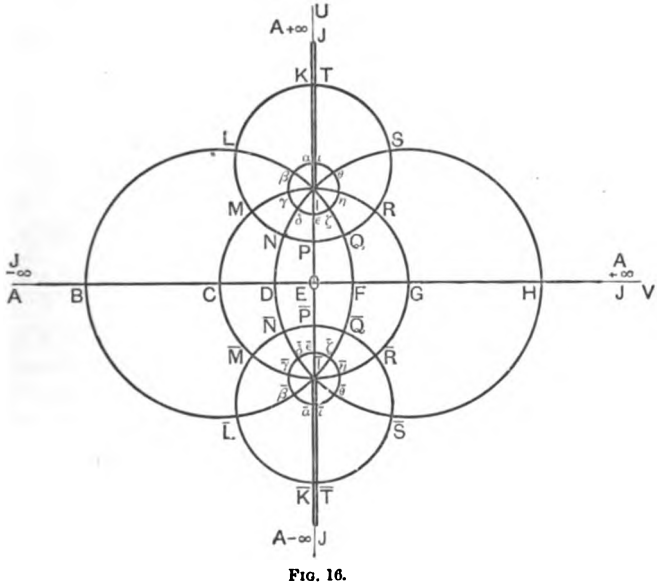

FIG. 16.

that to A and J in the z-plane correspond the points at ∞ on the u- and v-axes in the w-plane; also that there is no continuity directly across $\mathrm{IK}\infty$ or $\bar{\mathrm{I}}\mathrm{K}\bar{\infty}$ in the w-plane, except in passing from one branch of $\mathrm{Tan}^{-1}w$ to the next.

For the t-th branch of the inverse function we have

$$_t\mathrm{Tan}^{-1}w = z_t = t\pi + \mathrm{Tan}^{-1}w \qquad (5),$$

where the principal value $\mathrm{Tan}^{-1}w$ is given by $\mathrm{Tan}^{-1}w = x + yi$, x and y being determined, under the restrictions proper to the principal branch, by means of (3) and (4).

§ 35.] It will be a useful exercise for the student to discuss directly the graphical representation of $w = \text{Sec}\, z$, $w = \text{Cosec}\, z$, and $w = \text{Cot}\, z$. The figures in the w-plane for these functions may, however, be derived from those already given, by means of the following interesting general principle.

If Z *be any z-path,* W *and* W′ *the corresponding w-paths for* $w = f(x+yi)$ *and* $w' = 1/f(x+yi)$, *then* W′ *is the image with respect to the u-axis of the inverse of* W, *the centre of inversion being the origin of the w-plane and the radius of inversion being unity.*

This is easily proved; for, if (ρ, ϕ), (ρ', ϕ') be the polar co-ordinates of points on W and W′ corresponding to the point (x, y) on Z, then we have

$$\rho(\cos\phi + i\sin\phi) = f(x+yi),$$
$$\rho'(\cos\phi' + i\sin\phi') = 1/f(x+yi).$$

Hence
$$\rho(\cos\phi + i\sin\phi) = 1/\rho'(\cos\phi' + i\sin\phi'),$$
$$= (1/\rho')(\cos(-\phi') + i\sin(-\phi')).$$

Therefore $\rho = 1/\rho'$, $\phi = -\phi'$, which is the analytical expression of the principle just stated.

From this it appears at once that, if we choose for our standard z-paths a double system of orthotomic parallels to the x- and y-axes, then the w-paths for $w = \text{Cot}\, z$ will be a double system of orthotomic circles, and the w-paths for $w = \text{Sec}\, z$ and $w = \text{Cosec}\, z$ a double system of orthotomic Bicircular Quartics.

Example 1. If $u + vi = \text{Sec}(x+yi)$, show that

$$u = 2\cos x \cosh y/(\cos 2x + \cosh 2y);$$
$$v = 2\sin x \sinh y/(\cos 2x + \cosh 2y);$$
$$(u^2+v^2)^2 = u^2/\cos^2 x - v^2/\sin^2 x;$$
$$(u^2+v^2)^2 = u^2/\cosh^2 y + v^2/\sinh^2 y.$$

Discuss the graphical representation of the functional equation, and show how to deduce the t-th branch from the principal branch of the function.

The curves represented by the last two equations are most easily traced from their polar equations, which are

$$\rho^2 = 2(\cos 2\phi - \cos 2x)/\sin^2 2x,$$
$$\rho^2 = 2(\cosh 2y - \cos 2\phi)/\sinh^2 2y,$$

respectively.

Example 2. The same problem for $u + vi = \text{Cosec}(x+yi)$.

Example 3. The same problem for $u + vi = \text{Cot}(x+yi)$.

§ 36.] Before leaving the present part of our subject, it will be well to point out the general theorem which underlies the fact that to the orthogonal parallels in the z-plane in the six

cases just discussed correspond a system of orthogonal paths in the w-plane.

Let us suppose that $f(z)$ is a continuous function of the complex variable z, such that for a finite area round every point $z = a$ within a certain region in the z-plane $f(z)$ can always be expanded in a convergent series of powers of $z - a$, so that we have

$$f(z) = f(a) + A_1(z-a) + A_2(z-a)^2 + \ldots \qquad (1),$$

where $A_1, A_2, \ldots$ are functions of a and not of z.

When A_1 does not vanish, the point a is said to be an *ordinary* or non-singular point for the function $f(z)$. If $A_1 = 0$, $A_2 = 0, \ldots, A_{n-1} = 0, A_n \neq 0$, the point a is said to be a *multiple point* of the nth order.

Then we have the following general theorem, which is fundamental in the present subject.

If the point a be an ordinary point, the angle between any two z-paths emanating from a is the same as the angle between the corresponding w-paths emanating from the point in the w-plane which corresponds to a.

Proof.—Let z be any point on any path emanating from a, (r, θ) the polar co-ordinates of z with respect to a as origin, the prime radius being parallel to the x-axis. Let w and b be the w-points corresponding to z and a, (ρ, ϕ) the polar co-ordinates of w with respect to b. Then we have

$$\begin{aligned}\rho(\cos\phi + i\sin\phi) &= w - b = f(z) - f(a),\\ &= A_1(z-a) + A_2(z-a)^2 + \ldots, \qquad \text{by (1)},\\ &= A_1 r(\cos\theta + i\sin\theta) + A_2 r^2(\cos\theta + i\sin\theta)^2 + \ldots \qquad (2).\end{aligned}$$

Let now $A_1 = r_1(\cos\alpha_1 + i\sin\alpha_1)$, $A_2 = r_2(\cos\alpha_2 + i\sin\alpha_2)$, . . ., then (2) may be written

$$\begin{aligned}\rho(\cos\phi + i\sin\phi) = {} & r_1 r\{\cos(\alpha_1 + \theta) + i\sin(\alpha_1 + \theta)\}\\ & + r_2 r^2\{\cos(\alpha_2 + 2\theta) + i\sin(\alpha_2 + 2\theta)\} + \ldots \qquad (3).\end{aligned}$$

Whence

$$\rho\cos\phi = r_1 r\cos(\alpha_1 + \theta) + r_2 r^2\cos(\alpha_2 + 2\theta) + \ldots \qquad (4);$$

$$\rho\sin\phi = r_1 r\sin(\alpha_1 + \theta) + r_2 r^2\sin(\alpha_2 + 2\theta) + \ldots \qquad (5).$$

In the limit, when r and consequently ρ are made infinitely small, (4) and (5) reduce to

$$(\rho/r)\cos\phi = r_1\cos(\alpha_1+\theta), \quad (\rho/r)\sin\phi = r_1\sin(\alpha_1+\theta) \quad (6).$$

Since ρ and r are both positive, these equations lead to

$$\rho/r = r_1, \text{ and } \phi = 2k\pi + \alpha_1 + \theta \quad (7).$$

Hence, if we take any two paths emanating from a in directions determined by θ and θ', we should have $\phi - \phi' = \theta - \theta'$, which proves our theorem.

We see also, from the first of the equations in (7), that if we construct any infinitely small triangle in the z-plane, having its vertex at a, to it will correspond an infinitely small *similar* triangle in the w-plane having its vertex at b.

Hence, if we establish a unique correspondence between points (u, v) *and* (x, y) *in any two planes by means of the relation*

$$u + vi = f(x + yi) = \chi(x, y) + i\psi(x, y),$$

then to any diagram D *in the one plane corresponds a diagram* D′ *in the other which is similar to* D *in its infinitesimal detail.*

The propositions just stated show that, *if we have in the z-plane any two families of curves* A *and* B *such that each curve of* A *cuts each curve of* B *at a constant angle* α, *then to these correspond respectively in the w-plane families* A′ *and* B′ *such that each curve of* A′ *cuts each curve of* B′ *at an angle* α. Since the six circular functions satisfy the preliminary condition regarding the function $f(x + yi)$, the theorem regarding the u-v-curves for these functions which correspond to $x = \text{const.}$, $y = \text{const.}$ follows at once.

If the point $z = a$ be a multiple point of the nth order for $f(z)$, then the above conclusions fail. In fact, the equations (7) then become

$$\rho/r^n = r_n, \quad \phi = 2k\pi + \alpha_n + n\theta \quad (7');$$

and we have $\phi - \phi' = n(\theta - \theta')$.

In this case, as the point z circulates once round a, the point w circulates n times round b. That is to say, when a is a singular point of the nth order for w, b is a winding point of the nth order for z; and the Riemann's surface for the w-plane has an n-fold winding point at b. We have a simple example of this in

the case of $w = z^3$, already discussed, for which $z = 0$ is a triple point, and $w = 0$ a winding point of the third order. The points $w = \pm 1$ and $z = \pm 0$ are corresponding points of a similar character for $w = \cos z$.

The theorem of the present paragraph is of great importance in many parts of mathematics. From one point of view it may be regarded as the geometrical condition that $\phi(x, y) + i\chi(x, y)$ may be, according to a certain definition, a function of $x + yi$. In this way it first made its appearance in the famous memoir entitled *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*, in which Riemann laid the foundations of the modern theory of functions, which has borne fruit in so many of the higher branches of mathematics.

From another point of view the theorem is of great importance in geometry. When the points in one plane are connected with those in another in the manner above described, so that corresponding figures have infinitesimal similarity, the one plane is said by German mathematicians to be *conform abgebildet*, that is, *conformably represented* (Cayley has used the phrase "orthomorphic transformation") upon the other; and there is a corresponding theory for surfaces in general. Many of the ordinary geometrical transformations are particular cases of this; for example, the student will readily verify that the equation $w = a^2/z$ corresponds to inversion.

Lastly, the theory of conjugate functions, as expounded by Clerk-Maxwell in his work on electricity (vol. i. chap. xii.), depends entirely on the theorem which we have just established. In fact, the curves in Figs. 12, 13, 15, and 16 may be taken to represent lines of force and lines of equal potential; so that every particular case of the equation $u + vi = f(x + yi)$ gives the solution of one or more physical problems.

EXERCISES XVIII.

(1.) Discuss the variation of $\sin^{-1}u$ and $\sin^{-1}iv$, where u and v are real, and vary from $-\infty$ to $+\infty$.

Draw the Argand diagrams for the following, giving in each case, where they have not been given above, the w-paths when the z-paths are circles about the origin and parallels to the real and imaginary axes :—

(2.) $w = \log z$. (3.) $w = \exp z$.

(4.) $w = \cosh z$. (5.) $w = \tanh z$.

(6.) Show that $\cos^{-1}(u + iv) = \cos^{-1}\mathrm{U} - i\cosh^{-1}\mathrm{V}$;

$$\sin^{-1}(u + iv) = \sin^{-1}\mathrm{U} + i\cosh^{-1}\mathrm{V},$$

where $2\mathrm{U} = \sqrt{\{(u+1)^2 + v^2\}} - \sqrt{\{(u-1)^2 + v^2\}}$,

$$2\mathrm{V} = \sqrt{\{(u+1)^2 + v^2\}} + \sqrt{\{(u-1)^2 + v^2\}},$$

the principal branch of each function being alone in question.

(7.) Show that the principal branch of $\tan^{-1}(u+iv)$ is given by $x+yi$, where $y=\frac{1}{2}\tanh^{-1}\{2u/(u^2+v^2+1)\}$;
and $x=\frac{1}{2}\tan^{-1}\{2u/(1-u^2-v^2)\}$, if $u^2+v^2<1$;
$=\pm\frac{1}{2}\pi+\frac{1}{2}\tan^{-1}\{2u/(1-u^2-v^2)\}$, if $u^2+v^2>1$,
the upper or lower sign being taken according as u is positive or negative.

(8.) If $u+vi=\cot(x+yi)$, show that

$$u=\sin 2x/(\cosh 2y-\cos 2x),\quad v=-\sinh 2y/(\cosh 2y-\cos 2x);$$
$$u^2+v^2-2u\cot 2x-1=0,\quad u^2+v^2+2v\coth 2y+1=0.$$

(9.) If $u+vi=\operatorname{cosec}(x+yi)$, show that

$$u=2\sin x\cosh y/(\cosh 2y-\cos 2x),\quad v=-2\cos x\sinh y/(\cosh 2y-\cos 2x);$$
$$(u^2+v^2)^2=u^2/\cos^2 x-v^2/\sin^2 y,\quad (u^2+v^2)^2=u^2/\cosh^2 y+v^2/\sinh^2 y.$$

Express the following in the form $u+vi$, giving both the principal branch and the general branch when the function is multiple-valued:—

(10.) $\operatorname{Cosh}^{-1}(x+yi)$. (11.) $\operatorname{Tanh}^{-1}(x+yi)$.

(12.) $\frac{1}{2i}\operatorname{Log}\{(x+yi)/(x-yi)\}$. (13.) $\operatorname{Log}\operatorname{Sin}(x+yi)$.

(14.) $(\cos\theta+i\sin\theta)^t$. (15.) $\operatorname{Log}_{\alpha+i\beta}(x+yi)$.

(16.) Show that the general value of $\operatorname{Sin}^{-1}(\operatorname{cosec}\theta)$ is $(t+\frac{1}{2})\pi+i\log\cot\frac{1}{2}(t\pi+\theta)$, where t is any integer.

(17.) Show that the real part of $\operatorname{Exp}_i\{\operatorname{Log}(1+i)\}$ is $e^{-\pi^2/8}\cos(\frac{1}{4}\pi\log 2)$.

(18.) Prove, by means of the series for $\operatorname{Cos}\theta$ and $\operatorname{Sin}\theta$, that $\operatorname{Sin}2\theta=2\operatorname{Sin}\theta\operatorname{Cos}\theta$.

(19.) Deduce Abel's generalised form of the binomial theorem from §§ 20, 22.

(20.) Show that

$$1+{}_{m+ni}C_1x+{}_{m+ni}C_2x^2+\ldots \text{ ad } \infty$$
$$=(1+x)^m[\cos\{n\log(1+x)\}+i\sin\{n\log(1+x)\}].$$

(21.) Show that the families of curves represented by

$$\sin x\cosh y=\lambda,\quad \cos x\sinh y=\mu$$

are orthotomic.

(22.) Find the equation to the family of curves orthogonal to $r^n\cos n\theta=\lambda$.

(23.) Find the condition that the two families

$$Ax^2+2Bxy+Cy^2=\lambda,\quad A^1x^2+2B^1xy+C^1y^2=\mu$$

be orthotomic.

SPECIAL APPLICATIONS OF THE FOREGOING THEORY TO THE CIRCULAR FUNCTIONS.

§ 37.] In order to avoid breaking our exposition of the general theory of the elementary transcendents, we did not stop

to deduce consequences from the various fundamental theorems. To this part of the subject we now proceed; and we shall find that many of the ordinary theorems regarding series involving the circular functions are simple corollaries from what has gone before.

Let us take, in the first place, the generalised form of the binomial theorem given in § 15. So long as $1 + \Sigma_m C_n z^n$ is convergent, we have seen that it represents the principal value of $(1+z)^m$. Hence, if $z = r(\cos\theta + i\sin\theta)$, where r is positive, and $-\pi \not> \theta \not> +\pi$, we have

$$1 + \Sigma_m C_n r^n(\cos n\theta + i\sin n\theta) = (1 + 2r\cos\theta + r^2)^{m/2}(\cos m\phi + i\sin m\phi),$$

where $-\frac{1}{2}\pi \not> \phi = \tan^{-1}\{r\sin\theta/(1 + r\cos\theta)\} \not> +\frac{1}{2}\pi$.

Hence, equating real and imaginary parts, we must have

$$1 + \Sigma_m C_n r^n \cos n\theta = (1 + 2r\cos\theta + r^2)^{m/2}\cos m\phi \quad (1);$$

$$\Sigma_m C_n r^n \sin n\theta = (1 + 2r\cos\theta + r^2)^{m/2}\sin m\phi \quad (2).$$

These formulæ will hold for all values of m, provided $r < 1$.

When $r = 1$, we have

$$\phi = \tan^{-1}\{\sin\theta/(1 + \cos\theta)\} = \tfrac{1}{2}\theta,$$

and (1) and (2) become

$$1 + \Sigma_m C_n \cos n\theta = 2^m \cos^m\tfrac{1}{2}\theta \cos\tfrac{1}{2}m\theta \quad (1'),$$

$$\Sigma_m C_n \sin n\theta = 2^m \cos^m\tfrac{1}{2}\theta \sin\tfrac{1}{2}m\theta \quad (2').$$

These formulæ hold for all values of θ between $-\pi$ and $+\pi$,* when $m > -1$; and also for the limiting values $-\pi$ and $+\pi$ themselves, when $m > 0$.

§ 38.] *Series for $\cos m\phi$ and $\sin m\phi$, when m is not integral.*

If in (1) and (2) of last paragraph we put $\theta = \frac{1}{2}\pi$, and $r = \tan\phi$, so that ϕ must lie between $-\frac{1}{4}\pi$ and $+\frac{1}{4}\pi$, then $(1 + 2r\cos\theta + r^2)^{m/2} = \sec^m\phi$; and we find

$$\cos m\phi = \cos^m\phi(1 - {}_mC_2\tan^2\phi + {}_mC_4\tan^4\phi - \ldots) \quad (3),$$

$$\sin m\phi = \cos^m\phi({}_mC_1\tan\phi - {}_mC_3\tan^3\phi + \ldots) \quad (4).$$

* Since the left-hand sides of (1′) and (2′) are periodic, it is easy to see that, for $2\rho\pi - \pi \not> \theta \not> 2\rho\pi + \pi$, the right-hand sides will be $2^m\cos^m\frac{1}{2}\theta\cos\frac{1}{2}m(\theta - 2\rho\pi)$ and $2^m\cos^m\frac{1}{2}\theta\sin\frac{1}{2}m(\theta - 2\rho\pi)$ respectively, where $2^m\cos^m\frac{1}{2}\theta$, being the value of a modulus, must be made real and positive.

Whence

$$\tan m\phi = \frac{{}_mC_1 \tan\phi - {}_mC_3 \tan^3\phi + \ldots}{1 - {}_mC_2 \tan^2\phi + {}_mC_4 \tan^4\phi - \ldots}. \qquad (5).$$

These formulæ are the generalisations of formulæ (4), (5), (6) of § 12. They will hold even when ϕ has either of the limiting values $\pm \frac{1}{4}\pi$, provided $m > -1$; so that we have

$$2^{m/2} \cos \tfrac{1}{4}m\pi = 1 - {}_mC_2 + {}_mC_4 - \ldots;$$
$$2^{m/2} \sin \tfrac{1}{4}m\pi = {}_mC_1 - {}_mC_3 + \ldots .$$

Since

$$\cos^{m-2r}\phi = (1 - \sin^2\phi)^{(m-2r)/2} = 1 + \Sigma(-)^s {}_{(m-2r)/2}C_s \sin^{2s}\phi,$$

and the terms of this series are ultimately all positive, it follows that the double series deducible from (3), that is to say, from $\Sigma(-)^r {}_mC_{2r} \cos^{m-2r}\phi \sin^{2r}\phi$ by substituting expansions for the cosines, satisfies Cauchy's conditions (chap. xxvi., § 34), for there is obviously absolute convergency everywhere under our present restriction that $-\frac{1}{4}\pi \not> \phi \not> +\frac{1}{4}\pi$.

Hence we may arrange this double series according to powers of $\sin\phi$.

The coefficient of $(-)^r \sin^{2r}\phi$ is

$$\sum_{s=0}^{s=r} {}_{(m-2s)/2}C_{r-s} \; {}_mC_{2s}$$

$$= \frac{m(m-2) \ldots (m-2r+2)}{1.3 \ldots (2r-1)} \Sigma_{(m-1)/2}C_s \; {}_{(2r-1)/2}C_{r-s}$$

Now, by chap. xxiii., § 8, Cor. 5,

$$\Sigma_{(m-1)/2}C_s \; {}_{(2r-1)/2}C_{r-s} = {}_{(m+2r-2)/2}C_r.$$

Hence the coefficient of $(-)^r \sin^{2r}\phi$ is

$$\frac{m(m-2) \ldots (m-2r+2)(m+2r-2) \ldots (m+2)m}{1.3 \ldots (2r-1)2 \ldots (2r-2)2r}$$

$$= \frac{m^2(m^2-2^2) \ldots (m^2-2r-2^2)}{(2r)!}.$$

Hence

$$\cos m\phi = 1 - \frac{m^2}{2!}\sin^2\phi + \frac{m^2(m^2-2^2)}{4!}\sin^4\phi - \ldots \qquad (6).$$

In like manner, we can show that

$$\sin m\phi = \frac{m}{1!}\sin\phi - \frac{m(m^2-1^2)}{3!}\sin^3\phi + \frac{m(m^2-1^2)(m^2-3^2)}{5!}\sin^5\phi - \ldots \quad (7).$$

Also

$$\cos m\phi = \cos\phi\left\{1 - \frac{m^2-1}{2!}\sin^2\phi + \frac{(m^2-1^2)(m^2-3^2)}{4!}\sin^4\phi - \ldots\right\} \quad (8);$$

$$\sin m\phi = \cos\phi\left\{\frac{m}{1!}\sin\phi - \frac{m(m^2-2^2)}{3!}\sin^3\phi + \frac{m(m^2-2^2)(m^2-4^2)}{5!}\sin^5\phi - \ldots\right\} \quad (9).$$

The demonstration above given establishes these formulæ under the restriction $-\frac{1}{4}\pi > \phi > \frac{1}{4}\pi$. It can, however, be shown that they hold so long as $-\frac{1}{2}\pi > \phi > \frac{1}{2}\pi$; that is to say, so long as the series involved are convergent.

Cauchy, from whom the above is taken, shows that by expanding both sides in powers of m and equating coefficients we obtain expansions for ϕ, ϕ^2, ϕ^3, &c., in powers of $\sin\phi$.

Thus, for example, we deduce

$$\phi = \sin\phi + \frac{1}{2}\frac{\sin^3\phi}{3} + \frac{1.3}{2.4}\frac{\sin^5\phi}{5} + \frac{1.3.5}{2.4.6}\frac{\sin^7\phi}{7} + \ldots .$$

If we put $x = \sin\phi$, this gives

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots \quad (10).$$

In particular, if we put $x = \frac{1}{2}$, we obtain

$$\pi = 6\left\{\frac{1}{2} + \frac{1}{2.3.2^3} + \frac{1.3}{2.4.5.2^5} + \ldots\right\} \quad (11),$$

from which the value of π might be calculated with tolerable rapidity to a moderate number of places. The result to 10 places is $\pi = 3{\cdot}1415926536 \ldots .$

The important series (10) for expanding $\sin^{-1}x$ is here demonstrated for values of x lying between $-1/\sqrt{2}$ and $+1/\sqrt{2}$. It can be shown that it is valid between the limits $x=-1$ and $x=+1$.

The series was discovered by Newton, who gives it along with the series for $\sin x$ and $\cos x$ in powers of x in a small tract entitled *Analysis per Æquationes Numero Terminorum Infinitas*. Since this tract was shown by Newton to Barrow in 1669, the series (10) is one of the oldest examples of an infinite series applicable to the quadrature of the circle.

Example 1. If $m>0$, and

$$C=2^{-m}\sum_{n=0}^{\infty}{}_mC_n\cos(m-2n)x,$$

$$S=2^{-m}\sum_{n=0}^{\infty}{}_mC_n\sin(m-2n)x,$$

$$C'=2^{-m}\sum_{n=0}^{\infty}(-)^{n-1}{}_mC_n\cos(m-2n)x,$$

$$S'=2^{-m}\sum_{n=0}^{\infty}(-)^{n-1}{}_mC_n\sin(m-2n)x,$$

then, ρ being any integer,

1°. $C=(\cos x)^m\cos 2m\rho\pi$, $S=(\cos x)^m\sin 2m\rho\pi$,
from $x=(2\rho-\frac{1}{2})\pi$ to $x=(2\rho+\frac{1}{2})\pi$.

2°. $C=(-\cos x)^m\cos m(2\rho+1)\pi$, $S=(-\cos x)^m\sin m(2\rho+1)\pi$,
from $x=(2\rho+\frac{1}{2})\pi$ to $x=(2\rho+\frac{3}{2})\pi$.

3°. $C'=(\sin x)^m\cos m(2\rho+\frac{1}{2})\pi$, $S'=(\sin x)^m\sin m(2\rho+\frac{1}{2})\pi$,
from $x=2\rho\pi$ to $x=(2\rho+1)\pi$.

4°. $C'=(-\sin x)^m\cos m(2\rho+\frac{3}{2})\pi$, $S'=(-\sin x)^m\sin m(2\rho+\frac{3}{2})\pi$,
from $x=(2\rho+1)\pi$ to $x=(2\rho+2)\pi$.

These formulæ will also hold when m lies between -1 and 0, only that the extreme values of x in the various stretches must be excluded. (Abel, *Œuvres*, t. i., p. 249.)

If we multiply (1′) and (2′) above by $\cos\alpha$ and $\sin\alpha$ respectively, and add, we obtain the formulæ

$$\cos\alpha+\Sigma_m C_n\cos(\alpha-n\theta)=2^m\cos^m\tfrac{1}{2}\theta\cos(\alpha-\tfrac{1}{2}m\theta+m\rho\pi),$$

wherein it must be observed that $\cos^m\frac{1}{2}\theta$ is the modulus of $(1+2r\cos\theta+r^2)^{m/2}$ when $r=1$, and must therefore be always so adjusted as to have a real positive value.

From the equation just written, Abel's formulæ can at once be deduced by a series of substitutions.

Example 2. Show, by taking the limit when $m=0$ on both sides of (1) and (2) above, that the series (1) and (2) of § 40 can be deduced from the generalised form of the binomial theorem.

Example 3. Sum to infinity the series $\Sigma n^3{}_mC_n\sin^n\theta\cos n\theta$. This series is the real part of $\Sigma n^3{}_mC_n\sin^n\theta(\cos n\theta+i\sin n\theta)$. Hence

$$\begin{aligned}S&=R[\Sigma n^3{}_mC_n\sin^n\theta(\cos\theta+i\sin\theta)^n],\\&=R[\{m^3\sin^3\theta(\cos\theta+i\sin\theta)^3+m(3m-1)\sin^2\theta(\cos\theta+i\sin\theta)^2\\&\qquad+m\sin\theta(\cos\theta+i\sin\theta)\}\{1+\sin\theta(\cos\theta+i\sin\theta)\}^{m-3}].\end{aligned}$$

by Example 5 of chap. xxvii., § 5,

$$= [m^3 \sin^3\theta \cos\{3\theta + (m-3)\phi\} + m(3m-1)\sin^2\theta \cos\{2\theta + (m-3)\phi\} + m\sin\theta\cos\{\theta + (m-3)\phi\}](1 + 2\sin\theta\cos\theta + \sin^2\theta)^{(m-3)/2},$$

where $\phi = \tan^{-1}\{\sin^2\theta/(1 + \sin\theta\cos\theta)\}$.

§ 39.] *Formulæ deduced from the Exponential Series.*

From the equation

$$e^x(\cos y + i\sin y) = 1 + \Sigma(x + yi)^n/n!,$$

putting $x = r\cos\theta$, $y = r\sin\theta$, we deduce

$$e^{r\cos\theta}\{\cos(r\sin\theta) + i\sin(r\sin\theta)\} = 1 + \Sigma r^n(\cos n\theta + i\sin n\theta)/n!$$

Hence

$$e^{r\cos\theta}\cos(r\sin\theta) = 1 + \Sigma r^n\cos n\theta/n! \qquad (1);$$

$$e^{r\cos\theta}\sin(r\sin\theta) = \Sigma r^n\sin n\theta/n! \qquad (2);$$

which hold for all values of r and θ.

In like manner, many summations of series involving cosines and sines of multiples of θ may be deduced from series related to the exponential series in the way explained in chap. xxviii., § 8.

Thus, for instance, from the result of Example 3, in the paragraph just quoted, we deduce

$$\sum_1^\infty (1^3 + 2^3 + \ldots + n^3)x^n/n! = e^{r\cos\theta}\{r\cos(\theta + r\sin\theta) + \tfrac{1}{2}r^2\cos(2\theta + r\sin\theta) + 2r^3\cos(3\theta + r\sin\theta) + \tfrac{1}{4}\cos(4\theta + r\sin\theta)\}.$$

§ 40.] *Formulæ deduced from the Logarithmic Series.* Since the principal value of $\mathrm{Log}(1 + z)$ is given by $\mathrm{Log}(1 + z) = \log \mathrm{mod}(1 + z) + i\,\mathrm{amp}(1 + z)$, and since the series $z - z^2/2 + z^3/3 - \ldots$ represents the principal value of $\mathrm{Log}(1 + z)$, if we put $z = r(\cos\theta + i\sin\theta)$, we have

$$\log(1 + 2r\cos\theta + r^2)^{1/2} + i\tan^{-1}\{r\sin\theta/(1 + r\cos\theta)\} = \Sigma(-)^{n-1}r^n(\cos n\theta + i\sin n\theta)/n,$$

where $-\frac{1}{2}\pi \not> \tan^{-1}\{r\sin\theta/(1 + r\cos\theta)\} \not> \frac{1}{2}\pi$, that is, the principal value of the function $\tan^{-1}$ is to be taken.

Hence we have the following:—

$$\tfrac{1}{2}\log(1 + 2r\cos\theta + r^2) = \Sigma(-)^{n-1}r^n\cos n\theta/n \qquad (1);$$

$$\tan^{-1}\{r\sin\theta/(1 + r\cos\theta)\} = \Sigma(-)^{n-1}r^n\sin n\theta/n \qquad (2).$$

Although, strictly speaking, we have established these results for values of θ between $-\pi$ and $+\pi$ both inclusive, yet, since both sides are periodic functions of θ, they will obviously hold for all values of θ, provided $r < 1$.

If $r = 1$, (1) and (2) will still hold, provided $\theta \neq \pm\pi$; for the series in (1) and (2) are both convergent, and we have, by Abel's Theorem,

$$\cos\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{3}\cos 3\theta - \ldots = \mathrm{L}_{r=1} \tfrac{1}{2}\log(1 + 2r\cos\theta + r^2),$$
$$= \log(2\cos\tfrac{1}{2}\theta) \qquad (3);$$
$$\sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \ldots = \tan^{-1}\{\sin\theta/(1+\cos\theta)\},$$
$$= \tan^{-1}\{\tan\tfrac{1}{2}(\theta + 2k\pi)\},$$
$$= \tfrac{1}{2}\theta + k\pi \qquad (4),$$

where k must be so chosen that $\frac{1}{2}\theta + k\pi$ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Thus, if θ lie between $-\pi$ and $+\pi$, $k = 0$, and we have simply

$$\sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \ldots = \tfrac{1}{2}\theta \qquad (4').$$

In particular, if we put $\theta = \frac{1}{2}\pi$, we get

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13} + \ldots \qquad (5),$$

which is Gregory's quadrature; see § 41.

When $\theta = \pm(2p+1)\pi$, the series in (3) diverges to $-\infty$, and the right-hand side becomes $\log 0$, that is $-\infty$, so that (3) still holds in a certain sense.

The behaviour of the series in (4) when $\theta = \pm(2p+1)\pi$ is very curious. Let us take, for simplicity, the case $\theta = \pm\pi$. With this value of θ we have for values of r as near unity as we please $\tan^{-1}\{r\sin\theta/(1+r\cos\theta)\} = 0$. Hence, by Abel's Theorem, when $\theta = \pm\pi$, $\sin\theta - \frac{1}{2}\sin 2\theta + \ldots = 0$, as is otherwise sufficiently obvious.

On the other hand, for any value of θ differing from $\pm\pi$ by however little, we have $\mathrm{L}_{r=1} \tan^{-1}\{r\sin\theta/(1+r\cos\theta)\} = \frac{1}{2}\theta$. Hence, again, by Abel's Theorem, for $\theta = \pm\pi \mp \phi$, where ϕ is infinitely small, we have

$$\sin\theta - \tfrac{1}{2}\sin 2\theta + \ldots = \pm\tfrac{1}{2}\pi \mp \tfrac{1}{2}\phi.$$

The series $y = \sin\theta - \frac{1}{2}\sin 2\theta + \ldots$ is therefore discontinuous in the neighbourhood of $\theta = \pm\pi$; for, when $\theta = \pm\pi$, $y = 0$, and when θ differs infinitely little from $\pm\pi$, y differs infinitely little from $\pm\pi/2$. This discontinuity is accompanied by the phenomenon of infinitely slow convergence in the neighbourhood of $r = 1$, $\theta = \pm\pi$; and the sudden alteration of the value of the sum is associated with the fact that the values of the double limits

$\underset{r=1}{\mathrm{L}}\ \underset{\theta=\pm\pi}{\mathrm{L}} \tan^{-1}\{r\sin\theta/(1+r\cos\theta)\}$ and $\underset{\theta=\pm\pi}{\mathrm{L}}\ \underset{r=1}{\mathrm{L}} \tan^{-1}\{r\sin\theta/(1+r\cos\theta)\}$ are not alike.

When θ lies between π and 3π, we may put $\theta=2\pi+\theta'$, where θ' lies between $-\pi$ and $+\pi$, then, for such values of θ, we have

$$\begin{aligned} y &= \sin\theta' - \tfrac{1}{2}\sin 2\theta' + \ldots, \\ &= \tfrac{1}{2}\theta', \qquad \text{as we have already shown,} \\ &= \tfrac{1}{2}\theta - \pi. \end{aligned}$$

Hence, however small ϕ may be, we have, for $\theta=\pi+\phi$, $y=\frac{1}{2}\phi-\frac{1}{2}\pi$. But, as we have just seen, for $\theta=\pi-\phi$ we have $y=-\frac{1}{2}\phi+\frac{1}{2}\pi$. Hence, as θ varies from $\pi-\phi$ to $\pi+\phi$, y varies abruptly from $-\frac{1}{2}\phi+\frac{1}{2}\pi$ to $\frac{1}{2}\phi-\frac{1}{2}\pi$. In other words, as θ passes through the value π, y suffers an abrupt decrease amounting to π.*

We have discussed this case so fully because it is probably the first instance that the student has met with of a function having the kind of discontinuity figured in chap. xv., Fig. 5. It ought to be a good lesson regarding the necessity for care in handling limiting cases in the theory of infinite series.

§ 41.] *Gregory's Series.* If in equation (2) of last paragraph we put $\theta=\frac{1}{2}\pi$, we deduce the expansion

$$\tan^{-1}r = r - \tfrac{1}{3}r^3 + \tfrac{1}{5}r^5 - \ldots \qquad (6),$$

where $\tan^{-1}r$ represents, as usual, the principal value of the inverse function, and $-1 \not> r \not> 1$.

In particular, if $r=1$, we have

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \ldots).$$

The series (6), which is famous in the history of the quadrature of the circle, was first published by James Gregory in 1670; and independently, a few years later, by Leibnitz. About the beginning of the 18th century, two English calculators, Abraham Sharp and John Machin (Professor of Astronomy at Gresham College), used the series to calculate π to a large number of places. Sharp, using the formulæ $\frac{1}{6}\pi=\tan^{-1}1/\sqrt{3}=(1/\sqrt{3})\{1-1/3.3+1/5.3^2-\ldots\}$, suggested by Halley, carried the calculation to 71 places; that is, about twice as far as Ludolph van Ceulen had gone. Machin, using a formula of his own, for long the best that was known, namely, $\frac{1}{4}\pi=4\tan^{-1}1/5-\tan^{-1}1/239$, went to 100 places. Euler, apparently unaware of what the English calculators had done, used the far less effective formula $\frac{1}{4}\pi=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$. Gauss (*Werke*, Bd. ii., p. 501) found, by means of the theory of numbers, two remarkable formulæ of this kind, namely:—

$$\begin{aligned} \tfrac{1}{4}\pi &= 12\tan^{-1}1/18 + 8\tan^{-1}1/57 - 5\tan^{-1}1/239, \\ &= 12\tan^{-1}1/38 + 20\tan^{-1}1/57 + 7\tan^{-1}1/239 + 24\tan^{-1}1/268, \end{aligned}$$

* The reader should now draw the graph of the function y, for all real values of θ.

by means of which π could be calculated with great rapidity should its value ever be required beyond the 707th place, which was reached by Mr. Shanks in 1873!*

EXERCISES XIX.

Sum the following series to infinity, pointing out in each case the limits within which the summation is valid :—

(1.) $1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\ldots$.

(2.) $x\frac{\cos\theta}{1}+\frac{1}{2}\cdot x^3\frac{\cos 3\theta}{3}+\frac{1.3}{2.4}x^5\frac{\cos 5\theta}{5}+\ldots$.

(3.) $\frac{\cos\theta}{1}+\frac{1}{2}\frac{\cos 3\theta}{3}+\frac{1.3}{2.4}\frac{\cos 5\theta}{5}+\ldots$;
result $\frac{1}{2}\cos^{-1}(1-2\sin\theta)$.

(4.) $\Sigma(2n-1)(2n-3)\cos n\theta/n!$ (5.) $\Sigma\sin n\theta/(n+2)n!$

(6.) $e^{-\alpha}\sin\theta-\frac{1}{3}e^{-3\alpha}\sin 3\theta+\frac{1}{5}e^{-5\alpha}\sin 5\theta-\ldots$.

(7.) $\sin\theta-\frac{1}{2.3}\sin 2\theta+\frac{1}{3.4}\sin 3\theta-\ldots$.

(8.) $\sin^2\theta-\frac{1}{2}\sin^2 2\theta+\frac{1}{3}\sin^2 3\theta-\ldots$;
result $\frac{1}{2}\log\sec\theta$.

(9.) $\Sigma\cos 2n\theta/n(n-1)$. (10.) $\Sigma\sin n\theta/(n^2-1)$.

(11.) $\frac{1}{4}\sin\theta\sin\theta-\frac{1}{2}\sin 2\theta\sin^2\theta+\frac{1}{3}\sin 3\theta\sin^3\theta-\ldots$.

(12.) $\cos(\alpha+\beta)-\cos(\alpha+3\beta)/3!+\cos(\alpha+5\beta)/5!-\ldots$.

(13.) $\cos\theta-\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta-\ldots$;
result $\frac{1}{2}\log(2+2\cos\theta)$, except when $\theta=(2p+1)\pi$.

(14.) $\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{3}\cos 3\theta+\ldots$;
result $-\frac{1}{2}\log(2-2\cos\theta)$, except when $\theta=2p\pi$.

(15.) $\sin\theta+\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+\ldots$;
result $=0$, if $\theta=0$; $=\frac{1}{2}(\pi-\theta)$, if $0<\theta<\pi$; &c.

(16.) $\sin\theta-\frac{1}{3}\sin 3\theta+\frac{1}{5}\sin 5\theta-\ldots$.

(17.) $x\cos\theta-\frac{1}{3}x^3\cos 3\theta+\frac{1}{5}x^5\cos 5\theta-\ldots$;
result $\frac{1}{2}\tan^{-1}\{2x\cos\theta/(1-x^2)\}$.

(18.) $\cos\theta\cos\phi-\frac{1}{2}\cos 2\theta\cos 2\phi+\frac{1}{3}\cos 3\theta\cos 3\phi-\ldots$;
result $\frac{1}{2}\log\{4\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)\}$.

(19.) $x\cos\theta\cos\phi-\frac{1}{2}x^3\cos 3\theta\cos 3\phi+\frac{1}{5}x^5\cos 5\theta\cos 5\phi-\ldots$;
result $\frac{1}{2}\tan^{-1}[4x(1-x^2)\cos\theta\cos\phi/\{(1+x^2)^2-4x^2(\cos^2\theta-\cos^2\phi)\}]$.

(20.) Show that $\log(1+x+x^2)=2\Sigma(-)^{n-1}\cos\frac{1}{3}n\pi\,x^n/n$, provided mod $x<1$, and examine whether the result holds when mod $x=1$.

* For the history of this subject see *Ency. Brit.*, art. "Squaring the Circle," by Muir.

(21.) Show that, under certain restrictions upon θ,

$$\log(1+2\cos\theta) = -2\Sigma \cos \tfrac{1}{2}n\pi \cos n\theta/n ;$$
$$\theta = -\Sigma \cos \tfrac{1}{2}n\pi \sin n\theta/n.$$

(22.) Show that

$$\frac{\pi}{2\sqrt{2}} = 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{12}-\frac{1}{13}+\ldots .$$

(Newton, *Second Letter to Oldenburg*, 1676.)

Exercises XX.

(1.) Calculate π to 10 places by means of Machin's formula.

(2.) Show that, if $x<1$,

$$(\tan^{-1}x)^2 = x^2-(1+1/3)x^4/2+\ldots(-)^{n-1}\{1+1/3+\ldots+1/(2n-1)\}x^{2n}/n\ldots .$$

Does the formula hold when $x=1$?

(3.) Expand $\tan^{-1}(x+\cot\alpha)$ in powers of x.

(4.) Deduce the series for $\sin^{-1}x$ from Gregory's series by means of the addition theorem for the binomial coefficients.

(5.) If x lie between $1/\sqrt{2}$ and 1, show that

$$\sin^{-1}x = \pi - \frac{\sqrt{(1-x^2)}}{x}\left\{1-\frac{1}{3}\frac{1-x^2}{x^2}+\frac{1}{5}\frac{(1-x^2)^2}{x^4}-\ldots\right\}.$$

(6.) Show that § 38 (10) is merely a particular case of (7).

(7.) Show that

$$\frac{\theta}{\cos\theta} = \sin\theta+\frac{2}{3}\sin^3\theta+\frac{2.4}{3.5}\sin^5\theta+\frac{2.4.6}{3.5.7}\sin^7\theta+\ldots .$$

(Pfaff.)

(8.) $$\frac{1}{2}\theta^2 = \frac{\sin^2\theta}{2}+\frac{2}{3}\frac{\sin^4\theta}{4}+\frac{2.4}{3.5}\frac{\sin^6\theta}{6}+\ldots$$

(Stainville.)

(9.) $$\theta^3 = \sin^3\theta+\frac{3}{4}\cdot\frac{3}{5}\left(1+\frac{1}{3^2}\right)\sin^5\theta+\ldots$$
$$+\frac{3.5\ldots(2n-1)}{4.6\ldots 2n}\frac{3}{2n+1}\left(1+\frac{1}{3^2}+\ldots+\frac{1}{(2n-1)^2}\right)\sin^{2n+1}\theta+\ldots .$$

(10.) $$\theta^4 = \sin^4\theta+\frac{4}{3}\cdot\frac{2}{5}\left(1+\frac{1}{2^2}\right)\sin^6\theta+\ldots$$
$$+\frac{4.6\ldots(2n-2)}{5.7\ldots(2n-1)}\frac{2}{n}\left(1+\frac{1}{2^2}+\ldots+\frac{1}{(n-1)^2}\right)\sin^{2n}\theta+\ldots .$$

(11.) Deduce from § 38 (6) and (7) an expression for $\theta^m/\sin^m\theta$ in powers of $\sin\theta$.

(12.) If $\sin\theta = x\sin(\theta+\alpha)$, show that $\theta+r\pi = \Sigma x^n \sin n\alpha/n$.

(13.) If $c^2=a^2-2ab\cos C+b^2$, then

$$\log c = \log a-(b/a)\cos C-\tfrac{1}{2}(b/a)^2\cos 2C-\tfrac{1}{3}(b/a)^3\cos 3C-\ldots .$$

(14.) Show that

$$1-\frac{n-3}{2}+\frac{(n-4)(n-5)}{2.3}-\frac{(n-5)(n-6)(n-7)}{2.3.4}+\ldots = \frac{1+(-)^{n+1}2\cos\frac{2}{3}n\pi}{n}.$$

Show that

(15.)* $\theta^2 = \sin^2\theta + 2^2 \sin^4\frac{\theta}{2} + 2^4 \sin^4\frac{\theta}{2^2} + 2^6 \sin^4\frac{\theta}{2^3} + \ldots$.

(16.)* $u^2 = \sinh^2 u - 2^2 \sinh^4\frac{u}{2} - 2^4 \sinh^4\frac{u}{2^2} - 2^6 \sinh^4\frac{u}{2^3} + \ldots$.

(17.)* $\frac{3}{4}\theta = \sin\theta + 3 \sin^3\frac{\theta}{3} + 3^2 \sin^3\frac{\theta}{3^2} + \ldots$.

(18.)* $\frac{3}{4}\sin\theta = \frac{1}{4 \cdot 3^{m-1}} \sin 3^m\theta + \sum_1^m \frac{1}{3^{m-1}} \sin^3 3^{m-1}\theta$.

(19.)* $\frac{3}{4}\cos\theta = \sum_1^\infty \frac{(-1)^{m-1}}{3^{m-1}} \cos^3 3^{m-1}\theta$.

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* See Laisant, "Essai sur les Fonctions hyperboliques," *Mém. de la Soc. de Bordeaux*, 1875.

CHAPTER XXX.

General Theorems regarding the Expansion of Functions in Infinite Forms.

EXPANSION IN INFINITE SERIES.

§ 1.] *Cauchy's Theorem regarding the Expansion of a Function of a Function.*

If

$$y = a_0 + \Sigma a_n x^n \qquad (1),$$

the series being convergent so long as $\text{mod}\, x < \text{R}$, *and if*

$$z = b_0 + \Sigma b_n y^n \qquad (2),$$

this series being convergent so long as $\text{mod}\, y < \text{S}$, *then from* (1) *and* (2) *we can derive the expansion*

$$z = \text{C}_0 + \Sigma \text{C}_n x^n,$$

provided x *be such that* $\text{mod}\, x < \text{R}$, *and also*

$$\text{mod}\, a_0 + \Sigma\, \text{mod}\, a_n\, (\text{mod}\, x)^n < \text{S}.$$

This theorem follows readily from chap. xxvi., §§ 14 and 34. We have already used particular cases of it in previous chapters.

§ 2.] *Expansion of an Infinite Product in the form of an Infinite Series.*

If Σu_n *be an absolutely convergent series, and* ${}_n\Sigma u_1$, ${}_n\Sigma u_1 u_2$, . . ., ${}_n\Sigma u_1 u_2 \ldots u_r$, . . . *denote the sums of the products of its first* n *terms taken one, two,* . . ., r, . . ., *at a time, then*

$$\underset{n=\infty}{\text{L}}\, {}_n\Sigma u_1 = \text{T}_1, \quad \underset{n=\infty}{\text{L}}\, {}_n\Sigma u_1 u_2 = \text{T}_2, \quad \ldots, \quad \underset{n=\infty}{\text{L}}\, {}_n\Sigma u_1 u_2 \ldots u_r = \text{T}_r, \quad \ldots$$

where T_1, T_2, . . ., T_n . . . *are all finite.*

Also the infinite series $1 + \Sigma \text{T}_n$ *is convergent; and converges to the same limit as the infinite product* $\Pi(1 + u_n)$.

After what has been laid down in chap. xxvi., it will obviously be sufficient if we prove the above theorem on the assumption that all the symbols $u_1, u_2, \ldots, u_n, \ldots$ represent positive quantities. In the more general case where these are complex numbers the moduli alone would be involved in the statements of inequality, and the statements of equality would be true as under.

Since $u_1, u_2, \ldots, u_n, \ldots$ are all positive, we see, by the Multinomial Theorem (chap. xxiii, § 12), that

$$\begin{aligned} 0 < {}_n\Sigma u_1 u_2 \ldots u_r &< (u_1 + u_2 + \ldots + u_n)^r/r! \\ &< (u_1 + u_2 + \ldots + u_n + \ldots \text{ ad } \infty)^r/r! \\ &< S^r/r!, \qquad (1), \end{aligned}$$

where S is the finite limit of the convergent series Σu_n; and the inequality (1) obviously holds for all values of r up to $r = n$, however great n may be.

Therefore ${}_n\Sigma u_1 u_2 \ldots u_r$ has always a finite limit, T_r say, such that

$$0 \not> T_r \not> S^r/r! \qquad (2).$$

By (2), we have

$$0 < 1 + T_1 + T_2 + \ldots \text{ ad } \infty < 1 + S/1! + S^2/2! + \ldots \text{ ad } \infty,$$

that is,

$$0 < 1 + \sum_1^\infty T_n < e^S \qquad (3).$$

Hence $1 + \Sigma T_n$ is a convergent series, whose limit cannot exceed e^S.

Again, since $L_n \Sigma u_1 u_2 \ldots u_r = T_r$ when $n = \infty$, we may write

$${}_n\Sigma u_1 u_2 \ldots u_r = (1 + {}_rA_n)T_r \qquad (4).$$

where $L_r A_n = 0$ when $n = \infty$.

Hence, A_n being a mean among ${}_1A_n, {}_2A_n, \ldots, {}_nA_n$, and therefore such that $LA_n = 0$ when $n = \infty$, we have

$$\begin{aligned} \prod_1^n (1 + u_n) \equiv 1 + {}_n\Sigma u_1 + {}_n\Sigma u_1 u_2 + \ldots + {}_n\Sigma u_1 u_2 \ldots u_n \\ = 1 + (1 + A_n)\sum_1^n T_n \qquad (5). \end{aligned}$$

If in (5) we put $n = \infty$, we get

$$\prod_1^\infty(1 + u_n) = 1 + \mathrm{L}\{(1 + \mathrm{A}_n)\sum_1^n \mathrm{T}_n\},$$
$$= 1 + \sum_1^\infty \mathrm{T}_n \qquad (6),$$

since $\mathrm{LA}_n = 0$, and $\sum_1^\infty \mathrm{T}_n$ is finite.

This completes the proof of our proposition.

Cor. 1. *If* Σu_n *be absolutely convergent, then,* T_n *having the above meaning,* $1 + \Sigma x^n \mathrm{T}_n$ *will be convergent for all finite values of* x*; and we shall have*

$$\prod_1^\infty(1 + xu_n) = 1 + \sum_1^\infty x^n \mathrm{T}_n \qquad (7).$$

This follows at once by the above, and by chap. xxvi., § 27.

Cor. 2. *Let*

$$u_n = {}_nv_0 + {}_nv_1 x + {}_nv_2 x^2 + \ldots \qquad (8),$$

where ${}_nv_0$, ${}_nv_1$, *&c., are independent of* x, *and the series on the right of* (8) *may either terminate or not; and let*

$$u_n' = \mathrm{mod}\, {}_nv_0 + \mathrm{mod}\, {}_nv_1 (\mathrm{mod}\, x) + \mathrm{mod}\, {}_nv_2 (\mathrm{mod}\, x)^2 + \ldots \quad (9).$$

Then, if $\Sigma u_n'$ *be convergent for all values of* x *such that* $\mathrm{mod}\, x < \rho$, *it follows that for all such values* $\Pi(1 + u_n)$ *is convergent, and can be expanded in a convergent series of ascending powers of* x.

For, if T_n have the meaning above assigned to it, then it will obviously be possible to arrange T_n as an ascending series of powers of x. Moreover, if we consider the double series that thus arises from $1 + \Sigma \mathrm{T}_n$, we see that all Cauchy's conditions (see chap. xxvi., § 35) for the absolute convergence of this double series are satisfied. Hence we may arrange $1 + \Sigma \mathrm{T}_n$ as a convergent series of ascending powers of x.

Example 1. To expand $(1+x)(1+x^2)(1+x^4)(1+x^8) \ldots$ in an ascending series of powers of x. (Euler, *Introd. in Anal. Inf.*, § 328).

The series $\Sigma(\mathrm{mod}\, x)^{2^n}$ is obviously convergent so long as $\mathrm{mod}\, x < 1$. Hence, so long as $\mathrm{mod}\, x < 1$, we may write

$$(1+x)(1+x^2)(1+x^4)(1+x^8) \ldots = 1 + \mathrm{C}_1 x + \mathrm{C}_2 x^2 + \ldots + \mathrm{C}_n x^n + \ldots \quad (10).$$

To determine the coefficients C_1, C_2, C_n, we observe that, if we multiply both sides of (10) by $1 - x$, the left-hand side becomes $\mathrm{L}_{n=\infty}(1 - x^{2^n})$, that is, 1, since $\mathrm{mod}\, x < 1$. We must therefore have

$$1/(1 - x) = 1 + \mathrm{C}_1 x + \mathrm{C}_2 x^2 + \ldots + \mathrm{C}_n x^n + \ldots,$$

that is,

$$1+x+x^2+\ldots+x^n+\ldots=1+C_1x+C_2x^2+\ldots+C_nx^n+\ldots,$$

therefore $$C_1=C_2=\ldots=C_n=\ldots=1.$$

Another way is to put x^2 for x on both sides of (10), and then multiply by $(1+x)$. We thus get

$$1+\Sigma C_nx^n=1+x+C_1x^2+\ldots+C_nx^{2n}+C_nx^{2n+1}+\ldots;$$

whence $$C_{2n}=C_{2n+1}=C_n, \quad C_1=1,$$

from which it is easy to prove that all the coefficients are unity.

Example 2. To show that

$$(1+xz)(1+x^2z)\ldots(1+x^mz)$$
$$=1+\sum_{n=1}^{m}\frac{(1-x^m)(1-x^{m-1})\ldots(1-x^{m-n+1})}{(1-x)(1-x^2)\ldots(1-x^n)}x^{n(n+1)/2}z^n \quad (1).$$

(Cauchy, *Comptes Rendus*, 1840.)

Let

$$(1+xz)(1+x^2z)\ldots(1+x^mz)$$
$$=1+A_1z+A_2z^2+\ldots+A_nz^n+\ldots+A_mz^m \quad (2),$$

where $A_1, A_2, \ldots$ are functions of x which have to be determined.

Put xz in place of z on both sides of (2), then multiply on both sides by $(1+xz)/(1+x^{m+1}z)$, and we get

$$(1+xz)(1+x^2z)\ldots(1+x^mz)$$
$$=\{1+(1+A_1)xz+(A_1+A_2)x^2z^2+\ldots+(A_{n-1}+A_n)x^nz^n+\ldots+A_mx^{m+1}z^{m+1}\},$$
$$\times\{1-x^{m+1}z+x^{2(m+1)}z^2+\ldots(-)^nx^{n(m+1)}z^n+\ldots\} \quad (3).$$

Hence, arranging the right-hand side of (3) according to powers of z, replacing the left-hand side by its equivalent according to (2), and then equating the coefficients of z^n on the two sides, we get

$$A_n=(A_n+A_{n-1})x^n-x^{m+1}(A_{n-1}+A_{n-2})x^{n-1}$$
$$+x^{2(m+1)}(A_{n-2}+A_{n-3})x^{n-2}$$
$$\ldots\ldots\ldots\ldots$$
$$(-)^{n-1}x^{(n-1)(m+1)}(A_1+1)x$$
$$(-)^nx^{n(m+1)};$$

whence

$$\frac{1-x^n}{x^n(1-x^m)}A_n=A_{n-1}-A_{n-2}x^m+A_{n-3}x^{2m}-\ldots(-)^{n-1}x^{(n-1)m} \quad (4).$$

Putting $n-1$ in place of n in (4), we have

$$\frac{1-x^{n-1}}{x^{n-1}(1-x^m)}A_{n-1}=A_{n-2}-A_{n-3}x^m+A_{n-4}x^{2m}-\ldots(-)^{n-2}x^{(n-2)m} \quad (5).$$

If we multiply (5) by x^m and add (4), we derive, after an obvious reduction,

$$(1-x^n)A_n=(x^n-x^{m+1})A_{n-1} \quad (6_1).$$

In like manner,

$$(1-x^{n-1})A_{n-1}=(x^{n-1}-x^{m+1})A_{n-2} \quad (6_2),$$
$$(1-x^{n-2})A_{n-2}=(x^{n-2}-x^{m+1})A_{n-3} \quad (6_3),$$
$$\ldots\ldots\ldots\ldots$$
$$(1-x)A_1=(x-x^{m+1}) \quad (6_n).$$

Multiplying (6_1), (6_2), . . ., (6_n) together, we derive

$$A_n = \frac{(x-x^{m+1})(x^2-x^{m+1})\ldots(x^n-x^{m+1})}{(1-x)(1-x^2)\ldots(1-x^n)} \quad (7),$$

$$= \frac{(1-x^m)(1-x^{m-1})\ldots(1-x^{m-n+1})}{(1-x)(1-x^2)\ldots(1-x^n)} x^{n(n+1)/2} \quad (8),$$

which establishes our result.

If $\mod x < 1$, the product $(1+xz)(1+x^2z)$ . . . will be convergent when continued to infinity, and will, by the theorem of the present paragraph, be expansible in a series of powers of z. The series in question will be obtained by putting $m=\infty$ in (1). We thus get

$$(1+xz)(1+x^2z)\ldots \text{ad } \infty = 1 + \sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{(1-x)(1-x^2)\ldots(1-x^n)} z^n \quad (9),$$

an important theorem of Euler's (*Introd. in Anal. Inf.*, § 306).

§ 3.] *Expansion of Sech x and Sec x.*

We have, by the definition of Exp x,

$$2/(\text{Exp } x + \text{Exp } -x) = 1/(1 + \Sigma x^{2n}/(2n)!) \quad (1).$$

Hence, if $$y = \Sigma x^{2n}/(2n)! \quad (2),$$

$$2/(\text{Exp } x + \text{Exp } -x) = 1/(1+y),$$

$$= 1 + \Sigma(-)^n y^n \quad (3).$$

The expansion (3) will be valid provided $\mod y < 1$; and the series (2) is absolutely convergent for all finite values of x. Hence, if $\xi = \mod x$, it follows from § 1 that the series (3) can be converted into a series of ascending powers of x provided

$$\sum_{n=1}^{\infty} \xi^{2n}/(2n)! < 1 \quad (4).$$

This last condition involves that

$$\tfrac{1}{2}(e^{\xi} + e^{-\xi}) - 1 < 1;$$

that is, that $\xi < \log(2 + \sqrt{3})$.

This condition can obviously be satisfied; and we conclude that $2/(\text{Exp } x + \text{Exp } -x)$ can be expanded in a series of ascending powers of x provided $\mod x$ do not exceed a certain finite limit.

Since the function in question is obviously an even function of x, only even powers of x will occur in the expansion. We may therefore assume

$$2/(\text{Exp } x + \text{Exp } -x) = 1 + \Sigma(-)^n E_n x^{2n}/(2n)! \quad (5).$$

To determine E_1, E_2, . . ., we multiply one side of (5) by

$\frac{1}{2}(\mathrm{Exp}\, x + \mathrm{Exp} - x)$, and the other by its equivalent $1 + \Sigma x^{2n}/(2n)!$; we thus have

$$1 = \{1 + \Sigma(-)^n \mathrm{E}_n x^{2n}/(2n)!\}\{1 + \Sigma x^{2n}/(2n)!\} \qquad (6).$$

E_1, E_2, . . . must be so determined that (6) becomes an identity. We must therefore have

$$\frac{1}{(2n)!\,0!} - \frac{\mathrm{E}_1}{(2n-2)!\,2!} + \frac{\mathrm{E}_2}{(2n-4)!\,4!} - \ldots (-)^n \frac{\mathrm{E}_n}{0!\,(2n)!} = 0 \quad (7);$$

or,

$$\mathrm{E}_n = {}_{2n}\mathrm{C}_2 \mathrm{E}_{n-1} - {}_{2n}\mathrm{C}_4 \mathrm{E}_{n-2} + \ldots (-)^{n-2} {}_{2n}\mathrm{C}_{2n-2} \mathrm{E}_1 + (-1)^{n-1} \quad (8).$$

The last equation enables us to calculate E_1, E_2, E_3, . . . successively. We have, in fact,

$$\mathrm{E}_1 = 1\,; \quad \mathrm{E}_2 = 6\mathrm{E}_1 - 1\,; \quad \mathrm{E}_3 = 15\mathrm{E}_2 - 15\mathrm{E}_1 + 1\,;$$
$$\mathrm{E}_4 = 28\mathrm{E}_3 - 70\mathrm{E}_2 + 28\mathrm{E}_1 - 1\,; \quad \&c.$$

whence

$\mathrm{E}_1 = 1,$	$\mathrm{E}_6 = 2702765,$
$\mathrm{E}_2 = 5,$	$\mathrm{E}_7 = 199360981,$
$\mathrm{E}_3 = 61,$	$\mathrm{E}_8 = 19391512145,$
$\mathrm{E}_4 = 1385,$	$\mathrm{E}_9 = 2404879675441,$
$\mathrm{E}_5 = 50521,$	. . . . .

These numbers were first introduced into analysis by Euler;* and the above table contains their values so far as he calculated them.

Since the constants E_1, E_2, . . . are determined so as to make (6) an identity, (6), and therefore also (5), will be valid for all values of x, real or complex, which render all the series involved convergent. Hence, since $1 + \Sigma x^{2n}/(2n)!$ is convergent for all values of x, (5) will be valid for all values of x which render the series $1 + \Sigma(-)^n \mathrm{E}_n x^{2n}/(2n)!$ convergent. We shall determine the radius of convergency of this series presently. Meantime we observe that (5) as it stands may be written

$$\mathrm{Sech}\, x = 1 + \Sigma(-)^n \mathrm{E}_n x^{2n}/(2n)! \qquad (9);$$

and, if we put ix in place of x, it gives

$$\mathrm{Sec}\, x = 1 + \Sigma \mathrm{E}_n x^{2n}/(2n)! \qquad (10).$$

* See *Inst. Calc. Diff.*, § 224: the last five digits of E_9 are incorrectly given by Euler as 61671.

Cor. *Sech $^n x$ and Sec $^n x$ can each be expanded in a series of even powers of x.*

The possibility of such an expansion follows at once from the above. The coefficients may be expressed in terms of Euler's numbers. We may also use the identity $1 = (1 + \Sigma A_n x^{2n}/(2n)!) \cos^n x$; expand $\cos^n x$ first as a series of cosines of multiples of x; finally in powers of x; and thus obtain a recurrence formula for calculating $A_1, A_2, \ldots$ The convergency of any expansion thus obtained will obviously be co-extensive with the convergency of (10).

§ 4.] *Expansion of* Tanh x, x Coth x, Cosech x; Tan x, x Cot x, Cosec x.*

We have already shown, in chap. xxviii., § 6, for real values of x, that

$$x/(1 - e^{-x}) = 1 + \tfrac{1}{2}x + \Sigma(-)^{n-1} B_n x^{2n}/(2n)!,$$

the expansion being valid so long as the series on the right is convergent. In exactly the same way we can show, for any value of x real or complex, that

$$x/(1 - \mathrm{Exp} - x) = 1 + \tfrac{1}{2}x + \Sigma(-)^{n-1} B_n x^{2n}/(2n)! \qquad (1),$$

where $\mathrm{Exp} - x$ is defined as in chap. xxix., and x is such that mod x is less than the radius of convergency of the series in (1). From (1) we derive the following, all of which will be valid so long as the series involved are convergent

$$x(\mathrm{Exp}\, x - \mathrm{Exp} - x)/(\mathrm{Exp}\, x + \mathrm{Exp} - x) = 4x/(1 - \mathrm{Exp} - 4x) - 2x/(1 - \mathrm{Exp} - 2x) - x,$$
$$= \Sigma(-)^{n-1} 2^{2n}(2^{2n} - 1) B_n x^{2n}/(2n)! \qquad (2);$$

$$x(\mathrm{Exp}\, x + \mathrm{Exp} - x)/(\mathrm{Exp}\, x - \mathrm{Exp} - x) = x/(1 - \mathrm{Exp} - 2x) - x/(1 - \mathrm{Exp}\, 2x),$$
$$= 1 + \Sigma(-)^{n-1} 2^{2n} B_n x^{2n}/(2n)! \qquad (3);$$

$$2x/(\mathrm{Exp}\, x - \mathrm{Exp} - x) = 2x/(1 - \mathrm{Exp} - x) - 2x/(1 - \mathrm{Exp} - 2x),$$
$$= 1 + 2\Sigma(-)^{n}(2^{2n-1} - 1) B_n x^{2n}/(2n)! \qquad (4).$$

From these equations, we have at once

$$\mathrm{Tanh}\, x = \Sigma(-)^{n-1} 2^{2n}(2^{2n} - 1) B_n x^{2n-1}/(2n)! \qquad (5);$$
$$x\, \mathrm{Coth}\, x = 1 + \Sigma(-)^{n-1} 2^{2n} B_n x^{2n}/(2n)! \qquad (6);$$
$$x\, \mathrm{Cosech}\, x = 1 + 2\Sigma(-)^{n}(2^{2n-1} - 1) B_n x^{2n}/(2n)! \qquad (7).$$

* Euler, *l.c.*

If in (2), (3), and (4) we replace x by ix, we deduce

$$\operatorname{Tan} x = \Sigma 2^{2n}(2^{2n}-1)B_n x^{2n-1}/(2n)! \qquad (8);$$

$$x \operatorname{Cot} x = 1 - \Sigma 2^{2n} B_n x^{2n}/(2n)! \qquad (9);$$

$$x \operatorname{Cosec} x = 1 + 2\Sigma(2^{2n-1}-1)B_n x^{2n}/(2n)! \qquad (10).$$

Cor. *Each of the functions* $(\operatorname{Tanh} x)^n$, $(x \operatorname{Coth} x)^n$, $(x \operatorname{Cosech} x)^n$, $(\operatorname{Tan} x)^n$, $(x \operatorname{Cot} x)^n$, $(x \operatorname{Cosec} x)^n$ *can be expanded in an ascending series of powers of* x.

EXERCISES XXI.

(1.) If $\theta = \operatorname{gd} u$ (see chap. xxix., § 31), show that

$$\theta = a_1 u - a_3 u^3 + a_5 u^5 - \ldots,$$
$$u = a_1 \theta + a_3 \theta^3 + a_5 \theta^5 + \ldots,$$

where $a_{2n+1} = E_n/(2n+1)!$.

(2.) Find expressions for the coefficients in the expansions of $\operatorname{Sin}^n x$ and $\operatorname{Cos}^n x$.

(3.) Find recurrence-formulæ for calculating the coefficients in the expansions of $(x \operatorname{cosec} x)^n$ and $(\sec x)^n$.

In particular, show that

$$\operatorname{Sec}^{2p+1} x = \sum_{=0}^{\infty} n \frac{S_p E_n + S_{p-1} E_{n+1} + \ldots + S_1 E_{n+p-1} + E_{n+p}}{(2p)!} \cdot \frac{x^{2n}}{(2n)!},$$

where S_r denotes the sum of the products r at a time of $1^2, 3^2, 5^2, \ldots, (2p-1)^2$.

(Ely, *American Jour. Math.*, 1882.)

(4.) If $\operatorname{mod} x < 1$, show that

$$(1+x^2)(1+x^4)(1+x^6)\ldots \text{ ad } \infty = 1 + \Sigma x^{n^2+n}/(1-x^2)(1-x^4)\ldots(1-x^{2n}).$$

(5.) If $\operatorname{mod} x > 1$, and p be a positive integer, show that

$$1 + \sum_{n=1}^{\infty} \frac{x^{n(n+1-2p)/2}}{(x-1)(x^2-1)\ldots(x^n-1)} = 0.$$

(6.) Show that the Binomial Theorem for positive integral exponents is a particular case of § 2, Example 2.

(7.) Show that

$$(1+xz)(1+x^3z)\ldots(1+x^{2m-1}z) = 1 + \sum_{n=1}^{m} \frac{(1-x^{2m})(1-x^{2m-2})\ldots(1-x^{2m-2n+2})}{(1-x^2)(1-x^4)\ldots(1-x^{2n})} x^{n^2} z^n.$$

(Cauchy, *Comptes Rendus*, 1840.)

(8.) Show that

$$\frac{1}{(1-xz)(1-x^2z)\ldots(1-x^mz)} = 1 + \Sigma x^n z^n \frac{(1-x^m)(1-x^{m+1})\ldots(1-x^{m+n-1})}{(1-x)(1-x^2)\ldots(1-x^n)};$$

also that, if $\operatorname{mod} x < 1$, $\operatorname{mod} zx < 1$,

$$1/(1-xz)(1-x^2z)\ldots \text{ ad } \infty = 1 + \Sigma x^n z^n/(1-x)(1-x^2)\ldots(1-x^n).$$

(Euler, *Int. in Anal. Inf.*, § 313.)

(9.) If m be a positive integer $(1-x^m)(1-x^{m-1})\dots(1-x^{m-n+1})$ is exactly divisible by $(1-x)(1-x^2)\dots(1-x^n)$.

(Gauss, *Summatio quarumdam serierum singularium*, *Werke*, Bd. ii., p. 16.)

(10.) If $f(x, m)=1+\Sigma(-)^n\dfrac{(1-x^m)(1-x^{m-1})\dots(1-x^{m-n+1})}{(1-x)(1-x^2)\dots(1-x^n)}$, where mod $x>1$, show that

$$f(x, m)=f(x, m-2\lambda)(1-x^{m-1})(1-x^{m-3})\dots(1-x^{m-2\lambda+1})$$
$$=\frac{1-x^{m-1}}{1-x^{-1}}\cdot\frac{1-x^{m-3}}{1-x^{-3}}\cdot\frac{1-x^{m-5}}{1-x^{-5}}\dots \text{ ad } \infty.$$

Hence show that, if mod $x<1$, then

$$1+\Sigma x^{n(n+1)/2}=\frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^3}\cdot\frac{1-x^6}{1-x^5}\dots \text{ ad } \infty.$$

(Gauss, *Ib.*)

(11.) Show that, if m be a positive integer,

$$(1+x)(1+x^2)\dots(1+x^m)=1+\Sigma x^n\frac{(1-x^{2m})(1-x^{2m-2})\dots(1-x^{2m-2n+2})}{(1-x^2)(1-x^4)\dots(1-x^{2n})}.$$

(Gauss, *Ib.*)

(12.) Show that

$$\frac{1}{(1-xz)(1-x^3z)\dots(1-x^{2m-1}z)}$$
$$=1+\Sigma x^n z^n\frac{(1-x^{2m})(1-x^{2m+2})\dots(1-x^{2m+2n-2})}{(1-x^2)(1-x^4)\dots(1-x^{2n})}.$$

Also that, if mod $x<1$, and mod $zx<1$,

$$1/(1-xz)(1-x^3z)\dots \text{ ad } \infty=1+\Sigma x^n z^n/(1-x^2)(1-x^4)\dots(1-x^{2n}).$$

(13.) Show that, if mod $x<1$,

$$1/(1-x)(1-x^3)(1-x^5)\dots \text{ ad } \infty=(1+x)(1+x^2)(1+x^3)\dots \text{ ad } \infty.$$

(Euler, *l.c.*, § 325.)

(14.) If mod $x<1$,

$$(1-x)(1-x^2)(1-x^3)\dots \text{ ad } \infty=\sum_{-\infty}^{+\infty}(-)^n x^{(3n^2+n)/2}.$$

(Euler, *Nov. Comm. Pet.*, 1760.)

(15.) If mod $x<1$,

$$\log\{(1-x)(1-x^2)(1-x^3)\dots \text{ ad } \infty\}=-\sum_1^\infty f(n)x^n/n,$$

where $f(n)$ denotes the sum of all the divisors of the positive integer n; for example, $f(4)=1+2+4$.

Hence show that

$$\sum_1^\infty\frac{nx^n}{1-x^n}=\sum_1^\infty f(n)x^n.$$

(Euler, *Ib.*)

(16.) If $d(n)$ denote the number of the different divisors of the positive integer n, and mod $x<1$, show that

$$\sum_1^\infty d(n)x^n=\sum_1^\infty\frac{x^n}{1-x^n}.$$

(Lambert, *Essai d'Architectonique*, p. 507.)

Also that

$$\sum_1^\infty d(n)x^n = \sum_1^\infty x^{n^2}\left(\frac{1+x^n}{1-x^n}\right).$$

(Clausen, *Crelle's Jour.*, 1827.)

(17.) If $\mod x < 1$, show that

$$\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \ldots = \frac{x}{1+x^2} + \frac{x^2}{1+x^4} + \frac{x^3}{1+x^6} + \ldots .$$

(18.) $\Sigma x^{2n+1}/(1-x^{2n+1})^2 = \Sigma n x^n/(1-x^{2n})$.

$\Sigma(-1)^{n-1} n x^n/(1+x^n) = \Sigma(-)^{n-1} x^n/(1+x^n)^2$.

(19.) The sum of the products r at a time of x, x^2, . . ., x^n is

$$x^{r(r+1)/2}(x^{r+1}-1)(x^{r+2}-1)\ldots(x^n-1)/(x-1)(x^2-1)\ldots(x^{n-r}-1).$$

(20.) If S_r be the sum of the products r at a time of 1, x, . . ., x^{n-1}, then $S_r = S_{n-r}\, x^{-(n-1)(n-2r)/2}$.

(21.) Show that, if x lie between certain limits, and the roots of ax^2+bx+c be real, then $(px+q)/(ax^2+bx+c)$ can be expanded in the form $u_0 + \Sigma(u_n x^n + v_n x^{-n})$; and that, if the roots be imaginary, no expansion of this kind is possible for any value of x.

ON THE EXPRESSION OF CERTAIN FUNCTIONS IN THE FORM OF FINITE AND INFINITE PRODUCTS.

§ 5.] The following General Theorem covers a variety of cases in which it is possible to express a given function in the form of an infinite product; and will be of use to the student because it accentuates certain points in this delicate operation which are often left obscure if not misunderstood.

Let $f(n, p)$ be a function (with real or imaginary coefficients) of the integral variables n and p, such that $\mathrm{L}_{p=\infty} f(n, p)$ *is finite for all finite values of n, say* $\mathrm{L}_{p=\infty} f(n, p) = f(n)$; *and let us suppose that for all values of n and p $(n<p)$, however great, which exceed a certain finite value, $\mod f(n, p)/\mod f(n)$ is not infinite.*

Then

$$\mathrm{L}_{p=\infty} \prod_{n=1}^{p} \{1 + f(n, p)\} = \prod_1^\infty \{1 + f(n)\} \qquad (1),$$

provided $\Sigma \mod f(n)$ be convergent (that is, provided $\Pi\{1+f(n)\}$ be absolutely convergent).

Let us denote $\prod_{n=1}^{p} \{1 + f(n, p)\}$ by P_p; $\mathrm{L}_{p=\infty} \prod_{n=1}^{p} \{1 + f(n, p)\}$ by P; $\mod f(n, p)$ by $f_1(n, p)$; and $\mod f(n)$ by $f_1(n)$.

We may write

$$P_p = \prod_{n=1}^{m}\{1 + f(n, p)\} \prod_{n=m+1}^{p}\{1 + f(n, p)\},$$
$$= P_m Q_m, \text{ say,} \qquad (2).$$

Just as in chap. xxvi., § 26, we have

$$\mod (Q_m - 1) \not> \prod_{n=m+1}^{p}\{1 + f_1(n, p)\} - 1.$$

Now, by one of our conditions, if m, and therefore p, exceed a certain finite value, we may put $f_1(n, p)/f_1(n) = A_n$, where A_n is not infinite. If, therefore, A be an upper limit to A_n, and therefore finite and positive, we have $f_1(n, p) \not> A f_1(n)$. Hence

$$\mod (Q_m - 1) \not> \prod_{n=m+1}^{p}\{1 + A f_1(n)\} - 1,$$
$$\not> \prod_{m+1}^{\infty}\{1 + A f_1(n)\} - 1, \qquad (3).$$

Let us now put $p = \infty$ in (2). Since m is finite, and $\underset{p=\infty}{\mathrm{L}}\, f(n, p) = f(n)$, we have

$$\underset{p=\infty}{\mathrm{L}}\, P_m = \prod_{1}^{m}\{1 + f(n)\}.$$

Therefore

$$P = \prod_{1}^{m}\{1 + f(n)\} Q_m \qquad (4),$$

where Q_m is subject to the restriction (3).

Let us, finally, consider the effect of increasing m.

Since $\Pi\{1 + f_1(n)\}$ is absolutely convergent, $\Pi\{1 + A f_1(n)\}$ is absolutely convergent. It therefore follows that, by sufficiently increasing m, we can make $\prod_{m+1}^{\infty}\{1 + A f_1(n)\} - 1$, and, *a fortiori*, $\mod (Q_m - 1)$ as small as we please. Hence, by taking m sufficiently great, we can cause Q_m to approach 1 as nearly as we please. In other words, it follows from (4) that

$$P = \prod_{1}^{\infty}\{1 + f(n)\} \qquad (5).$$

In applying this theorem it is necessary to be very careful to see that both the conditions in the first part of the enunciation regarding the value of $f(n, p)$ are satisfied. Thus, for example, it is not sufficient that $\underset{p=\infty}{\mathrm{L}}\, f(n, p)$ have a finite definite value $f(n)$ for all finite values of n, and that $\Sigma f_1(n)$ be

absolutely convergent. This seems to be taken for granted by many mathematical writers; but, as will be seen from a striking example given below, such an assumption may easily lead to fallacious results.

§ 6.] *Factorisation of sinh pu, sinh u, sin pθ, and sin θ.**

From the result of chap. xii., § 20, we have, p being any positive integer,

$$x^{2p} - 1 \equiv (x^2 - 1) \prod_{n=1}^{p-1} \left(x^2 - 2x \cos \frac{n\pi}{p} + 1 \right) \qquad (1).$$

From this we have

$$\frac{x^{2p} - 1}{x^2 - 1} \equiv \prod_{n=1}^{p-1} \left(x^2 - 2x \cos \frac{n\pi}{p} + 1 \right);$$

whence, putting $x = 1$, and remembering that $\mathrm{L}(x^{2p}-1)/(x^2-1) = p$, we have

$$p = 2^{p-1} \prod_{1}^{p-1} (1 - \cos . n\pi/p) \qquad (2);$$

$$= 4^{p-1} \prod_{1}^{p-1} \sin^2 . n\pi/2p \qquad (3);$$

and, since $\sin . \pi/2p$, $\sin . 2\pi/2p$, . . ., $\sin . (p-1)\pi/2p$ are obviously all positive,

$$\sqrt{p} = 2^{p-1} \prod_{1}^{p-1} \sin . n\pi/2p \qquad (4).$$

If we divide both sides of (1) by x^p, we deduce

$$x^p - x^{-p} \equiv (x - x^{-1}) \Pi (x + x^{-1} - 2 \cos . n\pi/p) \qquad (5),$$

where for brevity we omit the limits for the product, which are as before.

If in (5) we put $x = e^u$, we get at once

$$\sinh pu = 2^{p-1} \sinh u \Pi (\cosh u - \cos . n\pi/p) \qquad (6),$$

$$= 4^{p-1} \sinh u \Pi (\sin^2 . n\pi/2p + \sinh^2 . u/2) \qquad (7).$$

Using (3), we can throw (7) into the following form:—

$$\sinh pu = p \sinh u \Pi \{ 1 + \sinh .^2 u/2/\sin .^2 n\pi/2p \} \qquad (8).$$

Finally, since (8) holds for all values of u, we may replace u by u/p, and thus derive

* The results in §§ 6-9 were all given in one form or another by Euler in his *Introductio in Analysin Infinitorum*. His demonstrations of the fundamental theorems were not satisfactory, although they are still to be found unaltered in many of our elementary text-books.

$$\sinh u = p \sinh \frac{u}{p} \prod_{n=1}^{p-1} \left\{ 1 + \frac{\sinh^2 . u/2p}{\sin^2 . n\pi/2p} \right\} \qquad (9).$$

We shall next apply to (9) the general theorem of § 5. Before doing so, we must, however, satisfy ourselves that the requisite conditions are fulfilled.

In the first place, so long as n is a finite integer, we have

$$\lim_{p=\infty} \frac{\sinh^2 . u/2p}{\sin^2 . n\pi/2p} = \frac{u^2}{n^2\pi^2} \qquad (10).$$

This can be deduced at once, for complex values of u, from the series for $\sinh . u/2p$ and $\sin . n\pi/2p$. When u is real it follows readily from chap. xxv., § 22.

The product $\Pi(1 + u^2/n^2\pi^2)$ is obviously absolutely convergent. We have, therefore, merely to show that, for all values of n and p exceeding a certain finite limit,

$$\operatorname{mod} \left\{ \frac{\sinh^2 . u/2p}{\sin^2 . n\pi/2p} \Big/ \frac{u^2}{n^2\pi^2} \right\} < A \qquad (11),$$

where A is a finite positive constant. That is to say, we have to show that

$$\operatorname{mod} \left(\frac{\sinh . u/2p}{u/2p} \right) \Big/ \left(\frac{\sin . n\pi/2p}{n\pi/2p} \right)$$

remains finite.

Now

$$\operatorname{mod} \left(\frac{\sinh . u/2p}{u/2p} \right) = \operatorname{mod} \left\{ 1 + \frac{1}{3!} \left(\frac{u}{2p} \right)^2 + \ldots \right\}$$

$$\geqslant 1 + \operatorname{mod} \left\{ \frac{1}{3!} \left(\frac{u}{2p} \right)^2 + \ldots \right\} \qquad (12).$$

Since the series within the bracket is absolutely convergent, its modulus can be made as small as we please by taking p sufficiently great.

Again we know, from chap. xxix., § 14, that, if $\theta \geqslant \sqrt{(6 \times 7)}$ $\geqslant 6{\cdot}48$, and, *a fortiori*, if $\theta \geqslant 2\pi$, then

$$\sin \theta < \theta - \tfrac{1}{6}\theta^3,$$

that is, if θ be positive,

$$\sin \theta/\theta < 1 - \tfrac{1}{6}\theta^2.$$

Now, since $n \geqslant p-1$, $n\pi/2p \geqslant \frac{1}{2}\pi$. Therefore

$$\frac{\sin.n\pi/2p}{n\pi/2p} \leqslant 1 - \tfrac{1}{6}\left(\frac{n\pi}{2p}\right)^2$$

$$\leqslant 1 - \frac{\pi^2}{24} \leqslant \cdot 58 \qquad (13).$$

From (12) and (13) it is abundantly evident that the condition (11) will be satisfied if only p be taken large enough; and it would be easy, if for any purpose it were necessary, to assign a numerical estimate for A. All the conditions for the applicability of the General Limit Theorem being fulfilled, we may make p infinite in (9). Remembering that $Lp \sinh.u/p = u$, we thus get

$$\sinh u = u \prod_{n=1}^{\infty} (1 + u^2/n^2\pi^2) \qquad (14).$$

To get the corresponding formulæ for $\sin p\theta$ and $\sin\theta$, we have simply to put in (5) $x = \exp i\theta$. The steps of the reasoning are, with a few trifling modifications, the same as before. It will therefore be sufficient to write down the main results with a corresponding numbering for the equations.

$$\sin p\theta = 2^{p-1} \sin\theta \prod_{n=1}^{p-1} (\cos\theta - \cos.n\pi/p) \qquad (6');$$

$$= 4^{p-1} \sin\theta \Pi(\sin^2.n\pi/2p - \sin^2.\theta/2) \qquad (7').$$

$$\sin p\theta = p \sin\theta \Pi(1 - \sin^2.\theta/2/\sin^2.n\pi/2p) \qquad (8').$$

$$\sin\theta = p \sin\frac{\theta}{p} \prod_{n=1}^{p-1} \left\{1 - \frac{\sin^2.\theta/2p}{\sin^2.n\pi/2p}\right\} \qquad (9').$$

$$\sin\theta = \theta \prod_{n=1}^{\infty} \{1 - \theta^2/n^2\pi^2\} \qquad (14').$$

It should be noticed that, inasmuch as (6), (7), (8), (9), and (14) were proved for all values of u, real and complex, we might have derived (6'), (7'), (8'), (9'), and (14') at once, by putting $u = i\theta$.

Cor. 1. *The following finite products for* $\sin p\theta$ *and* $\sinh pu$ should be noticed:—

$$\sin p\theta = 2^{p-1}\sin\theta\sin(\theta+\pi/p)\sin(\theta+2\pi/p)\ldots \sin(\theta+\overline{p-1}\pi/p) \quad (15);$$

$$\sinh pu = (-2i)^{p-1}\sinh u\sinh(u+i\pi/p)\sinh(u+2i\pi/p)\ldots \sinh(u+\overline{p-1}i\pi/p) \quad (16).$$

The first of these may be deduced from (6'), as follows :—

$$\sin p\theta = 2^{p-1}\sin\theta\Pi(\cos\theta-\cos.n\pi/p),$$
$$= 2^{p-1}\sin\theta\Pi\{2\sin(n\pi/2p+\theta/2)\sin(n\pi/2p-\theta/2)\},$$
$$= 2^{p-1}\sin\theta\Pi\{2\sin(n\pi/2p+\theta/2)\cos(\overline{p-n}\pi/2p+\theta/2)\}.$$

Hence, rearranging the factors, we get

$$\sin p\theta = 2^{p-1}\sin\theta\Pi\{2\sin(n\pi/2p+\theta/2)\cos(n\pi/2p+\theta/2)\},$$
$$= 2^{p-1}\sin\theta\prod_{n=1}^{p-1}\sin(\theta+n\pi/p).$$

We may deduce (16) from (15) by putting $\theta = -iu$.

Cor. 2. *Wallis's Theorem.*

If in (14') we put $\theta = \frac{1}{2}\pi$, we deduce

$$1 = \tfrac{1}{2}\pi\prod_{1}^{\infty}(1-1/2^2n^2) \quad (17);$$

whence
$$\frac{\pi}{2} = \frac{2^2}{1\cdot3}\cdot\frac{4^2}{3\cdot5}\cdot\ldots\frac{(2n)^2}{(2n-1)(2n+1)}\cdot\ldots \text{ ad } \infty,$$
$$= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\ldots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\ldots \text{ ad } \infty \quad (18).$$

This formula was given by Wallis in his *Arithmetica Infinitorum*, 1656. It is remarkable as the earliest expression of π by means of an infinite series of rational operations. Its publication probably led to the investigations of Brouncker, Newton, Gregory, and others, on the same subject.

§ 7.] *Factorisation of* $\cos p\theta$, $\cos\theta$, $\cosh pu$, $\cosh u$. Following the method of chap. xii., § 20, and using the roots of -1, we can readily establish the following identity :—

$$x^{2p}+1 \equiv \prod_{1=n}^{p}\left(x^2-2x\cos\frac{(2n-1)\pi}{2p}+1\right) \quad (1).$$

Putting herein $x=1$, we get

$$2 = 2^p\Pi(1-\cos.(2n-1)\pi/2p) \quad (2);$$
$$= 4^p\Pi\sin^2.(2n-1)\pi/4p \quad (3).$$

Hence, since all the sines are positive,

$$\sqrt{2} = 2^p \Pi \sin.(2n-1)\pi/4p \qquad (4).$$

From (1),

$$x^p + x^{-p} \equiv \Pi(x + x^{-1} - 2\cos.(2n-1)\pi/2p) \qquad (5);$$

whence, putting $x = \operatorname{Exp} i\theta$, we deduce

$$\cos p\theta = \tfrac{1}{2}.2^p\Pi(\cos\theta - \cos.(2n-1)\pi/2p) \qquad (6);$$

$$= \tfrac{1}{2}.4^p\Pi(\sin^2.(2n-1)\pi/4p - \sin^2.\theta/2) \qquad (7).$$

From (7), by means of (3), we derive

$$\cos p\theta = \Pi(1 - \sin^2.\theta/2/\sin^2.(2n-1)\pi/4p) \qquad (8).$$

From (8), putting θ/p in place of θ, we get

$$\cos\theta = \prod_{n=1}^{p}\left\{1 - \frac{\sin^2.\theta/2p}{\sin^2.(2n-1)\pi/4p}\right\} \qquad (9).$$

For any finite value of n we have

$$\mathrm{L}_{p=\infty} \frac{\sin^2.\theta/2p}{\sin^2.(2n-1)\pi/4p} = \frac{4\theta^2}{(2n-1)^2\pi^2} \qquad (10).$$

Also the product $\Pi(1 + 4\theta^2/(2n-1)^2\pi^2)$ is absolutely convergent.

Moreover,

$$\operatorname{mod}\left(\frac{\sin.\theta/2p}{\theta/2p}\right) = \operatorname{mod}\left\{1 - \frac{1}{3!}\left(\frac{\theta}{2p}\right)^2 + \ldots\right\}$$

$$\not> 1 + \operatorname{mod}\left\{\frac{1}{3!}\left(\frac{\theta}{2p}\right)^2 - \ldots\right\} \qquad (12);$$

So that mod $(\sin.\theta/2p/\theta/2p)$ can be brought as near to 1 as we please by sufficiently increasing p.

Also, since $(2n-1)\pi/4p \not> \frac{1}{2}\pi$, we have, exactly as in last paragraph,

$$\frac{\sin.(2n-1)\pi/4p}{(2n-1)\pi/4p} \not< \cdot 58 \qquad (13).$$

We may, therefore, put $p = \infty$ in (9); and we thus get

$$\cos\theta = \prod_{n=1}^{\infty}\{1 - 4\theta^2/(2n-1)^2\pi^2\} \qquad (14).$$

In like manner, putting $x = e^u$ in (5), we get

$$\cosh pu = \tfrac{1}{2}.2^p \prod_{n=1}^{p} (\cosh u - \cos.(2n-1)\pi/2p) \qquad (6');$$

$$= \tfrac{1}{2}.4^p \Pi(\sin^2.(2n-1)\pi/4p + \sinh^2.u/2) \qquad (7').$$

$$\cosh pu = \Pi\,(1 + \sinh^2.u/2/\sin.(2n-1)\pi/4p) \qquad (8').$$

$$\cosh u = \prod_{n=1}^{p} \left\{1 + \frac{\sinh^2.u/2p}{\sin^2.(2n-1)\pi/4p}\right\} \qquad (9').$$

$$\cosh u = \prod_{1}^{\infty} \{1 + 4u^2/(2n-1)^2\pi^2\} \qquad (14').$$

We might, of course, derive the hyperbolic from the circular formulæ by putting $\theta = iu$.

It is also important to observe that we might deduce (14) from the corresponding result of last paragraph, as follows:—

From (14') and (17) of last paragraph, we have

$$\sin\theta = \frac{2\theta}{\pi}\Pi\left\{\frac{1 - \theta^2/n^2\pi^2}{1 - 1/(2n)^2}\right\},$$

$$= \frac{2\theta}{\pi}\Pi\left\{\frac{2n\pi - 2\theta}{(2n-1)\pi} \cdot \frac{2n\pi + 2\theta}{(2n+1)\pi}\right\}.$$

Hence, putting $\frac{1}{2}\pi - \theta$ in place of θ, we deduce

$$\cos\theta = \frac{\pi - 2\theta}{\pi}\Pi\left\{\frac{(2n-1)\pi + 2\theta}{(2n-1)\pi} \cdot \frac{(2n+1)\pi - 2\theta}{(2n+1)\pi}\right\},$$

$$= (1 - 2\theta/\pi)\Pi\{(1 + 2\theta/(2n-1)\pi)(1 - 2\theta/(2n+1)\pi)\},$$

$$= (1 - 2\theta/\pi)(1 + 2\theta/\pi)(1 - 2\theta/3\pi)(1 + 2\theta/3\pi)\ldots\ .$$

Written in this last form the infinite product is only semi-convergent, and the order of its terms may not be altered without risk of changing its value; we may, however, associate them as they stand in groups of any finite number. Taking them in pairs, we have

$$\cos\theta = (1 - 4\theta^2/\pi^2)(1 - 4\theta^2/3^2\pi^2)\ldots,$$

$$= \prod_{n=1}^{\infty} \{1 - 4\theta^2/(2n-1)^2\pi^2\}.$$

§ 8.] From the above results we can deduce several others which will be useful presently.

We have, since all the products involved are absolutely convergent,

$$\frac{\sin(\theta+\phi)}{\sin\theta} = \frac{\theta+\phi}{\theta}\frac{\Pi\{1-(\theta+\phi)^2/n^2\pi^2\}}{\Pi\{1-\theta^2/n^2\pi^2\}},$$

provided $\theta \neq n\pi$.

Hence, provided $\theta \neq n\pi$,

$$\cos\phi + \sin\phi\cot\theta = \left(1+\frac{\phi}{\theta}\right)\Pi\left\{1-\frac{2\theta\phi+\phi^2}{n^2\pi^2-\theta^2}\right\} \quad (1).$$

In like manner, starting with $\cos(\theta+\phi)/\cos\theta$, we deduce

$$\cos\phi - \sin\phi\tan\theta = \Pi\left\{1-4\frac{2\theta\phi+\phi^2}{(2n-1)^2\pi^2-4\theta^2}\right\} \quad (2),$$

provided $\theta \neq \frac{1}{2}(2n-1)\pi$.

Also, from the identity

$$\frac{\sin\phi+\sin\theta}{\sin\theta} = \frac{\sin\frac{1}{2}(\phi+\theta)\cos\frac{1}{2}(\phi-\theta)}{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta},$$

we derive

$$\begin{aligned}1+\operatorname{cosec}\theta\sin\phi &= \left(1+\frac{\phi}{\theta}\right)\Pi\left[\frac{\{1-(\phi+\theta)^2/4n^2\pi^2\}\{1-(\phi-\theta)^2/(2n-1)^2\pi^2\}}{\{1-\theta^2/4n^2\pi^2\}\{1-\theta^2/(2n-1)^2\pi^2\}}\right]\\ &= \left(1+\frac{\phi}{\theta}\right)\Pi\left\{1-\frac{(-)^n 2\theta\phi+\phi^2}{n^2\pi^2-\theta^2}\right\} \quad (3),\end{aligned}$$

provided $\theta \neq n\pi$.

A great variety of other results of a similar character could be deduced; but these will suffice for our purpose.

§ 9.] Before leaving the present subject, it will be instructive to discuss an example which brings into prominence the necessity for one of the least obvious of the conditions for the applicability of the General Theorem of § 5.

We have, θ being neither 0 nor a multiple of π,

$$x^{2p} - 2x^p\cos\theta + 1 \equiv \{x^p - (\cos\theta + i\sin\theta)\}\{x^p - (\cos\theta - i\sin\theta)\}.$$

The pth roots of $\cos\theta + i\sin\theta$ are given by

$$\cos.(2n\pi+\theta)/p + i\sin.(2n\pi+\theta)/p, \quad n = 0, 1, \ldots, \overline{p-1} \quad (1).$$

The pth roots of $\cos\theta - i\sin\theta$, that is, of $\cos(-\theta)+i\sin(-\theta)$, by

$$\cos.(2n\pi-\theta)/p + i\sin.(2n\pi-\theta)/p, \quad n = 0, 1, \ldots, \overline{p-1} \quad (2).$$

Since $\cos.(2n\pi - \theta)/p = \cos.\{2(p-n)\pi + \theta\}/p,$

$\sin.(2n\pi - \theta)/p = -\sin.\{2(p-n)\pi + \theta\}/p,$

(2) may be replaced by

$$\cos.(2n\pi + \theta)/p - i\sin.(2n\pi + \theta)/p, \quad n = 0,\ 1,\ \ldots,\ p-1 \quad (2').$$

We have, therefore,

$$x^{2p} - 2x^p\cos\theta + 1 \equiv (x^2 - 2x\cos.\theta/p + 1)\prod_{n=1}^{p-1}\{x^2 - 2x\cos.(2n\pi+\theta)/p + 1\} \quad (3).$$

Since $\cos.(2n\pi+\theta)/p = \cos.\{2(p-n)\pi - \theta\}/p$, we may, if p be odd, arrange all the factors of the product on the right of (3) in pairs. Thus, if $p = 2q+1$, we have

$$x^{4q+2} - 2x^{2q+1}\cos\theta + 1 \equiv \left(x^2 - 2x\cos\frac{\theta}{2q+1} + 1\right)\prod_{n=1}^{q}\left\{\begin{array}{l}(x^2 - 2x\cos.(2n\pi+\theta)/(2q+1) + 1)\\ \times (x^2 - 2x\cos.(2n\pi-\theta)/(2q+1) + 1)\end{array}\right\} \quad (4).$$

If we now put $x = 1$, we get

$$4\sin^2\frac{\theta}{2} = 4^{2q+1}\sin^2\frac{\theta}{4q+2}\prod_{n=1}^{n=q}\left\{\sin^2.\frac{2n\pi+\theta}{4q+2}\sin^2.\frac{2n\pi-\theta}{4q+2}\right\} \quad (5).$$

If we divide both sides of (4) by x^{2q+1}, and put $x = \mathrm{Exp}\,i\phi$, we deduce

$$2(\cos(2q+1)\phi - \cos\theta) = 2^{2q+1}\{\cos\phi - \cos.\theta/(2q+1)\}\Pi\{\cos\phi - \cos.(2n\pi\pm\theta)/(2q+1)\} \quad (6),$$

where the double sign indicates that there are two factors to be taken.

Transforming (6), and using (5), &c., just as in the previous paragraphs, we get, finally,

$$\cos\phi - \cos\theta = 2\sin^2\tfrac{1}{2}\theta\left\{1 - \frac{\sin^2.\phi/(4q+2)}{\sin^2.\theta/(4q+2)}\right\}\prod_{n=1}^{q}\left\{1 - \frac{\sin^2.\phi/(4q+2)}{\sin^2.(2n\pi\pm\theta)/(4q+2)}\right\} \quad (7).$$

Since $n > q$, $(2n\pi\pm\theta)/(4q+2) > (2q\pi\pm\theta)/(4q+2)$; and the limit of this last when $q = \infty$ is $\frac{1}{2}\pi$. Hence, by taking q large enough we can secure that $(2n\pi\pm\theta)/(4q+2)$ shall have for its

upper limit a quantity which differs from $\frac{1}{2}\pi$ by as little as we please; and therefore (see § 6) that $\sin.(2n\pi \pm \theta)/(4q+2)/(2n\pi \pm \theta)/(4q+2)$ shall have for its lower limit a quantity not less than ·58.

We may, therefore, put $q = \infty$, &c., in (7). We then get

$$\cos\phi - \cos\theta = 2\sin^2\tfrac{1}{2}\theta(1 - \phi^2/\theta^2)\prod_{n=1}^{\infty}\{1 - \phi^2/(2n\pi \pm \theta)^2\} \quad (8),$$

that is,

$$\cos\phi - \cos\theta = 2\sin^2\tfrac{1}{2}\theta\left\{1 - \frac{\phi^2}{\theta^2}\right\}\left\{1 - \frac{\phi^2}{(2\pi-\theta)^2}\right\}\left\{1 - \frac{\phi^2}{(2\pi+\theta)^2}\right\}\cdots.$$

Putting $\phi = iu$ in (8), we deduce

$$\cosh u - \cos\theta = 2\sin^2\tfrac{1}{2}\theta(1 + u^2/\theta^2)\prod_{n=1}^{\infty}\{1 + u^2/(2n\pi \pm \theta)^2\} \quad (9).$$

The formula (8) might have been readily derived from those of previous paragraphs by using the identity $\cos\phi - \cos\theta = 2\sin\frac{1}{2}(\theta+\phi)\sin\frac{1}{2}(\theta-\phi)$ and proceeding as in the latter part of § 7.

Remark.—At first sight, it seems as if we might have dispensed with the transformation (4) and reasoned directly from (3), thus—

From (3) we deduce

$$2(\cos p\phi - \cos\theta) = 2^p(\cos\phi - \cos.\,\theta/p)\prod_{n=1}^{p-1}\{\cos\phi - \cos.\,(2n\pi+\theta)/p\},$$

Hence

$$\cos\phi - \cos\theta = 2\sin^2\tfrac{1}{2}\theta\left\{1 - \frac{\sin^2\phi/2p}{\sin^2\theta/2p}\right\}\prod_{n=1}^{p-1}\left\{1 - \frac{\sin^2.\,\phi/2p}{\sin^2.(2n\pi+\theta)/2p}\right\}.$$

Put now $p = \infty$, &c., and we get

$$\cos\phi - \cos\theta = 2\sin^2\tfrac{1}{2}\theta(1 - \phi^2/\theta^2)\prod_{1}^{\infty}\{1 - \phi^2/(2n\pi+\theta)^2\}.$$

This result is manifestly in contradiction with (8), although the reasoning by which it is established is the same as that often considered sufficient in such cases.

In point of fact, however, the condition of § 5, that $M = f_1(n, p)/f_1(n)$ must remain finite when n and p exceed certain limits, is not satisfied.

In the present case the upper limit of $(2n\pi + \theta)/2p$, namely, $\{2(p-1)\pi + \theta\}/2p$, can be made to approach as near to π as we please. Hence in this case M may become infinite. We have, in fact,

$$M = \mathrm{mod}\left\{\frac{\sin.(\phi/2p)/(\phi/2p)}{\sin.(2n\pi+\theta)/2p/(2n\pi+\theta)/2p}\right\}^2;$$

hence, if we give n its extreme value $p-1$, and put $p=\infty$, M becomes infinite. No finite upper limit to the modulus M can therefore be assigned; and the General Theorem of § 5 cannot be applied.

This is an instructive example of the danger of reasoning rashly concerning the limits of infinite products.

Exercises XXII.

(1.) If $(1+ix/a)(1+ix/b)(1+ix/c)\ldots = A+iB$, then
$$\Sigma \tan^{-1}(x/a) = \tan^{-1}(B/A).$$

Hence show that $\sum_1^\infty \tan^{-1}(2/n^2) = 3\pi/4$.

(Glaisher, *Quart. Jour. Math.*, 1878.)

(2.) Find the n roots of

$$x^n - nx^{n-2} + \frac{n(n-3)}{2!}x^{n-4} - \ldots$$
$$(-)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}x^{n-2r} + \ldots = 0.$$

(3.) If n be an odd integer, find the n roots of the equation

$$x + \frac{n^2-1^2}{3!}x^3 + \frac{(n^2-1^2)(n^2-3^2)}{5!}x^5 + \frac{(n^2-1^2)(n^2-3^2)(n^2-5^2)}{7!}x^7 + \ldots = a.$$

(4.) Solve completely

$$x^n + {}_nC_1\cos\alpha\, x^{n-1} + {}_nC_2\cos 2\alpha\, x^{n-2} + \ldots + \cos n\alpha = 0.$$

(Math. Trip., 1882.)

(5.) The roots of

$$x^n\sin n\theta - {}_nC_1x^{n-1}\sin(n\theta+\phi) + {}_nC_2x^{n-2}\sin(n\theta+2\phi) - \ldots = 0$$

are given by $x = \sin(\theta+\phi-k\pi/n)\,\mathrm{cosec}\,(\theta-k\pi/n)$, where $k = 0, 1, \ldots,$ or $(n-1)$.

If $\alpha = \pi/2p$, prove the following relations:—

(6.) $p = 2^{p-1}\sin 2\alpha \sin 4\alpha \ldots \sin(2p-2)\alpha$;

$1 = 2^{p-1}\sin\alpha \sin 3\alpha \ldots \sin(2p-1)\alpha$.

(7.) $\sqrt{p} = 2^{p-1} \cos\alpha \cos 2\alpha \dots \cos(p-1)\alpha$.

(8.) $1 = 2^{p-1} \sin.\, \alpha/2 \sin.\, 3\alpha/2 \dots \sin.\,(2p-1)\alpha/2$;

$= 2^{p-1} \cos.\, \alpha/2 \cos.\, 3\alpha/2 \dots \cos.\,(2p-1)\alpha/2$.

(9.) $\sin p\theta = 2^{p-1} \sin\theta \sin(2\alpha+\theta) \sin(4\alpha+\theta) \dots \sin(\overline{2p-2}\alpha+\theta)$;

$\cos p\theta = 2^{p-1} \sin(\alpha+\theta) \sin(3\alpha+\theta) \sin(5\alpha+\theta) \dots \sin(\overline{2p-1}\alpha+\theta)$.

(10.) $\tan p\theta = \tan\theta \tan(\theta+2\alpha) \dots \tan(\theta+(2p-2)\alpha)$, where p is odd.

(11.) $\tan\theta \tan(\theta+2\alpha) \dots \tan(\theta+(2p-2)\alpha) = (-1)^{p/2}$, where p is even.

(12.) Show that the modulus of

$$\cos(\theta+i\phi) \cos(\theta+i\phi+\pi/p) \dots \cos(\theta+i\phi+(p-1)\pi/p)$$

is $\{\cosh p\phi - \cos(p\pi+2p\theta)\}/2^{p-\frac{1}{2}}$.

(13.) If n be even, show that

$$\sin^2\frac{\theta}{2} = (-)^{n/2} 2^{n-2} \cos\frac{\theta}{n} \cos\frac{\theta+2\pi}{n} \cos\frac{\theta+4\pi}{n} \dots \cos\frac{\theta+(2n-2)\pi}{n}.$$

(14.) Show that $\prod_0^n (1+\sec 2^n\theta) = \tan 2^n\theta/\tan\theta$;

and evaluate $\prod_0^\infty \left\{\frac{1+\sec(\theta/2^n)}{2}\right\}$.

(15.) Show that

$$\prod_1^\infty \left(1-\frac{4}{3}\sin^2\frac{\theta}{3^n}\right) = \frac{\sin\theta}{\theta};$$

$$\prod_1^\infty \left(1-4\sin^2\frac{\theta}{3^n}\right) = \cos\theta ;$$

and write down the corresponding formulæ for the hyperbolic functions.

(Laisant.)

Prove the following results (Euler, *Int. in Anal. Inf.*, chap. ix.) :—

(16.) $$\frac{e^{b+x}+e^{c+x}}{e^b+e^c} = \Pi\left\{1+\frac{4(b-c)x+4x^2}{(2n-1)^2\pi^2+(b-c)^2}\right\};$$

$$\frac{e^{b+x}-e^{c-x}}{e^b-e^c} = \left(1+\frac{2x}{b-c}\right)\Pi\left\{1+\frac{4(b-c)x+4x^2}{(2n)^2\pi^2+(b-c)^2}\right\}.$$

(17.) $$\frac{\cosh y+\cosh c}{1+\cosh c} = \Pi\left\{1+\frac{\pm 2cy+y^2}{(2n-1)^2\pi^2+c^2}\right\};$$

$$\frac{\cosh y-\cosh c}{1-\cosh c} = \left(1-\frac{y^2}{c^2}\right)\Pi\left\{1-\frac{\pm 2cy+y^2}{(2n)^2\pi^2+c^2}\right\};$$

$$\frac{\sinh y+\sinh c}{\sinh c} = \left(1+\frac{y}{c}\right)\Pi\left\{1+\frac{(-)^n 2cy+y^2}{n^2\pi^2+c^2}\right\};$$

$$\frac{\sinh y-\sinh c}{\sinh c} = -\left(1-\frac{y}{c}\right)\Pi\left\{1+\frac{(-)^{n-1}2cy+y^2}{n^2\pi^2+c^2}\right\}.$$

Write down the corresponding formulæ for the circular functions, and deduce them by transformation from § 9.

(18.) $\dfrac{\cos\phi+\cos\theta}{1+\cos\theta}=\Pi\left\{1-\dfrac{\phi^2}{((2n-1)\pi\pm\theta)^2}\right\}$.

(19.) $\cos\phi+\tan\tfrac{1}{2}\theta\sin\phi=\Pi\left\{\left(1+\dfrac{2\phi}{(2n-1)\pi-\theta}\right)\left(1-\dfrac{2\phi}{(2n-1)\pi+\theta}\right)\right\}$.

(20.) $\dfrac{\cos(\theta-\phi)}{\cos\theta}=\Pi\left\{\left(1+\dfrac{2\phi}{(2n-1)\pi-2\theta}\right)\left(1-\dfrac{2\phi}{(2n-1)\pi+2\theta}\right)\right\}$;

$\dfrac{\sin(\theta-\phi)}{\sin\theta}=\left(1-\dfrac{\phi}{\theta}\right)\Pi\left\{\left(1+\dfrac{\phi}{2n\pi-\theta}\right)\left(1-\dfrac{\phi}{2n\pi+\theta}\right)\right\}$.

(21.) Show that

$$\cosh 2v-\cos 2u=2(u^2+v^2)\Pi\left\{\frac{(n\pi\pm u)^2+v^2}{n^2\pi^2}\right\};$$

$$\cosh 2v+\cos 2u=2\Pi\left\{\frac{((2n-1)\pi\pm 2u)^2+4v^2}{(2n-1)^2\pi^2}\right\};$$

$$\cosh 2u-\cos 2u=4u^2\Pi\left\{1+\frac{4u^4}{n^4\pi^4}\right\};$$

$$\cosh 2u+\cos 2u=2\Pi\left\{1+\frac{2^4u^4}{(2n-1)^4\pi^4}\right\}.$$

(Schlömilch, *Hanb. d. Alg. Anal.*, chap. xi.)

(22.) Evaluate $\prod_1^\infty\left(\dfrac{4n^2-4n+5}{4n^2-4n+1}\right)$.

(23.) If $\varpi=\log(1+\sqrt{2})$, show that

$$\sqrt{2}=\left(1+4\frac{\varpi^2}{1^2\pi^2}\right)\left(1+4\frac{\varpi^2}{3^2\pi^2}\right)\ldots \text{ ad } \infty.$$

EXPANSION OF THE CIRCULAR AND HYPERBOLIC FUNCTIONS IN AN INFINITE SERIES OF PARTIAL FRACTIONS.

§ 10.] By § 8 we have, provided $\theta\neq\frac{1}{2}(2n-1)\pi$,

$$\cos\phi-\sin\phi\tan\theta=\Pi\left\{1-4\frac{2\theta\phi+\phi^2}{(2n-1)^2\pi^2-4\theta^2}\right\} \qquad (1).$$

Now, referring to § 2, Cor. 2, we have here

$$u_n'=8\bmod\left\{\frac{\theta}{(2n-1)^2\pi^2-4\theta^2}\right\}(\bmod\phi)$$

$$+4\bmod\left\{\frac{1}{(2n-1)^2\pi^2-4\theta^2}\right\}(\bmod\phi)^2,$$

$$\mathrel{\dot{+}}\frac{8\theta'}{\bmod\{(2n-1)^2\pi^2-4\theta'^2\}}\phi'+\frac{4}{\bmod\{(2n-1)^2\pi^2-4\theta'^2\}}\phi'^2,$$

where $\theta'=\bmod\theta$, $\phi'=\bmod\phi$. It follows, therefore, that the product in (1) may be expanded as an ascending series of powers of ϕ.

Expanding also on the left of (1), we have

$$1 - \frac{\phi^2}{2!} + \ldots - \tan\theta\left(\phi - \frac{\phi^3}{3!} + \ldots\right)$$

$$= 1 - 4(2\theta\phi + \phi^2)\Sigma\frac{1}{(2n-1)^2\pi^2 - 4\theta^2}$$

$$+ 16(2\theta\phi + \phi^2)^2\Sigma\frac{1}{\{(2m-1)^2\pi^2 - 4\theta^2\}\{(2n-1)^2\pi^2 - 4\theta^2\}}$$

$$\ldots \qquad (2).$$

Since the two series in (2) must be identical, we have, by comparing the coefficients of ϕ,

$$\tan\theta = 8\theta\sum_1^\infty \frac{1}{(2n-1)^2\pi^2 - 4\theta^2} \qquad (3).$$

This series, which is analogous to the expansion of a rational function in partial fractions obtained in chap. viii., is absolutely convergent for all values of θ except $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, . . . It should be observed, however, that when θ lies between $\frac{1}{2}(2n-1)\pi$ and $\frac{1}{2}(2n+1)\pi$, the most important terms of the series are those in the neighbourhood of the nth term, so that the rate of convergence diminishes as θ increases.

We may, if we please, decompose $8\theta/\{(2n-1)^2\pi^2 - 4\theta^2\}$ into $2/\{(2n-1)\pi - 2\theta\} - 2/\{(2n-1)\pi + 2\theta\}$, and write the series (3) in the semi-convergent form

$$\tan\theta = \frac{2}{\pi - 2\theta} - \frac{2}{\pi + 2\theta} + \frac{2}{3\pi - 2\theta} - \frac{2}{3\pi + 2\theta}$$

$$+ \frac{2}{5\pi - 2\theta} - \frac{2}{5\pi + 2\theta} + \ldots \qquad (3').$$

In exactly the same way, we deduce from (1) and (3) of § 8 the following :—

$$\theta\cot\theta = 1 - 2\theta^2\Sigma\frac{1}{n^2\pi^2 - \theta^2} \qquad (4),$$

or

$$\theta\cot\theta = 1 - \frac{\theta}{\pi - \theta} + \frac{\theta}{\pi + \theta} - \frac{\theta}{2\pi - \theta} + \frac{\theta}{2\pi + \theta}$$

$$- \frac{\theta}{3\pi - \theta} + \frac{\theta}{3\pi + \theta} - \ldots \qquad (4'),$$

provided $\theta \neq \pi, 2\pi, 3\pi, \ldots$;
and

$$\theta \operatorname{cosec} \theta = 1 + 2\theta^2 \Sigma \frac{(-1)^{n-1}}{n^2\pi^2 - \theta^2} \qquad (5),$$

or

$$\theta \operatorname{cosec} \theta = 1 + \frac{\theta}{\pi - \theta} - \frac{\theta}{\pi + \theta} - \frac{\theta}{2\pi - \theta} + \frac{\theta}{2\pi + \theta} + \frac{\theta}{3\pi - \theta} - \frac{\theta}{3\pi + \theta} - \ldots \qquad (5'),$$

provided $\theta \neq \pi, 2\pi, 3\pi, \ldots$.

We might derive (4) from (3) by writing $(\frac{1}{2}\pi - \theta)$ for θ on both sides, multiplying by θ, decomposing into a semi-convergent form like (3'), and then reassociating the terms in pairs; also (5) might be deduced from (3) and (4) by using the identity $2 \operatorname{cosec} \theta = \tan \frac{1}{2}\theta + \cot \frac{1}{2}\theta$.

When we attempt to get a corresponding result for $\sec \theta$, the method employed above ceases to work so easily; and the result obtained is essentially different. We can reach it most readily by transformation from (5'). If we put (5') into the form

$$\operatorname{cosec} \theta = \frac{1}{\theta} + \frac{1}{\pi - \theta} - \frac{1}{\pi + \theta} - \frac{1}{2\pi - \theta} + \frac{1}{2\pi + \theta} + \frac{1}{3\pi - \theta} - \frac{1}{3\pi + \theta} - \ldots,$$

which we may do, provided $\theta \neq 0$, and then put $\frac{1}{2}\pi - \theta$ in place of θ, we get

$$\sec \theta = \frac{2}{\pi - 2\theta} + \frac{2}{\pi + 2\theta} - \frac{2}{3\pi - 2\theta} - \frac{2}{3\pi + 2\theta} + \frac{2}{5\pi - 2\theta} + \frac{2}{5\pi + 2\theta} - \ldots \qquad (6');$$

or, if we combine the terms in pairs,

$$\sec \theta = 4\Sigma(-)^{n-1} \frac{(2n-1)\pi}{(2n-1)^2\pi^2 - 4\theta^2} \qquad (6),$$

where $\theta \neq \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$.

The series (6), unlike its congeners (3), (4), and (5), *is only*

semi-convergent; for, when n is very large, its nth term is comparable with the nth term of the series $\Sigma 1/(2n-1)$.

We might, by pairing the terms differently, obtain an absolutely convergent series for sec θ, namely,

$$\sec\theta = \frac{2}{\pi - 2\theta} + 4(\pi - 2\theta)\sum_{1}^{\infty}\frac{(-1)^{n-1}}{4n^2\pi^2 - (\pi - 2\theta)^2} \qquad (7);$$

but this is essentially different in form from (3), (4), and (5).

Cor. 1. *The sum of all the products two and two of the terms of the series* $\Sigma 1/\{(2n-1)^2\pi^2 - 4\theta^2\}$ *is* $(\tan\theta - \theta)/128\theta^3$; *and the like sum for the series* $\Sigma 1/\{n^2\pi^2 - \theta^2\}$ *is* $(3 - \theta^2 - 3\theta\cot\theta)/8\theta^4$.

This may be readily established by comparing the coefficients of ϕ^3 in (2) above, and in the corresponding formula derived from § 8 (1).

Cor. 2. *The series* $\Sigma 1/\{(2n-1)^2\pi^2 - 4\theta^2\}^2$ *converges to the value* $(\theta\tan^2\theta - \tan\theta + \theta)/64\theta^3$; *and* $\Sigma 1/(n^2\pi^2 - \theta^2)^2$ *to the value* $(\theta^2\operatorname{cosec}^2\theta + \theta\cot\theta - 2)/4\theta^4$.

Since the above series have been established for all values of θ, real and imaginary, subject merely to the restriction that θ shall not have a value which makes the function to be expanded infinite, we may, if we choose, put $\theta = ui$. We thus get, *inter alia*,

$$\tanh u = 8u\Sigma 1/\{(2n-1)^2\pi^2 + 4u^2\} \qquad (8);$$

$$u\coth u = 1 + 2u^2\Sigma 1/\{n^2\pi^2 + u^2\} \qquad (9);$$

$$u\operatorname{cosech} u = 1 - 2u^2\Sigma(-1)^{n-1}/\{n^2\pi^2 + u^2\} \qquad (10);$$

$$\operatorname{sech} u = 4\Sigma(-)^{n-1}(2n-1)\pi/\{(2n-1)^2\pi^2 + 4u^2\} \qquad (11).$$

EXPRESSIONS FOR THE NUMBERS OF BERNOULLI AND EULER. RADIUS OF CONVERGENCY FOR THE EXPANSIONS OF TAN θ, COT θ, COSEC θ, AND SEC θ.

§ 11.] If mod $\theta < \pi$, then every term of the infinite series $\Sigma\theta^2/(n^2\pi^2 - \theta^2)$ can be expanded in an absolutely convergent series of ascending powers of θ. Also, when all the powers of θ are replaced by their moduli, the series arising from $1/(n^2\pi^2 - \theta^2)$ will simply become $1/\{n^2\pi^2 - (\text{mod }\theta)^2\}$, which is positive, since mod $\theta < \pi$. The double series

$$\sum_{n=1}^{\infty}\left\{\frac{\theta^2}{n^2\pi^2}+\frac{\theta^4}{n^4\pi^4}+\ldots+\frac{\theta^{2m}}{n^{2m}\pi^{2m}}+\ldots\right\}$$

therefore satisfies Cauchy's criterion, and may be arranged according to powers of θ. Hence, if

$$\sigma_{2m}=1/1^{2m}+1/2^{2m}+1/3^{2m}+\ldots \qquad (1),$$

we have, by § 10 (4),

$$\begin{aligned}\theta\cot\theta &= 1-2\Sigma\theta^2/(n^2\pi^2-\theta^2),\\ &= 1-2\Sigma\sigma_{2m}\theta^{2m}/\pi^{2m} \qquad (2).\end{aligned}$$

Since $\sigma_{2m}(<\sigma_2)$ is certainly finite,* the series (2) will be convergent so long as, and no longer than, $\theta<\pi$.

Now, by § 4 (9), we have

$$\theta\cot\theta=1-\Sigma 2^{2m}B_m\theta^{2m}/(2m)! \qquad (3),$$

provided θ be small enough.

The two series (2) and (3) must be identical. Hence we have

$$B_m=\frac{2(2m)!\sigma_{2m}}{(2\pi)^{2m}}=\frac{2(2m)!}{(2\pi)^{2m}}\left\{\frac{1}{1^{2m}}+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\ldots\right\} \qquad (4).$$

§ 12.] If, instead of using the expansion for $\theta\cot\theta$, we had used in a similar way the expansion for $\tan\theta$, we should have arrived at the formula

$$B_m=\frac{2(2m)!}{(1-1/2^{2m})(2\pi)^{2m}}\left\{\frac{1}{1^{2m}}+\frac{1}{3^{2m}}+\frac{1}{5^{2m}}+\ldots\right\} \qquad (5).$$

This last result may be deduced very readily from (4); it is, indeed, merely the first step in a remarkable transformation of the formula (4), which depends on a transformation of the series σ_m due to Euler.† We observe that the result of multiplying the convergent series σ_{2m} by $1-1/2^{2m}$ is to deprive the series of all terms whose denominators are multiples of 2. Thus

$$(1-1/2^{2m})\sigma_{2m}=1+1/3^{2m}+1/5^{2m}+\ldots\ .$$

* It may, in fact, be easily shown that $L\sigma_{2m}=1$ when $m=\infty$; for, by chap. xxv., § 25, we have the inequality $1/(2m-1)>1/2^{2m}+1/3^{2m}+1/4^{2m}+\ldots>1/(2m-1)2^{2m-1}$, which shows that $L(1/2^{2m}+1/3^{2m}+1/4^{2m}+\ldots)=0$, when $m=\infty$.

† See *Introd. in Anal. Inf.*, § 283.

If we take the next prime, namely 3, and multiply $(1 - 1/2^{2m})\sigma_{2m}$ by $1 - 1/3^{2m}$, we shall deprive the series of all terms involving multiples of 3; and so on. Thus we shall at last arrive at the equation

$$(1 - 1/2^{2m})(1 - 1/3^{2m})(1 - 1/5^{2m}) \ldots (1 - 1/p^{2m})\sigma_{2m} = 1 + 1/q^{2m} + \ldots \quad (6),$$

where $2, 3, 5, \ldots, p$ are the succession of natural primes up to p, and q is the next prime to p. We may, of course, make q as large as we please, and therefore $1/q^{2m} + \ldots$ (which is less than the residue after the $q - 1$th term of the convergent series σ_{2m}) as small as we please. Hence

$$\sigma_{2m} = 1/(1 - 1/2^{2m})(1 - 1/3^{2m})(1 - 1/5^{2m}) \ldots \quad (7),$$

where the succession of primes continues to infinity. Hence

$$B_m = 2(2m)!/(2\pi)^{2m}(1 - 1/2^{2m})(1 - 1/3^{2m})(1 - 1/5^{2m}) \ldots \quad (8).$$

§ 13.] *Bernoulli's Numbers are all positive; they increase after* B_3; *and have* ∞ *for an upper limit.*

That the numbers are all positive is at once apparent from § 11 (4). The latter part of the corollary may also be deduced from (4) by means of the inequality of chap. xxv., § 25. For we have

$$1/(2m-1) > 1/2^{2m} + 1/3^{2m} + 1/4^{2m} + \ldots > 1/(2m-1)2^{2m-1} \quad (9).$$

Hence

$$\frac{B_{m+1}}{B_m} = \frac{(2m+2)(2m+1)\sigma_{2m+2}}{(2\pi)^2\sigma_{2m}},$$

$$> \frac{(2m+2)(2m+1)\{1 + 1/(2m+1)2^{2m+1}\}}{(2\pi)^2\{1 + 1/(2m-1)\}},$$

$$> \frac{(2m)^2 - 1}{4\pi^2}.$$

Hence $B_{m+1}/B_m > 1$, provided $m > \sqrt{(\pi^2 + \frac{1}{4})}$, that is, if $m > 3 \cdot 16$. Now $B_4 > B_3$, hence $B_3 < B_4 < B_5 < \ldots$.

Again, it follows from (9) that $L\sigma_{2m} = 1$ when $m = \infty$, and $L(2m)!/(2\pi)^{2m}$ is obviously infinite; hence LB_m is infinite.

Cor. $B_m/(2m)!$ *ultimately decreases in a geometrical proportion having for its common ratio* $1/4\pi^2$. *From which it follows that the*

series for $\tan\theta$, $\theta\cot\theta$, *and* $\theta\operatorname{cosec}\theta$, *given in* § 4, *have for their radii of convergence* $\theta = \frac{1}{2}\pi$, π *and* π *respectively.*

§ 14.] Turning now to the secant series, we observe that $4\Sigma(-)^{n-1}(2n-1)\pi/\{(2n-1)^2\pi^2 - 4\theta^2\}$ does not, if treated in the above way as it stands, give a double series satisfying Cauchy's criterion, for, although when mod $\theta < \frac{1}{2}\pi$ the horizontal series are absolutely convergent after we replace θ by mod θ, yet the sum of the sums of the horizontal series, namely, $4\Sigma(-)^{n-1}(2n-1)\pi/\{(2n-1)^2\pi^2 - 4(\text{mod }\theta)^2\}$, is only semi-convergent. We can, however, pair the positive and negative terms together, and deal with the series in the form

$$4\Sigma\left\{\frac{(4n-3)\pi}{(4n-3)^2\pi^2 - 4\theta^2} - \frac{(4n-1)\pi}{(4n-1)^2\pi^2 - 4\theta^2}\right\} \qquad (10),$$

that is,
$$8\pi\Sigma\frac{(4n-3)(4n-1)\pi^2 + 4\theta^2}{\{(4n-3)^2\pi^2 - 4\theta^2\}\{(4n-1)^2\pi^2 - 4\theta^2\}} \qquad (11).$$

Since (11) remains convergent when for θ we substitute mod θ, it is clear that we may expand each term of (10) in ascending powers of θ, and rearrange the resulting double series according to powers of θ. In this way we get

$$\sec\theta = 4\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left\{\frac{1}{(4n-3)^{2m+1}} - \frac{1}{(4n-1)^{2m+1}}\right\}\right]\frac{2^{2m}\theta^{2m}}{\pi^{2m+1}},$$

$$= \sum_{m=0}^{\infty} 2^{2m+2}\tau_{2m+1}\theta^{2m}/\pi^{2m+1} \qquad (12),$$

where
$$\tau_{2m+1} = 1/1^{2m+1} - 1/3^{2m+1} + 1/5^{2m+1} - \ldots \qquad (13).$$

Comparing (12) with the series

$$\sec\theta = 1 + \Sigma E_m\theta^{2m}/(2m)!,$$

obtained in § 3, we see that

$$E_m = \frac{2^{2m+2}(2m)!\,\tau_{2m+1}}{\pi^{2m+1}},$$

$$= 2(2m)!\left(\frac{2}{\pi}\right)^{2m+1}\left\{\frac{1}{1^{2m+1}} - \frac{1}{3^{2m+1}} + \frac{1}{5^{2m+1}} - \ldots\right\} \qquad (14),$$

which may be transformed into

$$E_m = 2(2m)!\left(\frac{2}{\pi}\right)^{2m+1}\Big/\left(1 + \frac{1}{3^{2m+1}}\right)\left(1 - \frac{1}{5^{2m+1}}\right)\left(1 + \frac{1}{7^{2m+1}}\right)\ldots$$

in the same way as before. (15).*

* See again Euler, *Introd. in Anal. Inf.*, § 284.

Cor. 1. *Euler's numbers are all positive; they continually increase in magnitude, and have infinity for their upper limit.*

For we have

$$1 > \tau_{2m+1} > 1 - 1/3^{2m+1} \qquad (16).$$

Hence

$$\frac{E_{m+1}}{E_m} = \frac{(2m+2)(2m+1)4\tau_{2m+2}}{\pi^2 \tau_{2m+1}},$$

$$> \frac{(2m+2)(2m+1)4(1-1/3^{2m+3})}{\pi^2}.$$

But this last constantly increases with m, and is already greater than 1, when $m = 1$. Hence $E_1 < E_2 < E_3 < \ldots$ Also, from (16), we see that $L\tau_{2m+1} = 1$ when $m = \infty$, and $L(2m)!(2/\pi)^{2m+1} = \infty$, hence $LE_m = \infty$.

Cor. 2. $E_m/(2m)!$ *ultimately decreases in a geometrical progression whose common ratio is* $4/\pi^2$. *Hence the radius of convergence of the secant series is* $\theta = \frac{1}{2}\pi$.

§ 15.] We have, by § 11 (4),

$$\sigma_{2m} = \frac{1}{1^{2m}} + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \ldots = \frac{2^{2m-1}B_m \pi^{2m}}{(2m)!} \qquad (1);^*$$

and hence

$$\sigma'_{2m} = \frac{1}{1^{2m}} + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} + \ldots = \left(1 - \frac{1}{2^{2m}}\right)\frac{2^{2m-1}B_m \pi^{2m}}{(2m)!},$$

$$= \frac{(2^{2m}-1)B_m \pi^{2m}}{2(2m)!} \qquad (2);$$

and

$$\sigma''_{2m} = \frac{1}{1^{2m}} - \frac{1}{2^{2m}} + \frac{1}{3^{2m}} - \ldots = \left(1 - \frac{2}{2^{2m}}\right)\frac{2^{2m-1}B_m \pi^{2m}}{(2m)!},$$

$$= \frac{(2^{2m-1}-1)B_m \pi^{2m}}{(2m)!} \qquad (3).$$

Again, from (14) of last paragraph

$$\tau_{2m+1} = \frac{1}{1^{2m+1}} - \frac{1}{3^{2m+1}} + \frac{1}{5^{2m+1}} - \ldots = \frac{E_m}{2^{2m+2}(2m)!}\pi^{2m+1} \qquad (4).$$

* The remarkable summations involved in the formulæ (1), (2), (3) were discovered independently by John Bernoulli (see *Op.*, t. iv., p. 10), and by Euler (*Comm. Ac. Petrop.*, 1740).

Inasmuch as we have independent means of calculating the numbers B_m and E_m, the above formulæ enable us to sum the various series involved. It does not appear that the series σ_{2m+1} can be expressed by means of B_m or E_m; but Euler has calculated (to 16 decimal places) the numerical values of σ_{2m+1} in a number of cases, by means of Maclaurin's formula for approximate summation.* As the values of σ_m are often useful for purposes of verification, we give here a few of Euler's results. It must not be forgotten that the formulæ involving π for σ_m are accurate when m is even; but only approximations when m is odd.

$$\sigma_2 = 1{\cdot}6449340668\ \ldots = \pi^2/6.$$
$$\sigma_3 = 1{\cdot}2020569031\ \ldots = \pi^3/25{\cdot}79436\ \ldots\ .$$
$$\sigma_4 = 1{\cdot}0823232337\ \ldots = \pi^4/90.$$
$$\sigma_5 = 1{\cdot}0369277551\ \ldots = \pi^5/295{\cdot}1215\ \ldots\ .$$
$$\sigma_6 = 1{\cdot}0173430620\ \ldots = \pi^6/945.$$
$$\sigma_7 = 1{\cdot}0083492774\ \ldots = \pi^7/2995{\cdot}286\ \ldots\ .$$
$$\sigma_8 = 1{\cdot}0040773562\ \ldots = \pi^8/9450.$$
$$\sigma_9 = 1{\cdot}0020083928\ \ldots = \pi^9/29749{\cdot}35\ \ldots\ .$$

EXPANSIONS OF THE LOGARITHMS OF THE CIRCULAR FUNCTIONS.

§ 16.] From the formulæ of §§ 6 and 7, we get, by taking logarithms,

$$\log \sin\theta = \log\theta + \sum_{n=1}^{\infty} \log(1 - \theta^2/n^2\pi^2),$$
$$= \log\theta - \sum_{m=1}^{\infty} \sigma_{2m}\theta^{2m}/m\pi^{2m} \qquad (1),$$

since the double series arising from the expansions of the logarithms is obviously convergent, provided mod $\theta < \pi$.

If we express σ_{2m} by means of Bernoulli's numbers, (1) may be written

$$\log \sin\theta = \log\theta - \sum_{m=1}^{\infty} 2^{2m-1}B_m\theta^{2m}/m(2m)! \qquad (1').$$

* *Inst. Calc. Diff.*, chap. vi.

The corresponding formulæ for $\cos\theta$ are

$$\log\cos\theta = -\Sigma(2^{2m}-1)\sigma_{2m}\theta^{2m}/m\pi^{2m} \qquad (2);$$
$$= -\Sigma 2^{2m-1}(2^{2m}-1)B_m\theta^{2m}/m(2m)! \qquad (2').$$

The like formulæ for $\log\tan\theta$, $\log\cot\theta$, $\log\sinh u$, $\log\cosh u$, &c., can be derived at once from the above.

If a table of the values of σ_{2m} or of B_m be not at hand, the first few may be obtained by expanding $\log(\sin\theta/\theta)$, that is, $\log(1-\theta^2/3!+\theta^4/5!-\ldots)$, and comparing with the series $-\Sigma\sigma_{2m}\theta^{2m}/m\pi^{2m}$. For example, we thus find at once that $\sigma_2=\pi^2/6$.

STIRLING'S THEOREM.

§ 17.] Before leaving this part of the subject, we shall give an elementary proof of a theorem of great practical importance which was originally given by Stirling in his *Methodus Differentialis* (1730).

When n is very great, $n!$ approaches equality with $\sqrt{(2n\pi)}(n/e)^n$; or, more accurately, when n is a large number, we have

$$n! = \sqrt{(2\pi n)}(n/e)^n \exp\{1/12n+\theta\} \qquad (1),$$

where $-1/24n^2<\theta<1/24n(n-1)$.

Since $\log\{n/(n-1)\} = -\log(1-1/n)$, we have

$$\log\frac{n}{n-1} = \frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{1}{4n^4}+\ldots+\frac{1}{mn^m}+\ldots.$$

We can deprive this expansion of its second term by multiplying by $n-\frac{1}{2}$. We thus get

$$(n-\tfrac{1}{2})\log\frac{n}{n-1} = 1+\frac{1}{12n^2}+\frac{1}{12n^3}+\ldots+\frac{m-1}{2m(m+1)n^m}+\ldots.$$

Hence, taking the exponential of both sides, and writing successively n, $n-1$, $n-2$, . . ., 2 in the resulting equation, we deduce

$$\left(\frac{n}{n-1}\right)^{n-\frac{1}{2}} = \exp\left(1+\frac{1}{12n^2}+\frac{1}{12n^3}+\ldots+\frac{m-1}{2m(m+1)n^m}+\ldots\right),$$

$$\left(\frac{n-1}{n-2}\right)^{n-1-\frac{1}{2}} = \exp\left(1 + \frac{1}{12(n-1)^2} + \frac{1}{12(n-1)^3} + \ldots + \frac{m-1}{2m(m+1)(n-1)^m} + \ldots\right),$$

$$\left(\frac{n-2}{n-3}\right)^{n-2-\frac{1}{2}} = \exp\left(1 + \frac{1}{12(n-2)^2} + \frac{1}{12(n-2)^3} + \ldots + \frac{m-1}{2m(m+1)(n-2)^m} + \ldots\right),$$

. . . . . .

$$\left(\frac{3}{2}\right)^{3-\frac{1}{2}} = \exp\left(1 + \frac{1}{12.3^2} + \frac{1}{12.3^3} + \ldots + \frac{m-1}{2m(m+1)3^m} + \ldots\right),$$

$$\left(\frac{2}{1}\right)^{2-\frac{1}{2}} = \exp\left(1 + \frac{1}{12.2^2} + \frac{1}{12.2^3} + \ldots + \frac{m-1}{2m(m+1)2^m} + \ldots\right).$$

By multiplying all these together, we get

$$\frac{n^{n-\frac{1}{2}}}{(n-1)!} = \exp\left\{(n-1) + \frac{1}{12}S'_2 + \frac{1}{12}S'_3 + \ldots + \frac{m-1}{2m(m+1)}S'_m + \ldots\right\} \quad (2),$$

where $S'_m = 1/2^m + 1/3^m + 1/4^m + \ldots + 1/n^m$.

Now

$$S'_m = S_m - 1/(n+1)^m - 1/(n+2)^m - \ldots \quad (3),$$

where $S_m = 1/2^m + 1/3^m + \ldots + 1/n^m + \ldots$ ad ∞.

By the inequality (6) of chap. xxv., § 25, we have

$$1/(m-1)n^{m-1} > 1/(n+1)^m + 1/(n+2)^m + \ldots > 1/(m-1)(n+1)^{m-1}.$$

Hence

$$S_m - 1/(m-1)(n+1)^{m-1} > S'_m > S_m - 1/(m-1)n^{m-1}.$$

Therefore

$$\frac{1}{12}S'_2 + \frac{1}{12}S'_3 + \ldots + \frac{m-1}{2m(m+1)}S'_m + \ldots > \tfrac{1}{2}\sum_2^\infty \frac{(m-1)S_m}{m(m+1)} - \tfrac{1}{2}\sum_2^\infty \frac{1}{m(m+1)n^{m-1}} \quad (4);$$

$$< \tfrac{1}{2}\sum_2^\infty \frac{(m-1)S_m}{m(m+1)} - \tfrac{1}{2}\sum_2^\infty \frac{1}{m(m+1)(n+1)^{m-1}} \quad (5).$$

Since $S_m < 1/(m-1)$, the series $\Sigma(m-1)S_m/m(m+1)$ converges to a finite limit which is independent both of m and of n.

Again,

$$\sum_2^\infty \frac{1}{m(m+1)n^{m-1}}$$

$$= \frac{1}{2.3n} + \frac{1}{3.4n^2} + \frac{1}{4.5n^3} + \ldots \qquad (6);$$

$$< \frac{1}{6n} + \frac{1}{12n^2}\left\{1 + \frac{1}{n} + \frac{1}{n^2} + \ldots\right\},$$

$$< \frac{1}{6n} + \frac{1}{12n(n-1)} \qquad (7).$$

Also, by (6),

$$\sum_2^\infty \frac{1}{m(m+1)(n+1)^{m-1}}$$

$$= \sum_2^\infty \left(\frac{1}{m} - \frac{1}{m+1}\right)\frac{1}{(n+1)^{m-1}},$$

$$= (n+1)\sum_2^\infty \frac{1}{m(n+1)^m} - (n+1)^2 \sum_2^\infty \frac{1}{(m+1)(n+1)^{m+1}},$$

$$= (n+1)\left\{-\frac{1}{n+1} - \log\left(1 - \frac{1}{n+1}\right)\right\}$$

$$-(n+1)^2\left\{-\frac{1}{(n+1)} - \frac{1}{2(n+1)^2} - \log\left(1 - \frac{1}{n+1}\right)\right\},$$

$$= +\frac{1}{2} + n - (n^2+n)\log\left(1 + \frac{1}{n}\right);$$

$$= \frac{1}{2} + n - n + \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} - \ldots - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \ldots,$$

$$= \frac{1}{2.3n} - \frac{1}{3.4n^2} + \frac{1}{4.5n^3} - \ldots,$$

$$> \frac{1}{6n} - \frac{1}{12n^2} \qquad (8).$$

Combining (2), (4), (5), (7), and (8), we have

$$\frac{n^{n+\frac{1}{2}}}{n!} > \exp\left\{n - 1 + \tfrac{1}{2}\sum_2^\infty \frac{(m-1)S_m}{m(m+1)} - \frac{1}{12n} - \frac{1}{24n(n-1)}\right\} \qquad (9);$$

$$< \exp\left\{n - 1 + \tfrac{1}{2}\sum_2^\infty \frac{(m-1)S_m}{m(m+1)} - \frac{1}{12n} + \frac{1}{24n^2}\right\} \qquad (10).$$

Hence, putting

$$C = \exp\left\{1 - \tfrac{1}{2}\sum_{2}^{\infty}\frac{(m-1)\,S_m}{m(m+1)}\right\} \qquad (11),$$

so that C is a finite numerical constant, we have

$$n! > Ce^{-n}n^{n+\frac{1}{2}}\exp\left(\frac{1}{12n} - \frac{1}{24n^2}\right) \qquad (12),$$

$$< Ce^{-n}n^{n+\frac{1}{2}}\exp\left(\frac{1}{12n} + \frac{1}{24n(n-1)}\right) \qquad (13);$$

or, since the exponential function is continuous,

$$n! = Ce^{-n}n^{n+\frac{1}{2}}\exp\left(\frac{1}{12n} + \theta\right) \qquad (14),$$

where $-1/24n^2 < \theta < 1/24n(n-1)$.

Hence, putting $n = \infty$ on both sides of (14), we have

$$Ln! = CLe^{-n}n^{n+\frac{1}{2}} \qquad (15).$$

The constant C may be calculated numerically by means of the equation (11). Its value is, in fact, $\sqrt{(2\pi)}$, as may be easily shown by using Wallis's Theorem, § 6 (18).

Thus we have, when $n = \infty$,

$$\frac{\pi}{2} = L\frac{2^{2n}(n!)^2(2n+1)}{1^2 3^2 \ldots (2n+1)^2} = L\frac{2^{4n}(n!)^4(2n+1)}{\{(2n+1)!\}^2}.$$

Hence, using (15), we get

$$\frac{\pi}{2} = C^2 L\frac{2^{4n}e^{-4n}n^{4n+2}(2n+1)}{e^{-4n-2}(2n+1)^{4n+3}},$$

$$= \frac{C^2}{4}L\frac{e^2}{\{(1+1/2n)^{2n}\}^2\{1+1/2n\}^2},$$

$$= \frac{C^2}{4}\frac{e^2}{e^2}.$$

Therefore, since C is obviously positive,

$$C = \sqrt{(2\pi)} \qquad (16).$$

Using this value of C in (14), we get finally

$$n! = \sqrt{(2\pi n)}\,(n/e)^n \exp\{1/12n + \theta\} \ * \qquad (17),$$

where $-1/24n^2 < \theta < 1/24n(n-1)$.

* An elementary proof that $Ln! = L\sqrt{(2\pi n)}\,(n/e)^n$ was given by Glaisher (*Quart. Jour. Math.*, 1878). In an addition by Cayley a demonstration of the approximation (17) is also given; but inasmuch as it assumes that series

Cor. *By combining* (11) *and* (16), *we deduce that*

$$1 - \tfrac{1}{2}\sum_{2}^{\infty}\frac{(m-1)\,S_m}{m(m+1)} = \tfrac{1}{2}\log 2 + \tfrac{1}{2}\log\pi \qquad (18),$$

where $S_m = 1/2^m + 1/3^m + 1/4^m + \ldots$ ad ∞.

Exercises XXIII.

(1.) Show that, when $\mod x > \pi$, $x\cot x$ can be expanded in the form $A_0 + \Sigma(B_n x^{-n} + C_n x^n)$; and determine the coefficients in the particular case where $\pi < x < 2\pi$.

(2.) Show that the sum of the products r at a time of the squares of the reciprocals of all the integral numbers is $\pi^{2r}/(2r+1)!$; and find the like sum when the odd integers alone are considered.

(3.) Sum to n terms

$$\tan\theta + \tan(\theta+\pi/n) + \tan(\theta+2\pi/n) + \ldots;$$
$$\tan^2\theta + \tan^2(\theta+\pi/n) + \tan^2(\theta+2\pi/n) + \ldots.$$

Sum the following:—

(4.) $1/(1^2+x^2) + 1/(2^2+x^2) + 1/(3^2+x^2) \ldots$.

(5.) $1/x^2 - 1/(x^2-\pi^2) + 1/(x^2-2^2\pi^2) - \ldots$.

(6.) $1/x + 1/(x-1) + 1/(x+1) + 1/(x-2) + 1/(x+2) + \ldots$.

(7.) $1/(1-e) + 1/(4-e) + 1/(9-e) + \ldots + 1/(n^2-e) + \ldots$.

(8.) $1/1.2 + 1/2.4 + 1/3.6 + 1/4.8 + \ldots$.

Show that

(9.) $(\pi^2-6)/6 = 1/1^2.2 + 1/2^2.3 + 1/3^2.4 + \ldots$.

(10.) $\pi/8 - 1/3 = 1/1.3.5 - 1/3.5.7 + 1/5.7.9 - \ldots$.

(11.) If $f_r(n)$ be an integral function of n whose degree is r, show that $\sum^{\infty} f_r(n)/(2n-1)^{2m}$ can be expressed in terms of Bernoulli's numbers, provided $r \not> 2m-2$; and $\sum^{\infty}(-)^{n-1} f_r(n)/(2n-1)^{2m+1}$ in terms of Euler's numbers, provided $r \not> 2m-1$.

In particular, show that

$$\frac{1}{3^4} + \frac{1+2}{5^4} + \frac{1+2+3}{7^4} + \ldots = \frac{\pi^2}{64}\left(1 - \frac{\pi^2}{12}\right).$$

(12.) Show that

$$\sum_{-\infty}^{\infty} 1/(n\pi+\theta)^2 = \operatorname{cosec}^2\theta;$$

$$\sum_{-\infty}^{\infty} 1/(n\pi+\theta)^4 = \operatorname{cosec}^4\theta - \tfrac{2}{3}\operatorname{cosec}^2\theta,$$

$n=0$ being included among the values to be given to n. (Wolstenholme.)

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of the form $1/2^m + 1/3^m + \ldots$ can be expanded in powers of $1/m$, it cannot be said to be elementary. The proofs usually given by means of the Maclaurin-sum-formula are unsatisfactory, for they depend on the use of a series which does not in general converge when continued to infinity, and which can only be used in conjunction with its residue. See Raabe, *Crelle's Jour.*, XXV.

(13.) $$\sum_1^\infty \frac{1}{n^4+x^4} = \frac{\pi\sqrt{2}}{4x^3}\,\frac{\sinh.\pi x\sqrt{2}+\sin.\pi x\sqrt{2}}{\cosh.\pi x\sqrt{2}-\cos.\pi x\sqrt{2}} - \frac{1}{2x^4}.$$

(Math. Trip., 1888.)

(14.) Show that

$$\sum_{n=1}^\infty \frac{1}{\{(2n)^2-(2m-1)^2\}^2} = \frac{\pi^2}{16(2m-1)^2} - \frac{1}{2(2m-1)^4};$$

$$\sum_{n=1}^\infty \frac{1}{\{(2n-1)^2-(2m)^2\}^2} = \frac{\pi^2}{64m^2}.$$

Also that the sum of the reciprocals of the squares of all possible differences between the square of any even and the square of any odd number is $\pi^4/384$.

(15.) If $p<n$, show that

$$\frac{\cos p\theta}{\cos n\theta} = \frac{1}{n}\sum_{r=0}^{n-1}(-)^r \frac{\sin.(2r+1)\pi/2n\,.\cos.p(2r+1)\pi/2n}{\cos\theta-\cos.(2r+1)\pi/2n}.$$

(16.) Show that

$$\tan^{-1}\frac{v}{u} - \sum_{n=1}^\infty\left\{\tan^{-1}\frac{v}{n\pi-u} - \tan^{-1}\frac{v}{n\pi+u}\right\} = \tan^{-1}(\tanh v\cot u);$$

$$\sum_{n=1}^\infty\left\{\tan^{-1}\frac{2v}{(2n-1)\pi-2u} - \tan^{-1}\frac{2v}{(2n-1)\pi+2u}\right\} = \tan^{-1}(\tanh v\tan u).$$

(Schlömilch, *Handb. d. Alg. Anal.*, cap. xi.)

(17.) If $\lambda(x)\equiv x\prod_1^\infty\{1-(x/na)^2\}$, $\mu(x)\equiv\prod_1^\infty\{1-(2x/\overline{2n-1}.a)^2\}$, express $\lambda(x+a/2)$ in terms of $\mu(x)$, and also $\mu(x+a/2)$ in terms of $\lambda(x)$.

Hence evaluate $\mathrm{L}_{m=\infty}\ 1.3.5\ \ldots\ (2m-1)\sqrt{(2m+1)}/2^m m!$.

(Math. Trip., 1882.)

(18.) Show that, if r be a positive integer,

$$\mathrm{L}_{r=\infty}\left(1-\frac{1}{r}\right)^{1/1}\left(1-\frac{2}{r}\right)^{1/2}\cdots\left(1-\frac{r-1}{r}\right)^{1/(r-1)} = e^{-\pi^2/6}.$$

(19.) Show that

$$\mathrm{L}_{x=\infty}\left(\frac{x}{x^2+1^2}+\frac{x}{x^2+2^2}+\frac{x}{x^2+3^2}+\ldots\right) = \frac{\pi}{2}.$$

REVERSION OF SERIES—EXPANSION OF AN ALGEBRAIC FUNCTION.

§ 18.] The subject which we propose to discuss in this and the following paragraphs originated, like so many other branches of modern analysis, in the works of Newton, more especially in his tract *De Analysi per Æquationes Numero Terminorum Infinitas.*

Let us consider the function

$$\Sigma(m, n)x^m y^n \equiv (1, 0)x + (0, 1)y + (2, 0)x^2 + (1, 1)xy + (0, 2)y^2 + \ldots,$$

where the indices m and n are positive integers, and we use the symbol (m, n) to denote the coefficient of $x^m y^n$, so that (m, n) is a constant. We suppose the absolute term (0, 0) to be zero; but the coefficients (1, 0) (0, 1) are to be different from zero. The rest of the coefficients may or may not be zero; but, if the number of terms be infinite, we suppose the double series to be absolutely convergent when $\mod x = \mod y = 1$.* From this it follows that the coefficient (m, n) must become infinitely small when m and n become infinitely great; so that a positive quantity λ can in all cases be assigned such that $\mod (m, n) \not> \lambda$ whatever values we assign to m and n. It also follows (see chap. xxvi., § 37) that $\Sigma(m, n)x^m y^n$ is absolutely convergent for all values of x and y such that $\mod x \not> 1 \mod y \not> 1$.

We propose to show *that one value, and only one value, of y as a function of x can be found which has the following properties*:—

1°. *y is expansible in a convergent series of integral powers of x for all values of x lying within limits which are not infinitely narrow.*

2°. *y has the initial value* 0 *when* $x = 0$.

3°. *y makes the equation*

$$\Sigma(m, n)x^m y^n = 0 \qquad (1)$$

an intelligible identity.

Let us assume for a moment that a convergent series for y of the kind demanded can be found. Its absolute term must vanish by condition 2°. Hence the series will be of the form

$$y = b_1 x + b_2 x^2 + b_3 x^3 + \ldots \qquad (2).$$

In order that this value of y may make (1) an intelligible identity, it must be possible to find a value of $x < 1$ such that (2) gives a value of $y < 1$. The series (1), when transformed by means of (2), will then satisfy Cauchy's criterion, and may be arranged according to powers of x. All that is further necessary

* The more general case, when the series is convergent so long as $\mod x \not> \alpha$ and $\mod y \not> \beta$, can easily be brought under the above by a simple transformation.

to satisfy condition 3° is simply that the coefficients of all the powers of x shall vanish.

It will be convenient for what follows to assume that $(0, 1) = -1$ (which we may obviously do without loss of generality), and then put (1) into the form—

$$\begin{aligned} y = {} & \{(1,0)x + (2,0)x^2 + (3,0)x^3 + \dots\} \\ & + \{(1,1)x + (2,1)x^2 + (3,1)x^3 + \dots\}y \\ & + \{(0,2) + (1,2)x + (2,2)x^2 + (3,2)x^3 + \dots\}y^2 \\ & \dots\dots\dots\dots\dots\dots\dots \\ & + \{(0,n) + (1,n)x + (2,n)x^2 + (3,n)x^3 + \dots\}y^n \\ & \dots\dots\dots\dots\dots\dots\dots \end{aligned} \qquad (3).$$

Using (2), we get

$$\begin{aligned} & b_1x + b_2x^2 + b_3x^3 + \dots \\ & = \{(1,0)x + (2,0)x^2 + (3,0)x^3 + \dots\} \\ & \quad + \{(1,1)x + (2,1)x^2 + (3,1)x^3 + \dots\}\{b_1 + b_2x + b_3x^2 + \dots\}x \\ & + \{(0,2) + (1,2)x + (2,2)x^2 + (3,2)x^3 + \dots\}\{b_1 + b_2x + b_3x^2 + \dots\}^2x^2 \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + \{(0,n) + (1,n)x + (2,n)x^2 + (3,n)x^3 + \dots\}\{b_1 + b_2x + b_3x^2 + \dots\}^nx^n \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \qquad (4).$$

Hence, equating coefficients, we have

$$\begin{aligned} b_1 &= (1,0), \\ b_2 &= (2,0) + (1,1)b_1 + (0,2)b_1^2, \\ b_3 &= (3,0) + (1,1)b_2 + (2,1)b_1 + (1,2)b_1^2 + 2(0,2)b_1b_2 + (0,3)b_1^3, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_n &= (n,0) + (1,1)b_{n-1} + (2,1)b_{n-2} + \dots + (0,n)b_1^n \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \qquad (5).$$

Here it is important to notice that each equation assigns one of the coefficients as an integral function of all the preceding coefficients. Hence, since the first equation gives one and only one value for b_1, all the coefficients are uniquely determined. There is therefore only one value of y, if any.

In order to show that (5) really affords a solution, we have to show that for a value of x whose modulus is small enough, but not infinitely small, the conditions for the absolute convergency of (2) and (4) are satisfied when b_1, b_2, . . . have the values assigned by (5).

This, following a method invented by Cauchy, we may show by considering a particular case. The moduli of the coefficients of the series (3) have, as we have seen, a finite upper limit λ. Suppose that in (3) all the coefficients are replaced by λ, and that x has a positive real value < 1. Then we have

$$\begin{aligned} y &= \lambda\{x + x^2 + x^3 + \ldots\} \\ &\quad + \lambda\{x + x^2 + x^3 + \ldots\}y \\ &\quad + \lambda\{1 + x + x^2 + x^3 + \ldots\}y^2 \\ &\quad \ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{6}$$

This series is convergent so long as $x < 1$ and mod $y < 1$. It can, in fact, be summed; for, adding $\lambda + \lambda y$ to both sides, we get

$$(1 + \lambda)y + \lambda = \lambda/(1 - x)(1 - y),$$

that is,
$$(1 + \lambda)y^2 - y + \lambda x/(1 - x) = 0.$$

Hence, remembering that the value of y with which we are concerned vanishes when $x = 0$, we have

$$y = [1 - \sqrt{\{1 - 4\lambda(1 + \lambda)x/(1 - x)\}}]/2(\lambda + 1) \tag{7}$$

Now, provided $4\lambda(1 + \lambda)x/(1 - x) < 1$, that is, $x < 1/(2\lambda + 1)^2$, the right-hand side of (7) can be expanded in an absolutely convergent series of integral powers of x, the absolute term in which vanishes. Also, when $x < 1/(2\lambda + 1)^2$, the value of y given by (7) is positive and < 1, therefore the absolute convergency of (6) is assured.

It follows that the problem we are considering can be solved in the present particular case. If we denote the series for y in this case by

$$y = C_1x + C_2x^2 + C_3x^3 + \ldots \tag{8},$$

then the equations for determining C_1, C_2, C_3, . . . will be found by putting $(1, 0) = (2, 0) = (1, 1) = \ldots = \lambda$ in (5), namely,

$$\begin{aligned} C_1 &= \lambda, \\ C_2 &= \lambda(1 + C_1 + C_1^2), \\ C_3 &= \lambda(1 + C_2 + C_1 + C_1^2 + 2C_1C_2 + C_1^3), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ C_n &= \lambda(1 + C_{n-1} + C_{n-2} + \ldots + C_1^n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{9};$$

from which it is seen that C_1, C_2, C_3, . . . are all real and positive.

Returning now to the system (5), and denoting moduli by attaching dashes, we have, since $(1, 0)'$, $(2, 0)'$, &c., are all less than λ,

$$b' = (1, 0)' < \lambda < C_1,$$
$$b_2' \not> (2, 0)' + (1, 1)'b_1' + (0, 2)'b_1'^2 < \lambda(1 + C_1 + C_1^2) < C_2,$$
$$b_3' \not> (3,0)' + (1,1)'b_2' + (2,1)'b_1' + (1, 2)'b_1'^2 + 2(0,2)'b_1'b_2' + (0, 3)'b_1'^3,$$
$$< \lambda(1 + C_2 + C_1 + C_1^2 + 2C_1C_2 + C_1^3) < C_3,$$
. . . . . (10).

Hence the moduli of the coefficients in (2) are less than the moduli in the series (8), which is known to be absolutely convergent. It therefore follows that the series (2) will certainly be absolutely convergent, provided $\operatorname{mod} x < 1/(2\lambda + 1)^2$.

It only remains to show that x may be so chosen (and yet not infinitely small) that y as given by (2) shall be such that $y' < 1$. We have

$$y' < b_1'x' + b_2'x'^2 + b_3'x'^3 + \ldots,$$
$$< C_1x' + C_2x'^2 + C_3x'^3 + \ldots,$$
$$< [1 - \sqrt{\{1 - 4\lambda(1 + \lambda)x'/(1 - x')\}}]/2(\lambda + 1) \quad (11).$$

Now the right-hand side of (11) is less than 1, provided $x' < 1/(2\lambda + 1)^2$. If, therefore, $\operatorname{mod} x < 1/(2\lambda + 1)^2$, the absolute convergency of the double series (3) or (4) will be assured; and (2) will convert (1) into an intelligible identity.

We have thus completely established that one and only one value of y expansible within certain limits as a convergent series of integral powers of x can be found to satisfy the equation (1); and the like follows for x as regards y. *The functions of x and y thus determined, being representable by power-series, are of course continuous.* The limits assigned in the course of the demonstration for the admissibility of the solution are merely lower limits; and it is easy to see that the solution is valid so long as (2) itself and the double series into which it converts the left-hand side of (1) remain absolutely convergent.

It should be remarked that we have not shown that no other power-series *whose absolute term does not vanish* can be found to satisfy (1); nor have we shown that no other function having zero initial value, but *not expansible in integral* powers of x, can be found to satisfy (1). We shall settle these questions presently in the case where the series $\Sigma(m, n)x^m y^n$ terminates.

§ 19.] The problem of the *Reversion of Series* properly so called is as follows:—

Given the equation

$$x = a_0 + a_m y^m + a_{m+1} y^{m+1} + \ldots \qquad (1),$$

where $a_m \neq 0$, *but* a_0 *may or may not be zero, and the series* $a_m y^m + a_{m+1} y^{m+1} + \ldots$ *is absolutely convergent so long as* $\mod y \not> a$ *fixed positive quantity* ρ, *to find a convergent expansion or convergent expansions for* y *in ascending powers of* $x - a_0$.

Let ξ denote $\{(x - a_0)/a_m\}^{1/m}$, that is, the principal value of the mth root of $(x - a_0)/a_m$, and ω_m a primitive mth root of unity, then (1) is equivalent to m equations of which the following is a type:—

$$\omega_m{}^r \xi = y\left(1 + \frac{a_{m+1}}{a_m} y + \frac{a_{m+2}}{a_m} y^2 + \ldots\right)^{1/m} \qquad (2).$$

Now, the series inside the bracket in (2) being absolutely convergent for all values of y such that $\mod y \not> \rho$, it follows from the binomial theorem combined with § 1 that we can, by taking y within certain limits, expand the right-hand side of (2) in an ascending series of powers of y. We thus get, say,

$$-\omega_m{}^r \xi + y + C_2 y^2 + C_3 y^3 + \ldots = 0 \qquad (3).$$

It follows, therefore, from the general theorem of last paragraph that we have, provided $\mod \xi$ does not exceed a certain limit,

$$y = b_1 \omega_m{}^r \xi + b_2 \omega_m{}^{2r} \xi^2 + b_3 \omega_m{}^{3r} \xi^3 + \ldots \qquad (4).$$

We have, of course, m *such results, in which the coefficients* $b_1, b_2, b_3, \ldots$ *will be the same, but* r *will have the different values* $0, 1, 2, \ldots, (m-1)$.

Each of these solutions is, by chap. xxvi., § 19, a continuous function of x.

Cor. *In the particular case where* $a_0 = 0$, $m = 1$, *we get the single solution*

$$y = b_1x + b_2x^2 + b_3x^3 + \ldots \qquad (5).$$

Example. To reverse the series

$$z = 1 + y/1! + y^2/2! + y^3/3! + \ldots \qquad (6).$$

Let $z = 1 + x$, then we have

$$x = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots \qquad (7).$$

Hence, provided mod x lie within a certain limit, we must have by the general theorem

$$y = b_1x + b_2x^2 + b_3x^3 + \ldots \qquad (8).$$

Knowing the existence of the convergent expansion (8), we may determine the coefficients as follows.

Give y a small increment k, and let the corresponding increment of x be h; then, from (7), we have

$$h = \frac{(y+k)-y}{1!} + \frac{(y+k)^2-y^2}{2!} + \frac{(y+k)^3-y^3}{3!} + \ldots .$$

Hence, since $\mathrm{L}\{(y+k)^n - y^n\}/k = ny^{n-1}$ when $k=0$, and since, owing to the continuity of the series as a function of y, $h=0$ when $k=0$, we have

$$\mathrm{L}\frac{h}{k} = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots ,$$
$$= 1 + x \qquad (9).$$

Again, from (8), we have, in like manner,

$$\mathrm{L}\frac{k}{h} = b_1 + 2b_2x + 3b_3x^2 + \ldots \qquad (10).$$

Combining (9) and (10), we have

$$b_1 + 2b_2x + 3b_3x^2 + \ldots = 1/(1+x),$$
$$= 1 - x + x^2 - \ldots .$$

We must therefore have

$$b_1 = 1, \quad b_2 = -1/2, \quad b_3 = 1/3, \ldots .$$

Therefore

$$y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \ldots ,$$
$$= \frac{z-1}{1} - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \ldots \qquad (11).$$

It must be remembered that (11) gives only that branch of the function y which is expansible in powers of $z-1$ and which vanishes when $z=1$. We have, in fact, merely given another investigation of the expansion of the principal value of $\log z$.

§ 20.] *Expansions of the various branches of an Algebraic Function.*

The equation

$$\Sigma(m, n)x^m y^n + (0, 0) = 0 \qquad (1),$$

where the series on the left terminates, gives for any assigned value of x a finite number of values of y. If the highest power of y involved be the nth, we might, in fact, write the equation in the form

$$A_n y^n + A_{n-1} y^{n-1} + \ldots + A_1 y + A_0 = 0 \qquad (2),$$

where $A_0, A_1, \ldots, A_n$ are all integral functions of x. If, then, we give to x any particular value, a, real or complex, it follows from chap. xii., § 23, that we get from (2) n corresponding values of y, say $b_1, b_2, \ldots, b_n$. If we change x into a value $a+h$ differing slightly from a, then $b_1, b_2, \ldots, b_n$ will change into $b_1 + k_1, b_2 + k_2, \ldots, b_n + k_n$; that is to say, we shall get n values of y which will in general be different from the former set. We may therefore say that (2) defines y as an n-valued function of x; and we call y when so determined an *algebraic function* of x.

Since every equation of the form $y = F(x)$, where $F(x)$ is an ordinary irrational algebraic function (as defined in chap. xiv., § 1), can be rationalised, it follows that every ordinary irrational algebraic function is a branch of an algebraic function as now defined. Since, however, integral equations whose degree is above the 4th cannot in general be formally solved by means of radicals, it does not follow, conversely, that every algebraic function is expressible as an ordinary irrational algebraic function.

In what follows we assume that the equation (2) contains (so long as x and y are not specialised) no factor involving x alone or y alone. We also suppose that, so long as x is not assigned, the equation is *Irreducible*, that is to say, that it has not a root in common with an integral equation of lower degree in y whose coefficients are integral functions of x. If this were so, a factor could (by the process for obtaining the G.C.M. of two integral functions) be found having for its coefficients integral functions of x, and the roots of the equation formed by equating this factor to 0 would be the common root or roots in question. Therefore the equation (2) could be broken up into two integral

equations in y whose coefficients would be integral functions of x; and each of these would define a separate algebraic function of x.

The condition of irreducibility involves that (2) cannot have two or more of its roots equal for all values of x. For, if (2) had, say, r equal roots, then, denoting all the roots by $y_1, y_2, \ldots, y_n$, the equation

$$\Sigma(y-y_1)(y-y_2)\ldots(y-y_{s-1})(y-y_{s+1})\ldots(y-y_n)=0 \quad (3)$$

would have $r-1$ roots in common with (2), for $r-1$ equal factors would occur in each of the terms comprehended by Σ. Now the coefficients of (3) are symmetric functions of the roots of (2); therefore (3) could be exhibited as an equation whose coefficients are integral functions of $A_0, A_1, \ldots, A_n$, and therefore integral functions of x.* Hence (2) would be reducible, which is supposed not to be the case.

It must, however, be carefully noticed that irreducibility in general (that is, so long as x is not specialised) does not exclude reducibility or multiplicity of roots for particular values of x. In fact, we can in general determine a number of particular values of x for which (2) and (3) may have a root in common. In other words, *it may happen that the n branches of y have points in common; but it cannot happen that any two of the n branches wholly coincide.*

When, for $x=a$, the n values $b_1, b_2, \ldots, b_n$ are all different, a (or its representative point in an Argand-diagram) is called an *ordinary point* of the function y, and $b_1, b_2, \ldots, b_n$ single values. If $b_1=b_2=\ldots=b_r$, each $=b$, say, then a is called an *r-ple point of the function*, and b an *r-ple value.*

For every value of x (*zero point*) which makes $A_0=0$, one branch of y has a zero value; for every value of x (*double zero point*) which makes $A_0=0$ and $A_1=0$, two branches have a zero value; and so on. These are called *single, double, . . ., zero values.*

For every value of x (*pole*) which makes $A_n=0$, one branch of y has an infinite value; for every value of x (*double pole*) which makes $A_n=0$ and $A_{n-1}=0$, two branches have an infinite

* See chap. xviii., § 4.

value; and so on. These may be called *single, double, . . ., infinities of the function.*

The main object of what follows is to show that *every branch of an algebraic function is (within certain limits), in the neighbourhood of every point, expansible in an ascending or descending power series of a particular kind; and thus to show that every branch is, except at a pole, continuous for all finite values of x.*

§ 21.] *If, at the point $x = a$, the algebraic function y has a single value $y = b$, then $y - b$ is, within certain limits, expansible in an absolutely convergent series of the form*

$$y - b = C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \ldots \qquad (4).$$

Let $x = a + \xi$, $y = b + \eta$, then the equation (1) becomes, after rearrangement,

$$(0,\ 0) + (1,\ 0)\xi + (0,\ 1)\eta + (2,\ 0)\xi^2 + \&c. = 0 \qquad (5).$$

Since $y = b$ is a single root of (1) corresponding to $x = a$, it follows that when $\xi = 0$ (5) must give one and only one zero value for η. Therefore we must have $(0,\ 0) = 0$ and $(0,\ 1) \neq 0$.

It follows, from the general theorem of § 18, that within certain limits the following convergent expansion,

$$\eta = C_1\xi + C_2\xi^2 + C_3\xi^3 + \ldots,$$

and no other of the kind will satisfy the equation (5); that is,

$$y = b + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \ldots \qquad (6)$$

will satisfy (1).

The function y determined by (6) is continuous so long as mod $(x - a)$ is less than the radius of convergency of the series involved; and it has the value $y = b$ when $x = a$.

If we suppose all the values of y, say $b_1, b_2, \ldots, b_n$, corresponding to $x = a$ to be single, then we shall get in this way for each one of them a value of the function y of the form (6). Hence we infer that

Cor. *So long as no two of the branches of an algebraic function have a point in common, each branch is a continuous function of x; and the increment of y at any point of a particular branch is expan-*

sible in an ascending series of powers of the increment of x so long as the modulus of the increment of x does not exceed a certain finite value.

§ 22.] We proceed to discuss the modification to which the conclusions of last paragraph are subject when $x = a$ is a multiple point of the function y.

We shall prove that for every multiple point of the qth order, to which corresponds a q-ple value $y = b$, we can find q different convergent expansions for y of the form $y = b + \Sigma C_r(x-a)^r$, where the exponents r form a series of increasing positive fractional numbers.

It will probably help the reader to keep the thread of the somewhat delicate analysis that follows if we premise the following remarks regarding expansibility in ascending power-series in general :—

If η be expansible in an absolutely convergent ascending series of positive powers of ξ, of the form

$$\eta = C_1\xi^{\alpha_1} + C_2\xi^{\alpha_1+\alpha_2} + C_3\xi^{\alpha_1+\alpha_2+\alpha_3} + \ldots \qquad (A),$$

where $\alpha_1, \alpha_2, \ldots$ are all positive, then we can establish a series of transformations of the following kind :—

$$\eta = \xi^{\alpha_1}(C_1 + \eta_1), \quad \eta_1 = \xi^{\alpha_2}(C_2 + \eta_2), \quad \eta_2 = \xi^{\alpha_3}(C_3 + \eta_3), \quad \ldots,$$
$$\eta_{n-1} = \xi^{\alpha_n}(C_n + \eta_n) \qquad (B),$$

where $\eta_1, \eta_2, \ldots, \eta_n$ all vanish when $\xi = 0$; $C_1, C_2, \ldots, C_n$ are all independent of ξ, and all different from zero; and $C_1 = L\eta/\xi^{\alpha_1}$, $C_2 = L\eta_1/\xi^{\alpha_2}$, $\ldots$, $C_n = L\eta_{n-1}/\xi^{\alpha_n}$ when $\xi = 0$.

Conversely, if we can establish a series of transformations of the form (B), and if we can show that η_n is expansible in a series of ascending positive powers of ξ, it will obviously follow that η is expansible in the form (A).

Let now $y = b$ be a q-ple value of y corresponding to $x = a$, and put as before $x = a + \xi$, $y = b + \eta$, then the equation (1) becomes

$$\Sigma(m, n)\xi^m\eta^n = 0 \qquad (7).$$

Since q values of y become b when $x = a$, q values of η must become 0 when $\xi = 0$. Hence the lowest power of η in (7) which is not multiplied by a power of ξ must be η^q. There must also be a power of ξ which is not multiplied by a power of η, otherwise (7) would be divisible in general by some power of η, which is impossible since (1) is irreducible. Let the lowest such power of ξ be ξ^p.

Put now

$$\eta = \xi^\lambda(C_1 + \eta_1) = \xi^\lambda v \qquad (8),$$

and let us seek to determine a positive value of λ such that $C_1 = Lv = L\eta/\xi^\lambda$ is finite both ways* when $\xi = 0$.

The equation (7) gives

$$\Sigma(m, n)\xi^{m+\lambda n}v^n = 0 \qquad (9).$$

Now (9) will furnish values of v which are finite both ways when $\xi = 0$, provided we can so determine λ that at least two terms of (9) are of the same positive degree in ξ, and lower in degree than all the other terms.

Assume for the present that we can find a value of λ for which a group of r terms has the character in question, so that

$$\delta = m_1 + \lambda n_1 = m_2 + \lambda n_2 = \ldots = m_r + \lambda n_r \qquad (10),$$

where $$n_1 \not> n_2 \not> \ldots \not> n_r;$$

and $$\lambda = (m_1 - m_r)/(n_r - n_1) \qquad (11).$$

$$\delta = (m_1 n_r - m_r n_1)/(n_r - n_1).$$

Then, putting $\xi_1 = \xi^{1/(n_r - n_1)}$,† so that $\xi_1 = 0$ when $\xi = 0$, and dividing out $\xi_1^{m_1 n_r - m_r n_1}$, we deduce an equation of the form

$$\phi(\xi_1, v)\xi_1 + (m_r, n_r)v^{n_r} + (m_{r-1}, n_{r-1})v^{n_{r-1}} + \ldots + (m_1, n_1)v^{n_1} = 0 \qquad (12),$$

where $\phi(\xi_1, v)$ is an integral function of ξ_1 and v.

For our present purpose we are concerned only with those

* That is, neither zero nor infinite—a useful phrase of De Morgan's.

† It is sufficient for our purpose to take the principal value merely of the $(n_r - n_1)$th root of ξ. It must be observed that in these operations no factor is to be divided out of $n_r - n_1$; see Example 1.

roots of (12) whose initial values are finite both ways. There are evidently $n_r - n_1$ such roots, and their initial values are given by

$$(m_r, n_r)v^{n_r - n_1} + (m_{r-1}, n_{r-1})v^{n_{r-1} - n_1} + \ldots + (m_1, n_1) = 0 \qquad (13).$$

If the roots of (13) are all different, then we get $n_r - n_1$ transformations of the form (8); and the corresponding values of v, that is, of $C_1 + \eta_1$, are given by the algebraical equation (12). Moreover, since all the values of v are single, we shall get for each value of η_1 an expansion of the form

$$\begin{aligned} \eta_1 &= d_1\xi_1 + d_2\xi_1^2 + \ldots, \\ &= d_1\xi^{1/(n_r - n_1)} + d_2\xi^{2/(n_r - n_1)} + \ldots \qquad (14); \end{aligned}$$

and each of these will give for η a corresponding expansion of the form

$$\eta = C_1\xi^{(m_1 - m_r)/(n_r - n_1)} + d_1\xi^{(m_1 - m_r + 1)/(n_r - n_1)} + d_2\xi^{(m_1 - m_r + 2)/(n_r - n_1)} + \ldots \qquad (14').$$

If a group of the roots of (13) be equal, then we must proceed by means of a second transformation,

$$\eta_1 = \xi_1^{\mu}(C_2 + \eta_2) \qquad (15),$$

to separate those roots of (12) which have equal values. If the next step succeeds in finally separating all the initial values, then we have for each of the group of equal roots of (13) two transformations (8) and (15), and finally an expansion like (14'), the result being the final separation of all the $n_r - n_1$ roots of (12), with convergent expansions for each of them.

Moreover, we must in every case be able, by means of a finite number of transformations like (8) and (15), to separate the initial values, otherwise we should have two branches of y coincident up to any order of approximation, which is impossible, since (1) is irreducible.

All that now remains is to show that we can in all cases select a number of groups of terms satisfying the conditions (10) sufficient to give us q expansions corresponding to the q branches which meet at the q-ple point $x = a$.

The best way, both in theory and in practice, of settling this point is to use *Newton's Parallelogram*, which is constructed as follows:—Let OX and OY (Fig. 1) be a pair of rectangular axes, the first quadrant of which is ruled into squares (or rectangles) for convenience in plotting points whose co-ordinates are positive integers. For each term $(m, n)\xi^m\eta^n$ in equation (7) we plot a

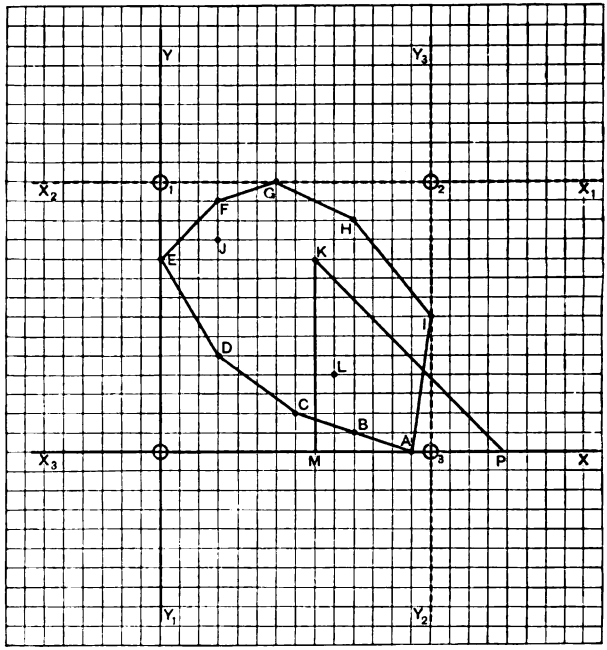

FIG. 1.

point K (*degree-point*) whose co-ordinates are OM = m, MK = n. We observe that, if KP be drawn so that cot KPO = λ, then OP = OM + MP = $m + n\lambda$. Hence OP is the degree in ξ of the term in (9) which corresponds to $(m, n)\xi^m\eta^n$. If, therefore, we select any group of terms whose degree-points lie on a straight line Λ, these will all have the same degree in ξ, namely, the intercept of Λ on OX.

The necessary and sufficient conditions, therefore, that a

group of two or more terms furnish the initial values of a group of expansions, let us say be an *effective group*, are :—

1°. That the line Λ containing the degree-points shall cut OX to the right of O, and OY above O. This secures that λ be positive.

2°. That all the other degree-points shall lie on the opposite side of Λ to the origin. This secures that all the other terms in (9) be of higher degree in ξ than those of the selected group.

We have thus the following rule for selecting the effective groups :—

Let A and E be the degree-points of the terms that contain ξ and η alone, so that $OA = p$, $OE = q$. Let a radius vector, coinciding originally with AX_1, turn clock-wise about A as centre until it passes through another of the degree-points B. If it passes through others at the same time as B, let the last of them taken in order from A be C. Next, let the radius turn about C as centre in the same direction as before, until it passes through another point or points, and let the last of this group taken in order from C be D. Then let the radius turn about D ; and so on, until at last it passes through E, or through a group of which E is the last.

We thus form a broken line convex towards O, beginning at A and ending at E, every part of which contains a group of degree-points the terms corresponding to which satisfy the conditions (10).

Now the degree of the equation (13) corresponding to any group CD is the difference between the degrees of η in the first and last terms C and D ; but this difference is the projection of CD on OY. The sum of all the projections of AC, CD, &c., on OY is OE, that is to say, q. Hence we shall get, by taking all the groups AC, CD, &c., q different expansions for y corresponding to the q different branches that meet at the multiple point $x = a$. Each one of these has the same initial value b, and each is represented by a separate expansion in positive ascending fractional powers of $x - a$.

Example. To separate the branches of the function η at the point $\xi=0$, η being determined by

$$D\xi^5\eta^5 + C\xi^7\eta^2 + E\eta^{10} + B\xi^{10}\eta + A\xi^{13} + L\xi^9\eta^4 + J\xi^3\eta^{11} + F\xi^3\eta^{13} + K\xi^8\eta^{10} + G\xi^6\eta^{14} + I\xi^{14}\eta^7 + H\xi^{10}\eta^{12} = 0. \quad (16).$$

The lowest term in η alone is η^{10}, so that $\xi=0$ is a multiple point of the 10th order. Plotting the degrees of the terms in Newton's diagram, and naming the points by affixing the coefficients, we find (see Fig. 1) that the effective groups are ABC, CD, DE. Taking, for simplicity of illustration,

$$A=+2, \quad B=-3, \quad C=+1, \quad D=-1, \quad E=+1,$$

we get from the group ABC

$$\lambda=6/2, \text{ and } v^2-3v+2=0,$$

that is, $v=1$, or 2, the corresponding expansions being

$$\eta=\xi^3(1+d_1\xi^{1/2}+d_2\xi^{2/2}+\ldots),$$
$$\eta=\xi^3(2+d_1'\xi^{1/2}+d_2'\xi^{2/2}+\ldots).$$

From the group CD, we get

$$\lambda=4/3, \quad v^3-1=0,$$

that is, $v=1$, ω, ω^2, where ω is a primitive imaginary cube root of 1, the corresponding expansions being

$$\eta=\xi^{4/3}(1+d_1\xi^{1/3}+d_2\xi^{2/3}+\ldots),$$
$$\eta=\xi^{4/3}(\omega+d_1'\xi^{1/3}+d_2'\xi^{2/3}+\ldots).$$
$$\eta=\xi^{4/3}(\omega^2+d_1''\xi^{1/3}+d_2''\xi^{2/3}+\ldots).$$

In like manner, DE gives five expansions of the type

$$\eta=\xi^{3/5}(\alpha+d_1\xi^{1/5}+d_2\xi^{2/5}+\ldots),$$

where α is any one of the five 5th roots of 1.

All the ten branches are thus accounted for.

It should be observed that, if we form an integral equation by selecting from any given one a series of terms which form an effective group, the new equation gives an algebraic function. Those branches of this function that have zero initial values coincide to a first approximation (that is, as far as the first term of the expansion) with certain of the branches of the algebraic function determined by the original equation which have initial zero values. Thus, reverting to the example just discussed, from the group ABC we have

$$A\xi^{13} + B\xi^{10}\eta + C\xi^7\eta^2 = 0.$$

This gives, when we drop out the irrelevant factor ξ^7,

$$C\eta^2 + B\xi^3\eta + A\xi^6 = 0,$$

which breaks up into two equations,

$$\eta + p\xi^3 = 0, \quad \eta + q\xi^3 = 0;$$

and thus determines two functions, each of which has a branch coincident *to a first approximation* with a branch of η (as determined by (16)) which has zero initial value.

In like manner, DE gives $C\xi^4 + D\eta^3 = 0$; and DE gives $D\xi^3 + E\eta^5 = 0$.

We thus get a number of binomial equations, each of which gives an approximation for a group of branches of the function η determined by (16). We shall return to this view of the matter in § 24.

§ 23.] Before leaving the general theory just established, we ought to point out that *Newton's Parallelogram enables us to obtain, at every point (singular or non-singular), convergent expansions for every branch of an algebraic function in ascending or descending power-series, as the case may be.*

To establish this completely, we have merely to consider the remaining cases where x or y or both become infinite.

1st. Let us suppose that the value of the function y tends towards a finite limit b when x tends towards ∞. Then, if we put $\eta = y - b$, $x = \xi$, we shall get an equation of the form

$$\Sigma(m, n)\xi^m\eta^n = 0 \qquad (17),$$

which gives $\eta = 0$ when $\xi = \infty$.

Let us suppose that Fig. 1, as originally constructed, is the Newton-diagram for (17), and let ξ^k be the highest power of ξ that occurs in (17) so that $OO_3 = k$. Now in (17) put $\xi = 1/\xi'$, and multiply the equation by ξ'^k; we then get the equation

$$\Sigma(m, n)\xi'^{k-m}\eta^n = 0 \qquad (18),$$

which is obviously equivalent to (17).

But the Newton-diagram for (18) is obviously still Fig. 1, provided O_3X_3 and O_3Y_3 be taken, instead of OX and OY, as the positive parts of the axes.

Hence, if we make a boundary convex towards O_3 in the same way as we did for O, we shall obtain a series of branches of η all of which are expansible in ascending powers of ξ', that is, in descending powers of ξ, and all of which give $\eta = 0$ when $\xi = \infty$. For each such branch we have

$$\eta = \xi^{\lambda}(c + d\xi^{\alpha} + e\xi^{\beta} + \ldots),$$

that is,

$$(y-b)x^{\lambda} = c + d/x^{\alpha} + e/x^{\beta} + \ldots \qquad (19),$$

where λ, α, β, . . . are all positive, and c is finite both ways.

2nd. Suppose that $x = a$ is a pole of y, so that $y = \infty$ when $x = a$; and put $\eta = y$, $\xi = x - a$, so that we derive an equation

$$\Sigma(m, n)\xi^{m}\eta^{n} = 0 \qquad (20),$$

for which Fig. 1 is the Newton-diagram with OX and OY as axes. Then, putting $\eta = 1/\eta'$, we get an equation of the form

$$\Sigma(m, n)\xi^{m}\eta'^{l-n} = 0 \qquad (21),$$

l being the highest exponent of η in (20).

The Newton-diagram for (21) is then Fig. 1 with O_1X_1 and O_1Y_1 as axes; and we construct, as before, a boundary, EFG say, convex towards O_1, every part of which gives a series of branches of η', that is, of $1/\eta$, expansible in ascending powers of ξ. For every such branch we shall have

$$\eta\xi^{\lambda} = 1/(c + d\xi^{\alpha} + e\xi^{\beta} + \ldots),$$

where λ, α, β, . . . are all positive, and c is finite both ways. Hence also, by the binomial theorem combined with § 1,

$$\eta\xi^{\lambda} = 1/c + d'\xi^{\alpha} + e'\xi^{\beta'} + \ldots,$$

that is,

$$y(x-a)^{\lambda} = 1/c + d'(x-a)^{\alpha} + e'(x-a)^{\beta'} + \ldots \qquad (22),$$

where λ, α, β', . . . are all positive, and c is finite both ways.

3rd. Suppose that y has an infinite value corresponding to $x = \infty$ (pole at infinity). Then, if we put $x = \xi = 1/\xi$, $y = \eta = 1/\eta'$, we shall get, by exactly the same kind of reasoning as before, a boundary GHI convex to O_2, each part of which will give a group of expansions of the form

$$\eta' = \xi^{\lambda}\{c + d\xi^{\alpha} + e\xi^{\beta} + \ldots\}.$$

Whence, as before, for every such branch

$$\begin{aligned} y/x^{\lambda} &= 1/(c + d/x^{\alpha} + e/x^{\beta} + \ldots), \\ &= 1/c + d'/x^{\alpha} + e'/x^{\beta} + \ldots \qquad (23), \end{aligned}$$

where λ, α, β', . . . are all positive, and c is finite both ways.

If we combine the results of the present with those of the foregoing paragraphs, we arrive at the following important general theorem regarding any algebraic function y :—

If $y = 0$ *when* $x = a$ $(a \neq \infty)$, *then* $\underset{x=a}{\mathrm{L}}\, y/(x-a)^{\lambda}$ *is finite both ways.*

If $y = 0$ *when* $x = \infty$, *then* $\underset{x=\infty}{\mathrm{L}}\, y/x^{-\lambda}$ *is finite both ways.*

If $y = \infty$ *when* $x = a$ $(a \neq \infty)$, *then* $\underset{x=a}{\mathrm{L}}\, y/(x-a)^{-\lambda}$ *is finite both ways.*

If $y = \infty$ *when* $x = \infty$, *then* $\underset{x=\infty}{\mathrm{L}}\, y/x^{\lambda}$ *is finite both ways.*

λ *is in all cases a finite positive commensurable number which may be called the* ORDER *of the particular zero or infinite value of* y.

This theorem leads us naturally to speak of *algebraical zero- or infinity-values* of functions in general, meaning such as have the property just stated. Thus $\sin x = 0$ when $x = 0$; but $\mathrm{L} \sin x/x = 1$ when $x = 0$; therefore we say that $\sin x$ has an algebraic zero of the first order when $x = 0$. Again, $\tan x = \infty$ when $x = \frac{1}{2}\pi$; but $\mathrm{L} \tan x/(x - \frac{1}{2}\pi)^{-1}$ is finite when $x = \frac{1}{2}\pi$; the infinity of $\tan x$ is therefore algebraical of the first order. On the other hand, $e^x = \infty$ when $x = \infty$; but this is not an algebraical infinity, since no finite value of λ can be found such that $\mathrm{L} e^x/x^{\lambda}$ is finite when $x = \infty$. (See chap. xxv., § 15.)

§ 24.] *Application of the method of successive approximation to the expansion of functions.* This method, when applied in conjunction with Newton's diagram, greatly increases the practical usefulness of the general theorems which have just been established. The method is, moreover, of great historical interest, because it appears from the scanty records left to us that it was in this form that the general theorems which we have been discussing originated in the mind of Newton.

Let us suppose that the terms of an equation (which may be an infinite series) have been plotted in Newton's diagram, and that an effective group of terms has been found lying on a line Λ; and let $\eta^m - \xi^n$ (the coefficients are taken to be unity for simplicity of illustration) be a factor in the group thus selected, repeated, say, α times, so that the whole group is $\phi_1(\xi, \eta)(\eta^m - \xi^n)^{\alpha}$. Let Λ be moved parallel to itself, until it meets a term or group of terms $\phi_2(\xi, \eta)$; then again until it meets a group $\phi_3(\xi, \eta)$; and so on.

The complete equation may now be arranged thus—

$$\phi_1(\xi, \eta)(\eta^m - \xi^n)^\alpha + \phi_2(\xi, \eta) + \phi_3(\xi, \eta) + \ldots = 0,$$

or thus—

$$(\eta^m - \xi^n)^\alpha + \frac{\phi_2(\xi, \eta)}{\phi_1(\xi, \eta)} + \frac{\phi_3(\xi, \eta)}{\phi_1(\xi, \eta)} + \ldots = 0 \qquad (24),$$

say,

$$(\eta^m - \xi^n)^\alpha + \tau_2 + \tau_3 + \ldots = 0.$$

Now, by the properties of the diagram, when $\eta = \xi^{n/m}$, $\phi_2(\xi, \eta)$, $\phi_3(\xi, \eta)$, . . . are in ascending or descending order as regards degree in ξ, and the same is true of τ_2, τ_3, . . . Let us suppose that ξ and η are small, so that τ_2, τ_3, . . . are in ascending order.

As we have seen, $\eta^m = \xi^n$, that is, $\eta = \xi^{n/m}$, gives a first approximation. To obtain a second, we may neglect τ_3, τ_4, . . ., and substitute in τ_2 the value of η as determined by the first approximation. To get a third approximation, neglect τ_4, . . ., substitute in τ_3 the value of η as given by the second approximation, and in τ_2 the value of η as given by the first approximation.

We may proceed thus by successive steps to any degree of approximation; the only points to be attended to are never to neglect any terms of higher degree than the highest retained, and not to waste labour in calculating at any stage the coefficients of terms of higher degree than those already neglected.

Example 1. Taking the equation (16), to find a third approximation to one of the branches of the group CD.

Next in order to C and D a parallel to CD meets successively B and A. Hence, putting, for simplicity, $D = +1$, $C = B = A = -1$, the equation (16) may be written

$$\xi^3\eta^2(\eta^3 - \xi^4) - \xi^{10}\eta - \xi^{13} + \ldots = 0.$$

Whence

$$\eta^3 - \xi^4 - \xi^7/\eta - \xi^{10}/\eta^2 + \ldots = 0 \qquad (25).$$

The first approximation is $\eta = \xi^{4/3}$; hence, neglecting ξ^{10}/η^2 in (25), we get for the second

$$\eta^3 - \xi^4 - \xi^7/\xi^{4/3} = 0.$$

Whence

$$\eta = \xi^{4/3}(1 + \xi^{5/3})^{1/3} = \xi^{4/3}(1 + \tfrac{1}{3}\xi^{5/3}) \qquad (26).$$

If we use this second approximation in ξ^7/η, and the first approximation in ξ^{10}/η^2 now to be retained, (25) gives for the third approximation

$$\eta^3 - \xi^4 - \xi^7/\xi^{4/3}(1 + \tfrac{1}{3}\xi^{5/3}) - \xi^{10}/\xi^{6/3} = 0.$$

Whence, if all terms higher than the last retained be neglected,

$$\eta^3 - \xi^4 - \xi^{17/3} - \tfrac{3}{8}\xi^{22/3} = 0,$$

which gives

$$\begin{aligned}\eta &= \xi^{4/3}(1 + \xi^{5/3} + \tfrac{3}{8}\xi^{10/3})^{\frac{1}{3}},\\ &= \xi^{4/3}(1 + \tfrac{1}{3}\xi^{5/3} + \tfrac{1}{3}\xi^{10/3}) \qquad (27),\end{aligned}$$

which is the required third approximation.

This might of course be obtained by successive applications of the method of transformation employed in the demonstration of § 22, or by the method of indeterminate coefficients, but the calculations would be laborious. It will be observed on comparing (27) with the theoretical result in § 22 that $d_1 = d_2 = d_3 = d_4 = d_6 = d_7 = d_8 = d_9 = 0$; a fact which, in itself, shows the advantages of the present method.

Example 2. To find a second approximation for the branches corresponding to ABC in equation (16), in the special case where $A = +1$, $B = -2$, $C = +1$, $D = -1$.

The terms concerned in this approximation are (ABC) and (D). We therefore write

$$\xi^7(\eta - \xi^3)^2 - \xi^3\eta^5 = 0,$$

or

$$(\eta - \xi^3)^2 - \eta^5/\xi^4 = 0.$$

The first approximation is $\eta = \xi^3$; hence the second is given by

$$(\eta - \xi^3)^2 - \xi^{15}/\xi^4 = 0,$$

that is,

$$(\eta - \xi^3)^2 - \xi^{11} = 0.$$

Whence

$$\eta - \xi^3 \pm \xi^{11/2} = 0,$$

which gives the two second approximations corresponding to the group. These are *two*, because to a first approximation the branches are coincident. This, therefore, is a case where a second approximation is necessary to distinguish the branches.

Example 3. To find a second approximation, for large values both of ξ and η, to the branch corresponding to HI in equation (16).

Referring to Fig. 1, we see that, if HI move parallel to itself towards O, the next point which it will meet is G. Hence, so far as the approximation in question is concerned, we may replace (16) by

$$(H\xi^{10}\eta^{12} + I\xi^{14}\eta^7) + G\xi^6\eta^{14} = 0.$$

For simplicity, let us put $H = 1$, $I = G = -1$, and write the above equation in the form

$$\eta^5 - \xi^4 - \eta^7/\xi^4 = 0.$$

Confining ourselves to *one* of the five first approximations, say $\eta = \xi^{4/5}$, we get for the second approximation

$$\eta^5 - \xi^4 - \xi^{8/5} = 0,$$

which gives

$$\eta = \xi^{4/5}(1 + \tfrac{1}{3}\xi^{-12/5}).$$

Example 4. Given

$$x = y + y^2/2! + y^3/3! + y^4/4! + \ldots$$

to find y to a fourth approximation. We have

$$y = x - y^2/2! - y^3/3! - y^4/4! - \ldots .$$

Hence 1st approx. $y=x$.

2nd ,, $y=x-\frac{1}{2}x^2$.

3rd ,, $y=x-\frac{1}{2}(x-\frac{1}{2}x^2)^2-\frac{1}{6}x^3$,
$=x-\frac{1}{2}x^2+\frac{1}{3}x^3$.

4th ,, $y=x-\frac{1}{2}(x-\frac{1}{2}x^2+\frac{1}{3}x^3)^2-\frac{1}{6}(x-\frac{1}{2}x^3)^3-\frac{1}{24}x^4$,
$=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$.

Historical Note.—As has already been remarked, the fundamental idea of the reversion of series, and of the expansion of the roots of algebraical or other equations in power-series, originated with Newton. His famous "Parallelogram" is first mentioned in the second letter to Oldenburg; but is more fully explained in the *Geometria Analytica* (see Horsley's edition of Newton's Works, t. i., p. 398). The method was well understood by Newton's followers, Stirling and Taylor; but seems to have been lost sight of in England after their time. It was much used (in a modified form of De Gua's) by Cramer in his well-known *Analyse des Lignes Courbes Algébriques* (1750). Lagrange gave a complete analytical form to Newton's method in his "Mémoire sur l'Usage des Fractions Continues," *Nouv. Mém. d. l'Ac. roy. d. Sciences d. Berlin* (1776). (See *Œuvres de Lagrange*, t. iv.)

Notwithstanding its great utility, the method was everywhere all but forgotten in the early part of this century, as has been pointed out by De Morgan in an interesting account of it given in the *Cambridge Philosophical Transactions*, vol. ix. (1855).

The idea of demonstrating, *a priori*, the possibility of expansions such as the reversion-formulæ of § 18 originated with Cauchy; and to him, in effect, are due the methods employed in §§ 18 and 19. See his memoirs on the Integration of Partial Differential Equations, on the Calculus of Limits, and on the Nature and Properties of the Roots of an Equation which contains a Variable Parameter, *Exercices d'Analyse et de Physique Mathématique*, t. i. (1840), p. 327; t. ii. (1841), pp. 41, 109. The form of the demonstrations given in §§ 18, 19 has been borrowed partly from Thomae, *El. Theorie der Analytischen Functionen einer Complexen Veränderlichen* (Halle, 1880), p. 107; partly from Stolz, *Allgemeine Arithmetik*, I. Th. (Leipzig, 1885), p. 296.

The Parallelogram of Newton was used for the theoretical purpose of establishing the expansibility of the branches of an algebraic function by Puiseaux in his Classical Memoir on the Algebraic Functions (*Liouv. Math. Jour.*, 1850). Puiseaux and Briot and Bouquet (*Théorie des Fonctions Elliptiques* (1875), p. 19) use Cauchy's Theorem regarding the number of the roots of an algebraic equation in a given contour; and thus infer the continuity of the roots. The demonstration given in § 21 depends upon the proof, *a priori*, of the possibility of an expansion in a power-series; and in this respect follows the original idea of Newton.

The reader who desires to pursue the subject further may consult Durège, *Elemente der Theorie der Functionen einer Complexen Veränderlichen Grösse*, for a good introduction to this great branch of modern function-theory.

The applications are very numerous, for example, to the finding of curvatures and curves of closest contact, and to curve-tracing generally. A number of beautiful examples will be found in that much-to-be-recommended text-book, Frost's *Curve Tracing*.

Exercises XXIV.

Revert the following series and find, so far as you can, expressions for the coefficient of the general term in the Reverse Series:—

(1.) $y=1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$.

(2.) $y=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots$.

(3.) $y=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$.

(4.) $y=x+x^2/2^2+x^3/3^2+x^4/4^2+\ldots$.

(5.) If $y=\sin x/\sin(x+a)$, expand x in powers of y.

x and y being determined as functions of each other by the following equations, find first and second approximations to those branches, real or imaginary, for which mod x or mod y, or both, become either infinitely small or infinitely great:—

(6.) $y^2-2y=x^4-x^2$.

(7.) $a^3(y+x)-2a^2x(y+x)+x^4=0$, (F. 69*).

(8.) $(x-y)^2-(x-y)x^3-\frac{1}{3}x^4-\frac{1}{3}y^4=0$, (F. 82).

(9.) $a(y^2-x^2)(y-2x)-y^4=0$, (F. 88).

(10.) $ax(y-x)^2-y^4=0$, (F. 96).

(11.) $x(y-x)^2-a^3=0$, (F. 115).

(12.) $x^3y^2-2a^2x^2y+a^4x-b^5=0$, (F. 121).

(13.) $y(y-x)^2(y+2x)=9cx^3$, (F. 131).

(14.) $\{x(y-x)-a^2\}^2y^3=a^7$, (F. 140).

(15.) $x^7-x^4y^3+a^3y^4-axy^5=0$, (F. 143).

(16.) $a(x^5+y^5)-a^2x^3y+x^2y^4=0$, (F. 143).

(17.) $x^3y^4+ax^2y^3+bx^4y+cx+dy^2=0$, where a, b, c, d are all positive, (F. 155).

(18.) If e_n be any constant whatever when n is a prime number, and such that $e_n=e_pe_qe_r\ldots$ when n is composite and has for its prime factors p, q, r, . . ., then show that

If a, b, c, . . . be a given succession of primes finite or infinite in number, s any integer of the form $a^\alpha b^\beta c^\gamma\ldots$, t any integer of the forms a, ab, abc, . . . (where none of the prime factors are powers), and if

$$F(x)=\Sigma e_s f(x^s),$$

then

$$f(x)=\Sigma(-)^u e_t F(x^t),$$

where u is the number of factors in t.

(This remarkable theorem was given by Möbius, *Crelle's Jour.*, ix. p. 105. For an elegant proof and many interesting consequences, see an article by J. W. L. Glaisher, *Phil. Mag.*, ser. 5, xviii., p. 518 (1884).)

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* F. 69 means that a discussion of the *real* branches of this function, with the corresponding graph, will be found in Frost's *Curve Tracing*, § 69.

CHAPTER XXXI.

Summation and Transformation of Series in General.

THE METHOD OF FINITE DIFFERENCES.

§ 1.] We have already touched in various connections upon the summation of series. We propose in the present chapter to bring together a few general propositions of an elementary character which will still further help to guide the student in this somewhat intricate branch of algebra.

It will be convenient, although for our immediate purposes it is not absolutely necessary, to introduce a few of the elementary conceptions of the Calculus of Finite Differences. We shall thus gain clearness and conciseness without any sacrifice of simplicity; and the student will have the additional advantage of an introduction to such works as Boole's *Finite Differences*, where he must look for any further information that he may require regarding the present subject.

Let, as heretofore, u_n be the nth term of any series; in other words, let u_n be any one-valued function of the integral variable n; $u_{n-1}, u_{n-2}, \ldots, u_1$ the same functions of $n-1, n-2, \ldots, 1$ respectively.

Farther, let $\Delta u_n, \quad \Delta u_{n-1}, \quad \ldots, \quad \Delta u_1$

denote $u_{n+1} - u_n, \quad u_n - u_{n-1}, \quad \ldots, \quad u_2 - u_1$;

also $\Delta(\Delta u_n), \quad \Delta(\Delta u_{n-1}), \quad \ldots, \quad \Delta(\Delta u_1)$,

which we may write, for shortness,

$$\Delta^2 u_n, \qquad \Delta^2 u_{n-1}, \quad \ldots, \qquad \Delta^2 u_1,$$

denote

$$\Delta u_{n+1} - \Delta u_n, \quad \Delta u_n - \Delta u_{n-1}, \quad \ldots, \quad \Delta u_2 - \Delta u_1 ;$$

and so on. Thus we have the successive series,

$$\begin{array}{llllll} u_1, & u_2, & u_3, & \ldots, & u_n, & \ldots \quad (1); \\ \Delta u_1, & \Delta u_2, & \Delta u_3, & \ldots, & \Delta u_n, & \ldots \quad (2); \\ \Delta^2 u_1, & \Delta^2 u_2, & \Delta^2 u_3, & \ldots, & \Delta^2 u_n, & \ldots \quad (3); \\ \Delta^3 u_1, & \Delta^3 u_2, & \Delta^3 u_3, & \ldots, & \Delta^3 u_n, & \ldots \quad (4); \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

where each term in any series is obtained by subtracting the one immediately above it from the one immediately above and to the right of it.

The series (2), (3), (4), . . . are spoken of as the series of 1st, 2nd, 3rd, . . . differences corresponding to the primary series (1).

Example 1. If $u_n = n^2$, the series in question are

$$\begin{array}{l} 1, 4, 9, 16, \ldots n^2, \ldots ; \\ 3, 5, 7, 9, \ldots 2n+1, \ldots ; \\ 2, 2, 2, 2, \ldots 2, \ldots ; \\ 0, 0, 0, 0, \ldots 0, \ldots ; \end{array}$$

where, as it happens, the second differences are all equal, and the third and all higher differences all vanish.

Cor. If we take for the primary series

$$\Delta^r u_1, \quad \Delta^r u_2, \quad \Delta^r u_3, \quad \ldots, \quad \Delta^r u_n, \quad \ldots,$$

then the series of 1st, 2nd, 3rd, . . . differences will be

$$\begin{array}{llllll} \Delta^{r+1} u_1, & \Delta^{r+1} u_2, & \Delta^{r+1} u_3, & \ldots, & \Delta^{r+1} u_n, & \ldots ; \\ \Delta^{r+2} u_1, & \Delta^{r+2} u_2, & \Delta^{r+2} u_3, & \ldots, & \Delta^{r+2} u_n, & \ldots ; \\ \Delta^{r+3} u_1, & \Delta^{r+3} u_2, & \Delta^{r+3} u_3, & \ldots, & \Delta^{r+3} u_n, & \ldots ; \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

In other words, *we have, in general,* $\Delta^r \Delta^s u_n = \Delta^{r+s} u_n$. This is sometimes expressed by saying that the difference operator Δ obeys the associative law for multiplication.

Although we shall only use it for stating formulæ in concise and easily-remembered forms, we may also introduce at this stage the operator E, which has for its office to increase by unity the variable in any function to which it is prefixed. Thus

$$\mathrm{E}\phi(n) = \phi(n+1); \quad \mathrm{E}u_n = u_{n+1}; \quad \mathrm{E}u_1 = u_2;$$

and so on.

In accordance with this definition we have $\mathrm{E}(\mathrm{E}u_n)$, which we contract into E^2u_n, $= \mathrm{E}u_{n+1} = u_{n+2}$; and, in general, $\mathrm{E}^m u_n = u_{n+m}$. We have also, as with Δ, $\mathrm{E}^r\mathrm{E}^s u_n = \mathrm{E}^{r+s}u_n$, for each of these is obviously equal to u_{n+r+s}.

Example 2. $\mathrm{E}^r n^3 = (n+r)^3$.

Example 3. The mth difference of an integral function of n of the rth degree is an integral function of the $(r-m)$th degree if $m<r$, a constant if $r=m$, zero if $m>r$.

Let

$$\phi_r(n) = an^r + bn^{r-1} + cn^{r-2} + \ldots;$$

then

$$\begin{aligned}\Delta\phi_r(n) &= a(n+1)^r + b(n+1)^{r-1} + c(n+1)^{r-2} + \ldots \\ &\qquad - an^r - \quad bn^{r-1} - \quad cn^{r-2} + \ldots, \\ &= ran^{r-1} + \{\tfrac{1}{2}r(r-1)a + (r-1)b\}n^{r-2} + \ldots, \\ &= \phi_{r-1}(n),\end{aligned}$$

say, where $\phi_{r-1}(n)$ is an integral function of n of the $(r-1)$th degree. Then, in like manner, we have $\Delta\phi_{r-1}(n) = \phi_{r-2}(n)$. But $\Delta\phi_{r-1}(n) = \Delta^2\phi_r n$; hence $\Delta^2\phi_r(n) = \phi_{r-2}(n)$. Similarly, $\Delta^3\phi_r(n) = \phi_{r-3}(n)$; and, in general, $\Delta^m\phi_r(n) = \phi_{r-m}(n)$. We see also that $\Delta^r\phi_r(n)$ will reduce to a constant, namely, $r!a$; and that all differences whose order exceeds r will be zero.

The product of a series of factors in arithmetical progression, such as $a(a+b)\ldots(a+(m-1)b)$, plays a considerable part in the summation of series. Such a product was called by Kramp a *Faculty*, and he introduced for it the notation $a^{m|b}$, calling a the base, m the exponent, and b the difference of the faculty. This notation we shall occasionally use in the slightly modified form $a^{|m|b}$, which is clearer, especially when the exponent is compound.

Since

$$a(a+b)\ldots(a+(m-1)b) = b^m(a/b)(a/b+1)\ldots(a/b+m-1),$$

any faculty can always be reduced to another whose difference is unity, that is, to another of the form $c^{|m|1}$, which, omitting the 1, we may write $c^{|m}$. In this notation the ordinary factorial $m!$ would be written $1^{|m|}$.

The reader should carefully verify and note the following properties of the differences of Facultics and Factorials. In all cases Δ operates as usual with respect to n.

Example 4.

$$\Delta(a+bn)^{|m|b} = mb\{a+b(n+1)\}^{|m-1|b}.$$

Example 5.

$$\Delta\{1/(a+bn)^{|m|b}\} = -mb/(a+bn)^{|m+1|b}.$$

Example 6.

$$\Delta\left\{\frac{a^{|n|b}}{c^{|n|b}}\right\} = \frac{a-c}{a-b}\,\frac{(a-b)^{|n+1|b}}{c^{|n+1|b}}.$$

Example 7.

$$\Delta \cos(\alpha+\beta n) = -2 \sin \tfrac{1}{2}\beta \sin(\alpha+\tfrac{1}{2}\beta+\beta n);$$
$$\Delta \sin(\alpha+\beta n) = +2 \sin \tfrac{1}{2}\beta \cos(\alpha+\tfrac{1}{2}\beta+\beta n).$$

§ 2.] *Fundamental Theorems.* The following pair of theorems* form the foundation of the methods of differences, both direct and inverse:—

I. $\Delta^m u_n = u_{n+m} - {}_mC_1 u_{n+m-1} + {}_mC_2 u_{n+m-2} + \ldots + (-)^m u_n.$

II. $u_{m+n} = u_m + {}_nC_1 \Delta u_m + {}_nC_2 \Delta^2 u_m + \ldots + \Delta^n u_m.$

To prove I. we observe that

$$\begin{aligned} \Delta u_n &= u_{n+1} - u_n; \\ \Delta^2 u_n &= u_{n+2} - u_{n+1} \\ &\quad - u_{n+1} + u_n, \\ &= u_{n+2} - 2u_{n+1} + u_n; \end{aligned}$$

hence

$$\begin{aligned} \Delta^3 u_n &= u_{n+3} - 2u_{n+2} + u_{n+1} \\ &\quad - u_{n+2} + 2u_{n+1} - u_n, \\ &= u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_n; \end{aligned}$$

and so on.

Here the numerical values of the coefficients are obviously being formed according to the addition rule for the binomial coefficients (see chap. iv., § 14); and the signs obviously alternate. Hence the first theorem follows at once.

To prove II. we observe that we have, by the definition of Δu_m, $u_{m+1} = u_m + \Delta u_m$. Hence, since the difference of a sum of functions is obviously the sum of their differences, we have, in like manner, $u_{m+2} = u_{m+1} + \Delta u_{m+1} = u_m + \Delta u_m + \Delta(u_m + \Delta u_m) = u_m + \Delta u_m + \Delta u_m + \Delta^2 u_m$. We therefore have in succession

* The second of these was given by Newton, *Principia*, lib. iii., lemma v. (1687); and is sometimes spoken of as Newton's Interpolation Formula. See his tract, *Methodus Differentialis* (1711); also Demoivre, *Miscellanea Analytica*, p. 152 (1730), and Stirling, *Methodus Differentialis*, &c., p. 97 (1730).

$$
\begin{aligned}
u_{m+1} &= u_m + \Delta u_m, \\
u_{m+2} &= u_m + \Delta u_m \\
&\quad + \Delta u_m + \Delta^2 u_m, \\
&\quad\quad u_m + 2\Delta u_m + \Delta^2 u_m; \\
u_{m+3} &= u_m + 2\Delta u_m + \Delta^2 u_m \\
&\quad + \Delta u_m + 2\Delta^2 u_m + \Delta^3 u_m, \\
&\quad\quad u_m + 3\Delta u_m + 3\Delta^2 u_m + \Delta^3 u_m;
\end{aligned}
$$

and so on.

The second theorem is therefore established by exactly the same reasoning as the first, the only difference being that the signs of the coefficients are now all positive.

If we use the symbol E, and separate the symbols of operation from the subjects on which they operate, the above theorems may be written in the following easily-remembered symbolical forms :—

$$\Delta^m u_n = (\mathrm{E} - 1)^m u_n \quad \text{(I.)}; \qquad u_{m+n} = (1 + \Delta)^n u_m \quad \text{(II.)}$$

§ 3.] The following theorem enables us to reduce the summation of any series to an inverse problem in the calculus of finite differences.

If v_n be any function of n such that $\Delta v_n = u_n$, then

$$\sum_{n=s}^{n} u_n = v_{n+1} - v_s \qquad (1).$$

This is at once obvious, if we add the equations

$$
\begin{aligned}
u_n &= \Delta v_n &&= v_{n+1} - v_n, \\
u_{n-1} &= \Delta v_{n-1} &&= v_n - v_{n-1}, \\
&\cdots\cdots\cdots\cdots \\
u_s &= \Delta v_s &&= v_{s+1} - v_s.
\end{aligned}
$$

The difficulty of the summation of any series thus consists entirely in finding a solution (any solution will do) of the finite difference equation $\Delta v_n = u_n$, or $v_{n+1} - v_n = u_n$. This solution can be effected in finite terms in only a limited number of cases, some of the more important of which are exemplified below.

On the other hand, the above theorem enables us to con-

struct an infinite number of finitely summable series. All we have to do is to take any function of n whatever and find its first difference; then this first difference is the nth term of a summable series. It was in this way that many of the ordinary summable series were first obtained by Leibnitz, James and John Bernoulli, Demoivre, and others.

Example 1. $\sum_{n=s}^{n} \{a+nb\}\{a+(n+1)b\}\dots\{a+(n+m-1)b\}$.

Using Kramp's notation, we have here to solve the equation

$$\Delta v_n = \{a+nb\}\,|\,m\,|\,b \qquad (2).$$

Now we easily find, by direct verification, or by putting $m+1$ for m and $n-1$ for n in § 1, Example 4, that

$$\Delta[\{a+(n-1)b\}\,|\,m+1\,|\,b/(m+1)b] = \{a+nb\}\,|\,m\,|\,b.$$

Hence $v_n = \{a+(n-1)b\}\,|\,m+1\,|\,b/(m+1)b$ is a value of v_n such as we require.

Therefore

$$\sum_{s}^{n}\{a+nb\}\,|\,m\,|\,b = \frac{\{a+nb\}\,|\,m+1\,|\,b - \{a+(s-1)b\}\,|\,m+1\,|\,b}{(m+1)b} \qquad (3).$$

Hence the well-known rule

$$\sum^{n}\{a+nb\}\{a+(n+1)b\}\dots\{a+(n+m-1)b\} = C + \{a+nb\}\{a+(n+1)b\}\dots\{a+(n+m-1)b\}\{a+(n+m)b\}/(m+1)b \qquad (4),$$

where C *is independent of* n, *and may be found in practice by making the two sides of* (4) *agree for a particular value of* n.

Example 2. To sum any series whose nth term is an integral function of n, say $f(n)$.

By the method of chap. v., § 22 (2nd ed.), we can express $f(n)$ in the form $a+bn+cn(n+1)+dn(n+1)(n+2)+\dots$ Hence

$$\sum^{n} f(n) = C + an + \tfrac{1}{2}bn(n+1) + \tfrac{1}{3}cn(n+1)(n+2) + \tfrac{1}{4}dn(n+1)(n+2)(n+3)+\dots \qquad (5),$$

where the constant C can be determined by giving n any particular value in (5).

Example 3. $\Sigma 1/\{a+bn\}\,|\,m\,|\,b$.

Proceeding exactly as in Example 1, and using § 1, Example 5, we deduce

$$\sum_{s}^{n}\frac{1}{\{a+bn\}\,|\,m\,|\,b} = \frac{1/\{a+bs\}\,|\,m-1\,|\,b - 1/\{a+b(n+1)\}\,|\,m-1\,|\,b}{(m-1)b} \qquad (6).$$

Hence a rule for this class of series like that given in Example 1.

Example 4. To sum the series $\Sigma f(n)/\{a+bn\}\,|\,m\,|\,b$, $f(n)$ being an integral function of n.

Decompose $f(n)$, as in Example 2, into

$$a+\beta(a+bn)^{\underline{1}\,\underline{b}}+\gamma(a+bn)^{\underline{2}\,\underline{b}}+\delta(a+bn)^{\underline{3}\,\underline{b}}+\ldots \qquad (7).$$

Then we have to evaluate

$$a\sum_{1}^{n}1/\{a+bn\}^{\underline{m}\,\underline{b}}+\beta\sum_{1}^{n}1/\{a+b(n+1)\}^{\underline{m-1}\,\underline{b}}+\ldots \qquad (8),$$

which can at once be done by the rule of Example 3.*

Example 5.

$$\sum_{s}^{n}\frac{a^{\underline{n}\,\underline{b}}}{c^{\underline{n}\,\underline{b}}}=\frac{a}{a-c+b}\left\{\frac{(a+b)^{\underline{n}\,\underline{b}}}{c^{\underline{n}\,\underline{b}}}-\frac{(a+b)^{\underline{s-1}\,\underline{b}}}{c^{\underline{s-1}\,\underline{b}}}\right\} \qquad (9).$$

This can be deduced at once from § 1, Example 6, by writing $a+b$ for b and $n-1$ for n.

Example 6. To sum the series whose terms are the *Figurate Numbers* of the mth rank.

The figurate numbers of the 1st, 2nd, 3rd, . . . ranks are the numbers in the 1st, 2nd, 3rd, . . . vertical columns of the table (II.) in chap. iv., § 25. Hence the $(n+1)$th figurate number of the mth rank is ${}_{n+m-1}C_{m-1}$ $={}_{n+m-1}C_{n}=m(m+1)\ldots(m+n-1)/n!$. Hence we have to sum the series

$$1+\sum_{1}^{n}\frac{m(m+1)\ldots(m+n-1)}{1.2\ldots n}.$$

Now if in (9), Example 5, we put $a=m$, $b=1$, $c=1$, we get

$$\sum_{2}^{n}\frac{m^{\underline{n}\,\underline{|}}}{1^{\underline{n}\,\underline{|}}}=\frac{(m+1)^{\underline{n}\,\underline{|}}}{1^{\underline{n}\,\underline{|}}}-\frac{m+1}{1}.$$

Hence

$$1+m+\frac{m(m+1)}{1.2}+\ldots+\frac{m(m+1)\ldots(m+n-1)}{1.2\ldots n}=\frac{(m+1)(m+2)\ldots(m+1+n-1)}{1.2\ldots n} \qquad (10);$$

that is to say, *the sum of the first n figurate numbers of the mth rank is the nth figurate number of the* $(m+1)$*th rank.*

This theorem is, however, merely the property of the function ${}_{m}H_{n}$, which we have already established in chap. xxiii., § 10, Cor. 4. The present demonstration of (10) is of course not restricted to the case where m is a positive integer.

Many other well-known results are included in the formula of Example 5, some of which will be found among the exercises below.

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* The methods of Examples 1 to 4 are all to be found in Stirling's *Methodus Differentialis*. He applies them in a very remarkable way to the approximate evaluation of series which cannot be summed. (See Exercises XXVII., 17.)

Example 7. To sum the series

$$S_n = \cos\alpha + \cos(\alpha+\beta) + \ldots + \cos(\alpha+(n-1)\beta);$$
$$T_n = \sin\alpha + \sin(\alpha+\beta) + \ldots + \sin(\alpha+(n-1)\beta).$$

From § 1, Example 7, we have $\cos(\alpha+\beta n) = \Delta\{\sin(\alpha-\frac{1}{2}\beta+\beta n)/2\sin\frac{1}{2}\beta\}$. Hence

$$S_n = \{\sin(\alpha-\tfrac{1}{2}\beta+\beta n) - \sin(\alpha-\tfrac{1}{2}\beta)\}/2\sin\tfrac{1}{2}\beta,$$
$$= \frac{\sin\frac{1}{2}\beta n}{\sin\frac{1}{2}\beta}\cos\{\alpha+\tfrac{1}{2}\beta(n-1)\}.$$

Similarly,

$$T_n = \frac{\sin\frac{1}{2}\beta n}{\sin\frac{1}{2}\beta}\sin\{\alpha+\tfrac{1}{2}\beta(n-1)\}.$$

§ 4.] *Expression for the sum of n terms of a series in terms of the first term and its successive differences.*

Let the series be $u_1 + u_2 + \ldots + u_n$; and let us add to the beginning an arbitrary term u_0. Then if we form the quantities

$$S_0 = u_0, \quad S_1 = u_0 + u_1, \quad S_2 = u_0 + u_1 + u_2,$$
$$\ldots, \quad S_n = u_0 + u_1 + u_2 + \ldots + u_n, \ldots,$$

we have

$$\Delta S_n = u_{n+1}, \quad \Delta^2 S_n = \Delta u_{n+1}, \quad \ldots, \quad \Delta^m S_n = \Delta^{m-1} u_{n+1}, \ldots.$$

Hence, putting $n = 0$,

$$\Delta S_0 = u_1, \quad \Delta^2 S_0 = \Delta u_1, \quad \ldots, \quad \Delta^m S_0 = \Delta^{m-1} u_1, \ldots \quad (1).$$

Now, by Newton's formula (§ 2, II.),

$$S_n = S_0 + {}_nC_1\Delta S_0 + {}_nC_2\Delta^2 S_0 + \ldots + \Delta^n S_0 \qquad (2).$$

If, therefore, we replace S_0, ΔS_0, $\Delta^2 S_0$, ... by their values according to (1), we have

$$\sum_0^n u_n = u_0 + {}_nC_1 u_1 + {}_nC_2\Delta u_1 + {}_nC_3\Delta^2 u_1 + \ldots + \Delta^{n-1}u_1 \quad (3);$$

or, if we subtract u_0 from both sides,

$$\sum_1^n u_n = {}_nC_1 u_1 + {}_nC_2\Delta u_1 + {}_nC_3\Delta^2 u_1 + \ldots + \Delta^{n-1}u_1 \quad (4).*$$

The formula (4) is simply an algebraical identity which may be employed to transform any series whatsoever; for example, in the case of the geometric series Σx^n it gives

* This formula, which, as Demoivre (*Miscell. An.*, p. 153) pointed out, is an immediate consequence of Newton's rule, seems to have been first explicitly stated by Montmort, *Journ. d. Savans* (1711). It was probably independently found by James Bernoulli, for it is given in the *Ars Conjectandi*, p. 98 (1713).

$$x^1 + x^2 + \ldots + x^n$$
$$= nx + \frac{n(n-1)}{2!}x(x-1) + \frac{n(n-1)(n-2)}{3!}x(x-1)^2 + \ldots + x(x-1)^{n-1},$$

which can be easily verified independently by transforming the right-hand side. The transformation (4) will, however, lead to the sum of the series, in the proper sense of the word *sum*, only when the mth differences of the terms become zero, m being a finite integer. The sum of the series will in that case be given by (4) as an integral function of n of the mth degree. Since the nth term of the series is the first difference of its finite sum, we see conversely that any series whose sum to n terms is an integral function of n of the mth degree must have for its nth term an integral function of n of the $m-1$th degree. We have thus reproduced from a more general point of view the results of chap. xx., § 10.

Example. Sum the series

$$\sum_1^n (n+1)(n+2)(n+3).$$

If we tabulate the first few terms and the successive differences, we get

	1,	2,	3,	4,	5
u_n	24,	60,	120,	210,	336,
Δu_n	36,	60,	90,	126,	
$\Delta^2 u_n$	24,	30,	36,		
$\Delta^3 u_n$	6,	6,			
$\Delta^4 u_n$	0.				

Hence, by (4),

$$\sum_1^n (n+1)(n+2)(n+3)$$
$$= n.24 + \frac{n(n-1)}{2}.36 + \frac{n(n-1)(n-2)}{6}.24 + \frac{n(n-1)(n-2)(n-3)}{24}.6,$$
$$= \tfrac{1}{4}(n^4 + 10n^3 + 35n^2 + 50n).$$

§ 5.] *Montmort's Theorem regarding the summation of* $\Sigma u_n x^n$.

An elegant formula for the transformation of the power-series $\Sigma u_n x^n$ may be obtained as follows. Let us in the first place consider $S = \sum_1^\infty u_n x^n$, which we suppose to be convergent when $\mod x < 1$; and let us further suppose that $\mod x < \mod (1-x)$. Put $x = y/(1+y)$; so that

$$\text{mod } y/(1+y) = \text{mod } x < 1,$$

and

$$\text{mod } y = \text{mod } x/(1-x) < 1.$$

Then, since

$$(1+y)^{-m} = 1 - {}_mC_1 y + {}_{m+1}C_2 y^2 - {}_{m+2}C_3 y^3 + \ldots,$$

we have

$$\begin{aligned} S = \sum_1^\infty u_n y^n/(1+y)^n, \\ = u_1 y - u_1 y^2 + u_1 y^3 - u_1 y^4 + u_1 y^5 - \ldots \\ + u_2 y^2 - {}_2C_1 u_2 y^3 + {}_3C_2 u_2 y^4 - {}_4C_3 u_2 y^5 + \ldots \\ + u_3 y^3 - {}_3C_1 u_3 y^4 + {}_4C_2 u_3 y^5 - \ldots \\ + u_4 y^4 - {}_4C_1 u_4 y^5 + \ldots \\ + u_5 y^5 - \ldots \end{aligned}$$

This double series evidently satisfies Cauchy's criterion, for both mod $y < 1$ and mod $y/(1+y) < 1$. Hence we may rearrange it according to powers of y. If we bear in mind § 2, I., we find at once

$$S = u_1 y + \Delta u_1 y^2 + \Delta^2 u_1 y^3 + \Delta^3 u_1 y^4 + \Delta^4 u_1 y^5 + \ldots.$$

Hence, replacing y by its value, namely, $x/(1-x)$, we get

$$\sum_1^\infty u_n x^n = \frac{u_1 x}{1-x} + \frac{\Delta u_1 x^2}{(1-x)^2} + \frac{\Delta^2 u_1 x^3}{(1-x)^3} + \ldots \qquad (1).^*$$

When the differences of a finite order m vanish, Montmort's formula gives a closed expression for the sum to infinity; and, if the differences diminish rapidly, it gives in certain cases a convenient formula for numerical approximation.

Cor. 1. *We have for the finite sum*

$$\sum_1^n u_n x^n = (u_1 - u_{n+1}x^n)\frac{x}{1-x} + (\Delta u_1 - x^n \Delta u_{n+1})\frac{x^2}{(1-x)^2} + (\Delta^2 u_1 - x^n \Delta^2 u_{n+1})\frac{x^3}{(1-x)^3} + \ldots \qquad (2).$$

For, if we start with the series $u_{n+1}x^{n+1} + u_{n+2}x^{n+2} + \ldots$, and proceed as before, we get

$$\sum_{n+1}^\infty u_n x^n = \frac{u_{n+1}x^{n+1}}{(1-x)^1} + \frac{\Delta u_{n+1}x^{n+2}}{(1-x)^2} + \frac{\Delta u_{n+2}x^{n+3}}{(1-x)^3} + \ldots \qquad (3).$$

From (1) and (3) we get (2) at once by subtraction.

* First given by Montmort, *Phil. Trans.* R.S.L. (1717). Demoivre gave in his *Miscellanea* a demonstration very much like the above.

The formula (2) will furnish a sum in the proper sense only when the differences vanish after a certain order. The summation of the integro-geometric series, already discussed in chap. xx., §§ 13 and 14, may be effected in this way. It should be observed that, inasmuch as (2) is an algebraic identity between a finite number of terms, its truth does not depend on the convergency of $\Sigma u_n x^n$, although that supposition was made in the above demonstration.

Cor. 2. *If u_n be a real positive quantity which constantly diminishes as n increases, and if* $\mathrm{L}u_n = 0$, *then*

$$u_1 - u_2 + u_3 - \ldots = \frac{1}{2}u_1 - \frac{1}{2^2}\Delta u_1 + \frac{1}{2^3}\Delta^2 u_1 - \ldots \quad (4).^*$$

This is merely a particular case of (1); for, if in (1) we put $-x$ for x, we get

$$\sum_1^\infty (-)^n u_n x^n = \sum_1^\infty (-)^n \Delta^{n-1} u_1 . \left(\frac{x}{1+x}\right)^n \quad (5).$$

Since the differences must ultimately remain finite, the right-hand side of (5) will be convergent when $x = 1$. Also, by Abel's Theorem (chap. xxvi., § 20), since $\Sigma(-)^n u_n$ is convergent, the limit of the left-hand side of (5) when $x = 1$ is $\sum_1^\infty (-)^n u_n$. Hence the theorem follows.

The transformation in formula (4) in general increases the convergency of the series, and it may of course, in particular cases, lead to a finite expression for the sum.

Cor. 3. *We get, by subtraction, the following formula* :—

$$u_1 - u_2 + \ldots (-)^{n-1} u_n = \frac{1}{2}(u_1 - (-)^n u_{n+1}) - \frac{1}{2^2}(\Delta u_1 - (-)^n \Delta u_{n+1}) + \frac{1}{2^3}(\Delta^2 u_1 - (-)^n \Delta^2 u_{n+1}) - \ldots \quad (6),$$

in which the restrictions on u_n will be unnecessary if the right-hand side be a closed expression, which it will be if the differences of u_n vanish after a certain order.

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* Euler, *Inst. Diff. Calc.*, Part II., cap. i. (1787).

Example 1. We have (Gregory's Series)

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots \qquad (7).$$

If we apply (4), we have $u_n=1/(2n-1)$. Hence

$$\Delta^r u_n=(-)^r 2.4\ldots 2r/(2n-1)(2n+1)(2n+3)\ldots(2n+2r-1);$$
$$\Delta^r u_1=(-)^r 2.4\ldots 2r/1.3.5\ldots(2r+1),$$
$$=(-)^r 2^r.1.2\ldots r/1.3.5\ldots(2r+1).$$

Therefore
$$\frac{\pi}{2}=1+\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\ldots \qquad (8).$$

Example 2. To sum the series

$$S_n=1^2-2^2+3^2-\ldots(-)^{n-1}n^2.$$

Since
$$\Delta u_{n+1}=2n+3, \quad \Delta u_1=3,$$
$$\Delta^2 u_{n+1}=2, \quad \Delta^2 u_1=2,$$
$$\Delta^3 u_{n+1}=0, \quad \Delta^3 u_1=0,$$

we have, by (6),

$$S_n=\tfrac{1}{2}\{1-(-)^n(n+1)^2\}-\tfrac{1}{4}\{3-(-)^n(2n+3)\}+\tfrac{1}{8}\{2-(-)^n 2\},$$
$$=(-)^{n-1}\tfrac{1}{2}n(n+1).$$

Exercises XXV.

(1.) Sum to n terms the series whose nth term is the nth r-gonal number.*

Sum the following series to n terms, and, where possible, also to infinity:—

(2.) $\Sigma n(n+2)(n+4)$. (3.) $\sum_2^n 1/(n^2-1)$.

(4.) $1/3.8+1/8.13+1/13.18+\ldots$.

(5.) $1/1.3.5+1/3.5.7+1/5.7.9+\ldots$.

(6.) $1/1.2.3.4+1/2.3.4.5+1/3.4.5.6+\ldots$.

(7.) $\Sigma(an+b)/n(n+1)(n+2)$.

(8.) $1/1.3.5+2/3.5.7+3/5.7.9+\ldots$.

(9.) $1/1.2.4+1/2.3.5+1/3.4.6+\ldots$.

(10.) $1/1.3.7+1/3.5.9+1/5.7.11+\ldots$.

(11.) $\Sigma(n+1)^2/n(n+2)$.

(12.) $4/1.3.5.7+9/2.4.6.8+16/3.5.7.9+\ldots$.

(13.) $\Sigma \sec n\theta \sec(n+1)\theta$. (14.) $\Sigma \tan(\theta/2^n)/2^n$.

(15.) $\Sigma \tan^{-1}\{(na-n+1)a^{n-1}/(1+n(n-1)a^{2n-1})\}$.

(16.) $\Sigma \tan^{-1}\{2/n^2\}$.

(17.) $m!+(m+1)!/1!+(m+2)!/2!+\ldots$.

(18.) $1!/m!+2!/(m+1)!+3!/(m+2)!+\ldots$.

* The sums to n terms of arithmetical progressions whose first terms are all unity, and whose common differences are 0, 1, 2, . . ., $(r-1)$, . . . respectively, are called the nth polygonal numbers of the 1st, 2nd, 3rd, . . ., rth, . . . order. The numbers of the first, second, third, fourth, . . . orders are spoken of as linear, triangular, square, pentagonal, . . . numbers.

(19.) $1-{}_mC_1+{}_mC_2-\ldots(-)^n{}_mC_n$.

(20.) Show that the figurate numbers of a given rank can be summed by the formula of § 3, Example 1.

(21.) $1+\frac{1}{m}+\frac{1.2}{m(m+1)}+\frac{1.2.3}{m(m+1)(m+2)}+\ldots$.

(22.) $\frac{a(a+1)\ldots(a+r)}{c}+\frac{a(a+1)\ldots(a+r+1)}{c(c+1)}+\ldots$.

(23.) $\frac{a}{c(c+1)\ldots(c+r)}+\frac{a(a+1)}{c(c+1)\ldots(c+r+1)}+\ldots$.

(24.) $\Sigma(a+n)^{\lfloor m-2\rfloor}/(c+n)^{\lfloor m\rfloor}$.

(25.) $\frac{1.3}{1.2.3.4}+\frac{1.3.5}{1.2.3.4.5}+\frac{1.3.5.7}{1.2.3.4.5.6}+\ldots$.

(26.) $\frac{(1+r)(1+2r)}{1.2.3.4.5}+\frac{(1+r)(1+2r)(1+3r)}{1.2.3.4.5.6}+\ldots$.

(27.) $\frac{2}{1}m-\frac{2^2}{1.3}m(m-1)+\frac{2^3}{1.3.5}m(m-1)(m-2)-\ldots$.

(28.) Show that

$$1\Big/\frac{1}{2}\cdot\frac{3}{2}\cdots\left(n+\frac{1}{2}\right)-\frac{1}{1!}\Big/\frac{1}{2}\cdot\frac{3}{2}\cdots\left(n-\frac{1}{2}\right)+\frac{1}{2!}\Big/\frac{1}{2}\cdot\frac{3}{2}\cdots\left(n-\frac{3}{2}\right)-\ldots$$
$$(-)^n\frac{1}{n!}\Big/\frac{1}{2}=(-)^n\frac{1}{n!}\Big/\left(n+\frac{1}{2}\right).$$

(Glaisher.)

(29.) Show that

$$1+2(1-a)+3(1-a)(1-2a)+\ldots+n(1-a)(1-2a)\ldots(1-(n-1)a)$$
$$=a^{-1}\{1-(1-a)(1-2a)\ldots(1-na)\}.$$

(30.) $$\frac{1}{x+1}=\frac{1}{x-1}-\frac{2!}{(x-1)(x-2)}+\frac{3!}{(x-1)(x-2)(x-3)}-\ldots$$
$$\frac{(-)^{n+1}n!}{(x-1)(x-2)\ldots(x-n)}\left(1-\frac{n+1}{x+1}\right).$$

(31.) If $a+b+2=c+d$, then

$$\sum_s^n\frac{a^{\lfloor n\rfloor}b^{\lfloor n\rfloor}}{c^{\lfloor n\rfloor}d^{\lfloor n\rfloor}}=\frac{ab}{(a+1)(b+1)-cd}\left\{\frac{(a+1)^{\lfloor n\rfloor}(b+1)^{\lfloor n\rfloor}}{c^{\lfloor n\rfloor}d^{\lfloor n\rfloor}}-\frac{(a+1)^{\lfloor s-1\rfloor}(b+1)^{\lfloor s-1\rfloor}}{c^{\lfloor s-1\rfloor}d^{\lfloor s-1\rfloor}}\right\}.$$

(32.)

$$1-\frac{q.r}{(p-q+1).(p+r-1)}+\frac{q(q-1).r(r-1)}{(p-q+1)(p-q+2).(p+r-1)(p+r-2)}-\ldots$$
$$=\frac{(p-q).(p+r)}{p.(p-q+r)}.$$

(*Educational Times Reprint*, vol. xli., p. 98.)

(33.) Transform the equation

$$\log 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots$$

by § 5, Cor. 2.

(34.) Show, by means of § 2, I., that, if m be a positive integer, then

$$1-{}_mC_1\frac{a}{b}+{}_mC_2\frac{a(a-1)}{b(b-1)}-{}_mC_3\frac{a(a-1)(a-2)}{b(b-1)(b-2)}+\ldots$$
$$=\left(1-\frac{a}{b}\right)\left(1-\frac{a}{b-1}\right)\cdots\left(1-\frac{a}{b-m+1}\right).$$

RECURRING SERIES.

§ 6.] We have already seen that any proper rational fraction such as $(a + bx + cx^2)/(1 + px + qx^2 + rx^3)$* can always be expanded in an ascending series of powers of x. In fact, if mod x be less than the modulus of that root of $rx^3 + qx^2 + px + 1 = 0$ which has the least modulus, we have (see chap. xxvii., §§ 6 and 7)

$$\frac{a + bx + cx^2}{1 + px + qx^2 + rx^3} = u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n + \ldots \quad (1).$$

We propose now to study for a little the properties of the series (1).

If we multiply both sides of the equation (1) by $1 + px + qx^2 + rx^3$, we have

$$a + bx + cx^2 = (1 + px + qx^2 + rx^3)(u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n + \ldots) \quad (2).$$

Hence, equating coefficients of powers of x, we must have

$$\begin{aligned}
u_0 &= a && (3_1);\\
u_1 + pu_0 &= b && (3_2);\\
u_2 + pu_1 + qu_0 &= c && (3_3);\\
u_3 + pu_2 + qu_1 + ru_0 &= 0 && (3_4);\\
&\ldots\ldots\ldots\ldots\\
u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} &= 0 && (3_{n+1}).\\
&\ldots\ldots\ldots\ldots
\end{aligned}$$

Any power-series which has the property indicated by the equation (3_{n+1}) is called a *Recurring Power-Series*;† and the equation (3_{n+1}) is spoken of as its *Scale of Relation*, or, briefly, its *Scale*. The quantities p, q, r, which are independent of n, may be called the *Constants of the Scale*. According as the scale has 1, 2, 3, . . ., r, . . . constants, the recurring series is said to be of the 1st, 2nd, 3rd, . . ., rth, . . . *order*. When $x = 1$, so that we have simply the series $u_0 + u_1 + u_2 + \ldots + u_n + \ldots$, with a relation such as (3_{n+1}) connecting its terms, we speak of

* For simplicity, we confine our exposition to the case where the denominator is of the 3rd degree; but all our statements can at once be generalised.

† The theory of Recurring Series was originated and largely developed by Demoivre.

the series as a *recurring series* simply;* so that every recurring series may be regarded as a particular case of a recurring power-series.

It is obvious from our definition that all the coefficients of a recurring power-series of the rth order can be calculated when the values of the first r are given. Hence *a recurring series of the rth order depends upon $2r$ constants; namely, the r constants of its scale, and r others.*

From this it follows that if the first $2r$ terms of a series be given, it can be continued as a recurring series of the rth order in one way only; as a recurring series of the $(r+1)$th order in a two-fold infinity of ways; and so on.

On the other hand, if the first $2r$ terms of the series be given, two conditions must be satisfied in order that it may be a recurring series of the $(r-1)$th order; four in order that it may be a recurring series of the $(r-2)$th order; and so on.

Example. Show that

$$x+2x^2+3x^3+4x^4+5x^5+6x^6+\ldots$$

is a recurring series of the 2nd order. Let the scale be $u_n+pu_{n-1}+qu_{n-2}=0$. Then we must have

$$3+2p+q=0, \quad 4+3p+2q=0, \quad 5+4p+3q=0, \quad 6+5p+4q=0.$$

The first two of these equations give $p=-2$, $q=+1$; and these values are consistent with the remaining two equations. Hence the theorem.

§ 7.] The rational fraction $(a+bx+cx^2)/(1+px+qx^2+rx^3)$, of which the recurring power-series $u_0+u_1x+u_2x^2+\ldots$ is the development when mod x is less than a certain value, is called the *Generating Function* of the series. We may think of the series and its generating function without regarding the fact that the one is the equivalent of the other under certain restrictions. If we take this view, we must look at the denominator of the function as furnishing the scale, and consider the co-

* We might of course regard a recurring power-series as a particular case of a recurring series in general. Thus, if we put $U_n=u_nx^n$, we might regard the series in (1) as a recurring series whose scale is

$$U_n+pxU_{n-1}+qx^2U_{n-2}+rx^3U_{n-3}=0.$$

efficients as determined by the equations (3_1), (3_2), . . ., (3_{n+1}).* No question then arises regarding the convergence of the series.

Given the scale and the first r terms of a recurring power-series of the rth order, we can always find its generating function.

Taking the case $r = 3$, we see, in fact, from the equations (3_1), (3_2), . . ., (3_{n+1}), . . . of § 6, that

$$\{u_0 + (u_1 + pu_0)x + (u_2 + pu_1 + qu_0)x^2\}/\{1 + px + qx^2 + rx^3\}$$

is the generating function of the series $u_0 + u_1x + u_2x^2 + \ldots$, whose scale is $u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0$.

Cor. 1. *Every recurring power-series may, if mod x be small enough, be regarded as the expansion of a rational fraction.*

Cor. 2. *The general term of any recurring series can always be found when its scale is given and a sufficient number of its initial terms.*

For we can find the generating function of the series itself or of a corresponding power-series; decompose the generating function into partial fractions of the form $A(x-\alpha)^{-s}$; expand each of these in ascending powers of x; and finally collect the coefficient of x^n from the several expansions.

Example. Find the general term of the recurring series whose scale is $u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 0$, and whose first three terms are $1 + 0 - 5$. Consider the corresponding power-series. Here $p = -4$, $q = 5$, $r = -2$; so that

$$a = u_0 = 1, \quad b = u_1 + pu_0 = -4, \quad c = u_2 + pu_1 + qu_0 = 0.$$

The generating function is therefore

$$\frac{1-4x}{1-4x+5x^2-2x^3} \equiv \frac{1-4x}{(1-x)^2(1-2x)} \equiv \frac{2}{1-x} + \frac{3}{(1-x)^2} - \frac{4}{(1-2x)}.$$

Expanding, we have

$$\frac{1-4x}{1-4x+5x^2-2x^3} = 2\{1+\Sigma x^n\} + 3\{1+\Sigma(n+1)x^n\} - 4\{1+\Sigma 2^n x^n\},$$
$$= 1 + \Sigma(3n+5-2^{n+2})x^n.$$

The general term in question is therefore $3n + 5 - 2^{n+2}$.

§ 8.] If u_n be any function of an integral variable n which satisfies an equation of the form

$$u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0,$$

or, what comes to the same thing,

$$u_{n+3} + pu_{n+2} + qu_{n+1} + ru_n = 0 \qquad (1),$$

* We might also regard the series as deduced from the generating function by the process of ascending continued division (see chap. v., § 20).

we see from the reasoning of last paragraph that u_n is uniquely determined by the equation (1), provided its three initial values u_0, u_1, u_2 are given; and we have found a process for actually determining u_n.

It is not difficult to see that we might assign any three values of u_n whatever, say $u_\alpha, u_\beta, u_\gamma$, and the solution would still be determinate. We should, in fact, by the process § 7, determine u_n as a function of n linearly involving three arbitrary constants u_0, u_1, u_2, say $f(u_0, u_1, u_2, n)$; and u_0, u_1, u_2 would be uniquely determined by the three linear equations

$$f(u_0, u_1, u_2, \alpha) = u_\alpha, \quad f(u_0, u_1, u_2, \beta) = u_\beta, \quad f(u_0, u_1, u_2, \gamma) = u_\gamma \quad (2).$$

An equation such as (1) is called *a Linear Difference-Equation of the 3rd order with constant coefficients*; and we see generally that *a linear difference-equation of the rth order with constant coefficients has a unique solution when the values of the function involved are given for r different values of its integral argument.*

Example. Find a function u_n such that $u_{n+3} - 4u_{n+2} + 5u_{n+1} - 2u_n = 0$; and $u_0 = 1$, $u_1 = 0$, $u_2 = -5$.

We have simply to repeat the work of the example in § 7.

§ 9.] *To sum a recurring series to $n+1$ terms, and (when convergent) to infinity.*

Taking the case of a power-series of the 3rd order, let

$$S_n = u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n,$$

then

$$\begin{aligned} px S_n &= pu_0 x + pu_1 x^2 + \ldots + pu_{n-1} x^n + pu_n x^{n+1}, \\ qx^2 S_n &= qu_0 x^2 + \ldots + qu_{n-2} x^n + qu_{n-1} x^{n+1} + qu_n x^{n+2}, \\ rx^3 S_n &= \ldots + ru_{n-3} x^n + ru_{n-2} x^{n+1} + ru_{n-1} x^{n+2} + ru_n x^{n+3} \end{aligned}$$

Hence adding, and remembering that $u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0$ for all values of n which exceed 2, we have

$$\begin{aligned} (1 + px + qx^2 + rx^3) S_n = u_0 + (u_1 + pu_0)x + (u_2 + pu_1 + qu_0)x^2 \\ + (pu_n + qu_{n-1} + ru_{n-2})x^{n+1} + (qu_n + ru_{n-1})x^{n+2} + ru_n x^{n+3} \quad (1); \end{aligned}$$

whence S_n can in general be at once determined by dividing by $1 + px + qx^2 + rx^3$.

The only exceptional case is that where for the particular value of x in question, say $x = \alpha$, it happens that

$$1 + p\alpha + q\alpha^2 + r\alpha^3 = 0.$$

In this case the right hand of (1) must, of course, also vanish, and S_n takes the indeterminate form 0/0. S_n may in cases of this kind be found by evaluating the indeterminate form by means of the principles of chap. xxv. This, however, is often much more troublesome than some more special process applicable to the particular case.

If the series $\Sigma u_n x^n$ be convergent, then $Lu_n x^n = 0$ when $n = \infty$; therefore the last three terms on the right of (1) will become infinitely small when $n = \infty$. We therefore have for the sum to infinity in any case where the series is convergent

$$S_\infty = \frac{u_0 + (u_1 + pu_0)x + (u_2 + pu_1 + qu_0)x^2}{1 + px + qx^2 + rx^3} \qquad (2).$$

The particular cases

$$u_0 + u_1 + u_2 + \ldots + u_n + \ldots \qquad (3),$$

$$u_0 - u_1 + u_2 - \ldots + (-)^n u_n + \ldots \qquad (4),$$

are of course deducible from (1) and (2) by putting $x = +1$ and $x = -1$. Exceptional cases will arise if $1 + p + q + r = 0$, or if $1 - p + q - r = 0$.

It is needless to give an example of the above process, for Examples 1 and 2, chap. xx., § 14, are particular instances, $\Sigma n^2 x^n$ and $1 + \Sigma(-)^{n-1} 2n x^n$ being, in fact, recurring series whose scales are $u_n - 3u_{n-1} + 3u_{n-2} - u_{n-3} = 0$ and $u_n + 2u_{n-1} + u_{n-2} \pm 0$ respectively.

Exercises XXVI.

Sum the following recurring series to $n+1$ terms, and, where admissible, to infinity :—

(1.) $2 + 5 + 13 + 35 + 97 + \ldots$.

(2.) $2 + 10 + 12 - 24 + 2 + 10 + 12 + \ldots$.

3.) $2 + 17x + 95x^2 + 461x^3 + \ldots$.

(4.) $5 + 12x + 30x^2 + 78x^3 + 210x^4 + \ldots$.

(5.) $1 + 4x + 17x^2 + 76x^3 + 353x^4 + \ldots$.

(6.) $1 + 4x + 10x^2 + 22x^3 + 46x^4 + \ldots$.

(7.) If a series has for its rth term the sum of r terms of a recurring series, it will itself be a recurring series with one more term in the scale of relation.

Find the sum of the series whose rth term is the sum of r terms of the recurring series $1 + 6 + 40 + 288 + \ldots$.

(8.) If T_n, T_{n+1}, T_{n+2} be consecutive terms of the recurring series whose scale is $T_{n+2}=aT_{n+1}-bT_n$, then

$$(T_{n+1}^2-aT_nT_{n+1}+bT_n^2)/(T_{n-r+1}^2-aT_{n-r}T_{n-r+1}+bT_{n-r}^2)=b^r.$$

(9.) Form and sum to n terms the *series* each term in which is half the difference of the two preceding terms.

(10.) Show that every integral series (chap. xx., § 4) is a recurring series; and find its scale.

(11.) If $u_n=u_{n-1}+u_{n-2}$, and $u_2=au_1$, show that

$$u_n^2-u_{n+1}u_{n-1}=(-)^n(a^2-a-1)u_1^2.$$

(12.) If the series u_1, u_2, u_3, . . ., u_n, . . . be such that in every four consecutive terms the sum of the extremes exceeds the sum of the means by a constant quantity c, find the law of the series; and show that the sum of $2n$ terms is

$$\tfrac{1}{2}m(m-1)(4m-5)c-m(m-2)u_1+mu_2+m(m-1)u_3.$$

(13.) If $u_{n+2}=u_{n+1}+u_n$, $u_1=1$, $u_2=1$, sum the series

$$\frac{1}{1.2}+\frac{2}{1.3}+\ldots+\frac{u_{n+2}}{u_{n+1}u_{n+3}}.$$

(14.) By French law an illegitimate child receives one-third of the portion of the inheritance that he would have received had he been legitimate. If there be l legitimate and n illegitimate children, show that the portion of inheritance 1 due to a legitimate child is

$$\frac{1}{l}-\frac{n}{3l(l+1)}+\frac{n(n-1)}{3^2l(l+1)(l+2)}-\ldots(-)^n\frac{n(n-1)\ldots2.1}{3^nl(l+1)\ldots(l+n)}.$$

(Catalan, *Nouv. Ann.*, ser. ii., t. 2.)

WARING'S METHOD FOR SUMMING THE SERIES FORMED BY TAKING EVERY kTH TERM FROM ANY POWER-SERIES WHOSE SUM IS KNOWN.

§ 10.] This method depends on the theorem that *the sum of the pth powers of the kth roots of unity is k if p be a multiple of k, but otherwise zero.*

This is easily seen to be true; for, if ω be a primitive kth root of 1, then the k roots are ω^0, ω^1, ω^2, . . ., ω^{k-1}. If $p=\mu k$, then $(\omega^s)^p=\omega^{s\mu k}=(\omega^k)^{\mu s}=1$. If p be not a multiple of k, then we have

$$\begin{aligned}(\omega^0)^p+(\omega^1)^p+\ldots+(\omega^{k-1})^p&=1+(\omega^p)^1+(\omega^p)^2+\ldots+(\omega^p)^{k-1},\\&=\{1-(\omega^p)^k\}/(1-\omega^p),\\&=0,\end{aligned}$$

for $(\omega^p)^k=(\omega^k)^p=1$, and $\omega^p\neq1$.

Let us suppose now that $f(x)$ is the sum of n terms of the power-series $u_0 + \Sigma u_n x^n$, n being finite, or, it may be, if the series is convergent, infinite.

Consider the expression

$$U_m \equiv \frac{(\omega^0)^{k-m} f(\omega^0 x) + (\omega^1)^{k-m} f(\omega^1 x) + (\omega^2)^{k-m} f(\omega^2 x) + \ldots + (\omega^{k-1})^{k-m} f(\omega^{k-1} x)}{k} \quad (1),$$

where m is 0 or any positive integer $< k$.

The coefficient of x^r in the equivalent series is

$$u_r\{(\omega^0)^{k-m+r} + (\omega^1)^{k-m+r} + (\omega^2)^{k-m+r} + \ldots + (\omega^{k-1})^{k-m+r}\}/k \quad (2).$$

Now, by the above theorem regarding the kth roots of unity, the quantity within the crooked brackets vanishes if $k - m + r$ be not a multiple of k, and has the value k if $k - m + r$ be a multiple of k. Therefore we have

$$U_m = u_m x^m + u_{m+k} x^{m+k} + u_{m+2k} x^{m+2k} + \ldots \quad (3),$$

where the series extends until the last power of x is just not higher than the nth, and, in particular, to infinity if $f(x)$ be a sum to infinity.*

If we put $m = 0$, we get

$$\{f(x) + f(\omega^1 x) + f(\omega^2 x) + \ldots + f(\omega^{k-1} x)\}/k = u_0 + u_k x^k + u_{2k} x^{2k} + u_{3k} x^{3k} + \ldots \quad (4).$$

Example 1.

$$1 + x + x^2 + \ldots + x^n = (1 - x^{n+1})/(1 - x).$$

Hence, if ω be a primitive cube root of 1, we have

$$1 + x^3 + x^6 + \ldots + x^{3s} = \tfrac{1}{3}\left\{\frac{1 - x^{n+1}}{1 - x} + \frac{1 - \omega^{n+1} x^{n+1}}{1 - \omega x} + \frac{1 - \omega^{2n+2} x^{n+1}}{1 - \omega^2 x}\right\},$$

where $3s$ is the greatest multiple of 3 which does not exceed n.

Example 2. To sum the series

$$\frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \ldots \text{ ad } \infty.$$

We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ ad } \infty.$$

* This method seems to be due to Waring. See *Phil. Trans.* R.S.L. (1784).

Hence, if ω be a primitive 4th root of unity, say $\omega = i$, then, since here $k=4$, $m=3$, $k-m=1$, $\omega^2=-1$, $\omega^3=-i$, we get

$$\tfrac{1}{4}(e^x + ie^{ix} - e^{-x} - ie^{-ix}) = \frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \ldots;$$

that is,

$$\tfrac{1}{2}(\sinh x - \sin x) = \frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \ldots .$$

MISCELLANEOUS METHODS.

§ 11.] When the nth term of a series is a rational fraction, the finite summation may often be effected by merely breaking up this term into its constituent partial fractions; and even when summation cannot be effected, many useful transformations can be thus obtained. In dealing with infinite series by this method, close attention must be paid to the principles laid down in chap. xxvi., especially § 13; otherwise the tyro may easily fall into mistakes. As an instance of this method of working, see chap. xxviii., § 14, Examples 1 and 2.

Example. Show that

$$\left\{\frac{1}{(x+1)^2(x+2)} + \frac{1}{(x+2)^2(x+3)} + \frac{1}{(x+3)^2(x+4)} + \ldots\right\}$$
$$+ \left\{\frac{1}{(x+1)(x+2)^2} + \frac{1}{(x+2)(x+3)^2} + \frac{1}{(x+3)(x+4)^2} + \ldots\right\} = \frac{1}{(x+1)^2}.$$

Denote the sums of n terms of the two given series by S_n and T_n respectively, and their nth terms by u_n and v_n respectively. Then

$$u_n = -1/(x+n) + 1/(x+n)^2 + 1/(x+n+1);$$
$$v_n = 1/(x+n) - 1/(x+n+1)^2 - 1/(x+n+1).$$

Whence we get at once

$$S_n + T_n = 1/(x+1)^2 - 1/(x+n+1)^2.$$

Therefore

$$S_\infty + T_\infty = 1/(x+1)^2.$$

§ 12.] *Euler's Identity.* The following obvious identity *

$$1 - a_1 + a_1(1-a_2) + a_1a_2(1-a_3) + \ldots + a_1a_2\ldots a_n(1-a_{n+1}) = 1 - a_1a_2\ldots a_{n+1} \quad (1)$$

is often useful in the summation of series. It contains, in fact,

* Used in the slightly different form,

$$(1+a_1)(1+a_2)(1+a_3)(1+a_4)\ldots$$
$$= 1 + a_1 + a_2(1+a_1) + a_3(1+a_1)(1+a_2) + a_4(1+a_1)(1+a_2)(1+a_3) + \ldots$$

by Euler, *Nov. Comm. Petrop.* (1760).

as particular cases a good many of the results already obtained above.

If in (1) we put

$$a_1 = \frac{x}{y}, \quad a_2 = \frac{x+p_1}{y+p_1}, \quad a_3 = \frac{x+p_2}{y+p_2}, \quad \ldots, \quad a_{n+1} = \frac{x+p_n}{y+p_n},$$

and multiply on both sides by $y/(y-x)$, we get

$$1 + \frac{x}{y+p_1} + \frac{x(x+p_1)}{(y+p_1)(y+p_2)} + \ldots + \frac{x(x+p_1)\ldots(x+p_{n-1})}{(y+p_1)(y+p_2)\ldots(y+p_n)}$$
$$= \frac{y}{y-x} - \frac{x}{y-x} \cdot \frac{(x+p_1)(x+p_2)\ldots(x+p_n)}{(y+p_1)(y+p_2)\ldots(y+p_n)} \quad (2).$$

If the quantities involved be such that

$$\lim_{n=\infty} \frac{(x+p_1)(x+p_2)\ldots(x+p_n)}{(y+p_1)(y+p_2)\ldots(y+p_n)} = 0 \quad (3),$$

then

$$1 + \frac{x}{y+p_1} + \frac{x(x+p_1)}{(y+p_1)(y+p_2)} + \ldots \text{ ad } \infty = \frac{y}{y-x} \quad (4).$$

If in (2) we put $y = 0$, we get

$$1 + \frac{x}{p_1} + \frac{x(x+p_1)}{p_1 p_2} + \ldots + \frac{x(x+p_1)\ldots(x+p_{n-1})}{p_1 p_2 \cdots p_n}$$
$$= \left(1 + \frac{x}{p_1}\right)\left(1 + \frac{x}{p_2}\right)\ldots\left(1 + \frac{x}{p_n}\right) \quad (5).$$

From (5) a variety of particular cases may be derived by putting $n = \infty$, and giving special values to $p_1, p_2, \ldots$. Thus, for instance, if the infinite series $\Sigma 1/p_n$ diverge to $+\infty$, then (see chap. xxvi., § 24) we have

$$1 - \frac{x}{p_1} + \frac{x(x-p_1)}{p_1 p_2} - \ldots \text{ ad } \infty = 0 \quad (6).$$

In general, *if the continued product* $\prod_1^\infty (1 + x/p_n)$ *converge to any definite limit, then the series* $1 + \sum_1^\infty x(x+p_1)\ldots(x+p_{n-1})/p_1 p_2 \ldots p_n$ *converges to the same limit.*

Example. Find when the infinite series

$$S \equiv 1 + \frac{x}{y+p} + \frac{x(x+p)}{(y+p)(y+2p)} + \frac{x(x+p)(x+2p)}{(y+p)(y+2p)(y+3p)} + \ldots \quad (7)$$

converges, and the limit to which it converges.

If in (2) above we put $p_1=p$, $p_2=2p$, &c., . . ., we have

$$S=\frac{y}{y-x}-\frac{x}{y-x}\,\mathrm{L}_{n=\infty}\frac{(x+p)(x+2p)\ldots(x+np)}{(y+p)(y+2p)\ldots(y+np)} \qquad (8).$$

Now the limit in question may be written

$$\prod_1^\infty\left\{1+\frac{(x-y)/np}{1+y/np}\right\};$$

but this diverges to ∞ if $(x-y)/p$ be positive, and converges to 0 if $(x-y)/p$ be negative (chap. xxvi., § 24).

Hence, if p denote in all cases a positive quantity, we see that

$$1+\frac{x}{y+p}+\frac{x(x+p)}{(y+p)(y+2p)}+\ldots \text{ ad } \infty=\frac{y}{y-x},$$

if $y>x$; and

$$1+\frac{x}{y-p}+\frac{x(x-p)}{(y-p)(y-2p)}+\ldots \text{ ad } \infty=\frac{y}{y-x},$$

if $y<x$.

Exercises XXVII.

(1.) Given $1/(1-x)^2=1+2x+3x^2+4x^3+\ldots$,

sum $1+4x^3+7x^6+10x^9+\ldots$.

(2.) Sum the series

$$1+x^3/4+x^6/7+\ldots;$$
$$1+x^3/3!+x^6/6!+\ldots.$$

(3.) If $f(x)=u_0+u_1x+u_2x^2+\ldots$, and $\alpha, \beta, \gamma, \ldots$ be the nth roots of -1, show that

$$\frac{1}{n}\{\alpha^{3n-m}f(\alpha x)+\beta^{2n-m}f(\beta x)+\ldots\}=u_mx^m-u_{m+n}x^{m+n}+u_{m+2n}x^{m+2n}-\ldots$$

where $m<n$. (De Morgan, *Diff. Calc.*, p. 319 (1839).)

Sum the following series, and point out the condition for convergency when the summation extends to infinity:—

(4.) $1-x^3/4+x^6/7-\ldots$ ad ∞;
$x-x^4/4!+x^7/7!-\ldots$ ad ∞.

(5.) $1+{}_mC_3+{}_mC_6+{}_mC_9+\ldots$ ad ∞;
$1-{}_mC_3+{}_mC_6-{}_mC_9+\ldots$ ad ∞.

(6.) $1/1.3+1/1.2.4+1/1.2.3.5+\ldots$ to n terms.

(7.) $1/1.2.3+{}_mC_1/2.3.4+{}_mC_2/3.4.5+\ldots$ ad ∞.

(8.) $1-2x/1+3x^2/2-4x^3/3+\ldots$ ad ∞.

(9.) $\cos\theta/1.2.3+\cos 2\theta/2.3.4+\cos 3\theta/3.4.5+\ldots$ ad ∞.

(10.) $1/1^2.2^2+7/2^2.3^2+\ldots+(2n^2+4n+1)/(n+1)^2(n+2)^2$.

(11.) $1/1^2.2^2-1/2^2.3^2+\ldots(-)^{n-1}1/n^2(n+1)^2+\ldots$ ad ∞.

(12.) If n be a positive integer, show that

$$\frac{n}{m+n}+\frac{1}{2}\frac{n(n-1)}{(m+n)(m+n-1)}+\frac{1}{3}\frac{n(n-1)(n-2)}{(m+n)(m+n-1)(m+n-2)}+\ldots$$
$$=\frac{n}{m+1}-\frac{1}{2}\frac{n(n-1)}{(m+1)(m+2)}+\frac{1}{3}\frac{n(n-1)(n-2)}{(m+1)(m+2)(m+3)}-\ldots.$$

(13.) Show that

$$\frac{{}_nC_1}{1-x/1}-\frac{{}_nC_2}{(1-x/1)(1-x/2)}+\frac{{}_nC_3}{(1-x/1)(1-x/2)(1-x/3)}-\ldots=\frac{n}{n-x};$$

and hence show that

$${}_nC_1\sigma_1-{}_nC_2\sigma_2+\ldots(-)^n{}_nC_n\sigma_n=1/n,$$

where $\sigma_r=1/1+1/2+\ldots+1/r$.

(14.) Sum the series

$$1-\frac{m^2}{1^2}+\frac{m^2(m^2-1^2)}{1^2.2^2}-\frac{m^2(m^2-1^2)(m^2-2^2)}{1^2.2^2.3^2}+\ldots \text{ ad } \infty;$$

$$1+\frac{m^2}{1^2}+\frac{m^2(m^2+1^2)}{1^2.3^2}+\frac{m^2(m^2+1^2)(m^2+3^2)}{1^2.3^2.5^2}+\ldots \text{ ad } \infty.$$

(15.) Show that

$$\frac{a_1}{a_1+p_1}+\frac{p_1a_2}{(a_1+p_1)(a_2+p_2)}+\frac{p_1p_2a_3}{(a_1+p_1)(a_2+p_2)(a_3+p_3)}+\ldots$$

$$+\frac{p_1p_2\ldots p_{n-1}a_n}{(a_1+p_1)(a_2+p_2)\ldots(a_n+p_n)}=1-\frac{p_1p_2\ldots p_n}{(a_1+p_1)(a_2+p_2)\ldots(a_n+p_n)}.$$

(16.) Show that

$$\tan^2\frac{1}{2}\pi x=\frac{1^4-(1^2-x^2)^2}{(1^2-x^2)^2}+\frac{3^4-(3^2-x^2)^2}{(1^2-x^2)^2(3^2-x^2)^2}+\ldots.$$

(Glaisher, *Math. Mess.*, 1873, p. 138.)

(17.) Show that

$$\frac{1}{n^2}=\frac{1}{n(n+1)}+\frac{1}{n(n+1)(n+2)}+\frac{1.2}{n(n+1)(n+2)(n+3)}+\ldots;$$

and apply this result to the approximate calculation of π^2 by means of the formula

$$\pi^2/6=1/1^2+1/2^2+1/3^2+\ldots.$$

(Stirling, *Methodus Differentialis*, p. 28.)

(18.) Show that $\Sigma 1/(m^n-1)=1$ and $\Sigma 1/(a^n-1)=\log 2$, where m and n have all possible positive integral values differing from unity, a is any even positive integer, and each distinct fraction is counted only once.

(Goldbach's Theorem, see *Liouv. Math. Jour.*, 1842.)

(19.) If n have any positive integral value except unity, and r be any positive integer which is not a perfect power, show that $\Sigma(n-1)/(r^n-1)=\pi^2/6$; and, if $d(n)$ denote the number of divisors of n, that $\Sigma(d(n)-1)/r^n=1$; also that $\Sigma(n-1)/r=\Sigma 1/(r-1)^2$. (*Ib.*)

CHAPTER XXXII.

Simple Continued Fractions.

NATURE AND ORIGIN OF CONTINUED FRACTIONS.

§ 1.] By a *continued fraction* is meant a function of the form

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4} \cdots}} \qquad (1);$$

the primary interpretation of which is that b_2 is the antecedent of a quotient whose consequent is all that lies under the line immediately beneath b_2, and so on.

There may be either a finite or an infinite number of links in the chain of operations; that is to say, we may have either a *terminating* or *non-terminating* continued fraction.

In the most general case the *component fractions* $\frac{b_2}{a_2}$, $\frac{b_3}{a_3}$, $\frac{b_4}{a_4}$, . . ., as they are sometimes called, may have either positive or negative numerators and denominators, and succeed each other without recurrence according to any law whatever. If they do recur, we have what is called a *recurring* or *periodic* continued fraction.

For shortness, the following abbreviative notation is often used instead of (1),

$$a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \frac{b_4}{a_4 +} \cdots \qquad (2),$$

the signs + being written below the lines to prevent confusion with

$$a_1 + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \frac{b_4}{a_4} + \ldots \quad .*$$

Examples have already been given (see chap. iii., Exercises III., 15) of the reduction of terminating continued fractions; and from these examples it is obvious that *every terminating continued fraction whose constituents* $a_1, a_2, \ldots, b_2, b_3, \ldots$ *are commensurable numbers reduces to a commensurable number.*

§ 2.] In the present chapter we shall confine ourselves mainly to the most interesting and the most important kind of continued fraction, that, namely, in which each of the numerators of the component fractions is + 1, and each of the denominators a positive integer. When distinction is necessary, this kind of continued fraction, namely,

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \ldots \qquad (1),$$

may be called a *simple continued fraction.* Unless it is otherwise stated, we suppose the continued fraction to terminate.

In this case, for a reason that will be understood by and by, the numbers $a_1, a_2, a_3, \ldots$ are called the first, second, third, . . . *partial quotients* of the continued fraction.

§ 3.] *Every number, commensurable or incommensurable, may be expressed uniquely as a simple continued fraction, which may or may not terminate.*

For, let X be the number in question, and a_1 the greatest integer which does not exceed X; then we may write

$$X = a_1 + \frac{1}{X_1} \qquad (1),$$

where $X_1 > 1$, but is not necessarily integral, or even commensurable.

Again, let a_2 be the greatest integer in X_1, so that $a_2 \not< 1$; then we have

$$X_1 = a_2 + \frac{1}{X_2} \qquad (2),$$

where $X_2 > 1$, as before.

---

* The notation $a_1 + \frac{b_2}{a_2} \vdots \frac{b_3}{a_3} \vdots \frac{b_4}{a_4} \vdots \ldots$ is frequently used by Continental writers.

Again, let a_2 be the greatest integer in X_2; then

$$X_2 = a_2 + \frac{1}{X_3} \qquad (3);$$

and so on.

This process will terminate if one of the quantities X, say X_{n-1}, is an integer; for we should then have

$$X_{n-1} = a_n \qquad (n).$$

Now, using (2), we get from (1)

$$X = a_1 + \cfrac{1}{a_2 + \cfrac{1}{X_2}}.$$

Thence, using (3), we get

$$X = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{X_3}}};$$

and so on.

Finally, then,

$$X = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \frac{1}{a_n} \qquad (\alpha).$$

It may happen that none of the quantities X comes out integral. In this case, the quotients a_1, a_2, . . . either recur, or go on continually without recurrence; and we then obtain in place of (α) a non-terminating continued fraction, which may be periodic or not according to circumstances.

To prove that the development is unique, we have to show that, if

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots = a_1' + \frac{1}{a_2' +} \frac{1}{a_3' +} \cdots \qquad (\beta),$$

then $a_1 = a_1'$, $a_2 = a_2'$, $a_3 = a_3'$, &c.

Now, since a_2 and a_2' are positive integers, and $\frac{1}{a_3 +} \cdots$ and $\frac{1}{a_3' +} \cdots$ are both positive, it follows that $\frac{1}{a_2 +} \frac{1}{a_3 +} \cdots$ and $\frac{1}{a_2' +} \frac{1}{a_3' +} \cdots$ are both proper fractions. Hence, by chap. iii., § 12,

we must have

$$a_1 = a_1' \qquad (\gamma),$$

and

$$\frac{1}{a_2+}\,\frac{1}{a_3+}\,\cdots = \frac{1}{a_2'+}\,\frac{1}{a_3'+}\,\cdots \qquad (\delta).$$

Again, from (δ), we have

$$a_2 + \frac{1}{a_3+}\,\frac{1}{a_4+}\,\cdots = a_2' + \frac{1}{a_3'+}\,\frac{1}{a_4'+}\,\cdots \qquad (\epsilon).$$

From (ϵ), by the same reasoning as before, we have

$$a_2 = a_2' \qquad (\zeta),$$

and

$$\frac{1}{a_3+}\,\frac{1}{a_4+}\,\frac{1}{a_5+}\,\cdots = \frac{1}{a_3'+}\,\frac{1}{a_4'+}\,\frac{1}{a_5'+}\,\cdots \qquad (\eta).$$

Proceeding in this way, we can show that each partial quotient in the one continued fraction is equal to the partial quotient of the same order in the other.*

This demonstration is clearly applicable even when the continued fraction does not terminate, provided we are sure that the fractions in (β), (δ), (η), &c. have always a definite meaning. This point will be settled when we come to discuss the question of the convergency of an infinite continued fraction.

Cor. *If* $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ *be all positive integers,* x_{n+1} *and* y_{n+1} *any positive quantities rational or irrational each of which is greater than unity, and if*

$$a_1 + \frac{1}{a_2+}\,\cdots\,\frac{1}{a_n+}\,\frac{1}{x_{n+1}} = b_1 + \frac{1}{b_2+}\,\cdots\,\frac{1}{b_n+}\,\frac{1}{y_{n+1}},$$

then must

$$a_1 = b_1,\ a_2 = b_2,\ \ldots,\ a_n = b_n,\ \textit{and also}\ x_{n+1} = y_{n+1}.$$

§ 4.] As an example of the general proposition of § 3, we may show that *every commensurable number may be converted into a terminating continued fraction.*

Let the number in question be A/B, where A and B are integers prime to each other. Let a_1 be the quotient and C the remainder when A is divided by B; a_2 the quotient and D the

* We suppose, as is clearly allowable, that, if the fraction terminates, the last quotient is > 1. It should also be noticed that the first partial quotient may be zero, but that none of the others can be zero, as the process is arranged above.

remainder when B is divided by C; a_2 the quotient and E the remainder when C is divided by D; and so on, just as in the arithmetical process for finding the G.C.M. of A and B. Since A and B are prime to each other, the last divisor will be 1, the last quotient a_n, say, and the last remainder 0. We then have

$$\frac{A}{B} = a_1 + \frac{C}{B} = a_1 + \frac{1}{B/C};$$

$$\frac{B}{C} = a_2 + \frac{D}{C} = a_2 + \frac{1}{C/D};$$

$$\frac{C}{D} = a_3 + \frac{E}{D} = a_3 + \frac{1}{D/E};$$

&c.

Hence

$$\frac{A}{B} = a_1 + \frac{1}{a_2 +}\ \frac{1}{a_3 +}\ \cdots\ \frac{1}{a_n}.$$

It should be noticed that, if A<B, the first quotient a_1 will be zero.

Example 1.

To convert 167/81 into a continued fraction.

Going through the process of finding the G.C.M. of 167 and 81, we have

```
81)167(2
   162
    5)81(16
      80
       1)5(5
         5
         0
```

Hence

$$\frac{167}{81} = 2 + \frac{1}{16 +}\ \frac{1}{5}.$$

Example 2.

Consider ·23 = 23/100.

We have

```
100)23(0
    0
   23)100(4
      92
       8)23(2
         16
          7)8(1
            7
            1)7(7
              7
              0
```

Hence

$$\cdot 23 = 0 + \frac{1}{4 +}\ \frac{1}{2 +}\ \frac{1}{1 +}\ \frac{1}{7}.$$

Cor. *If we remove the restriction that the last partial quotient shall be greater than unity, we may develop any commensurable number as a continued fraction which has, at our pleasure, an even or an odd number of partial quotients.*

For example, $2+\cfrac{1}{16+\cfrac{1}{5}}$ has an odd number of partial quotients; but we may write it $2+\cfrac{1}{16+\cfrac{1}{4+\cfrac{1}{1}}}$, which has an even number.

§ 5.] *Any single surd, and, in fact, any simple surd number, such as* $A+Bp^{1/n}+Cp^{2/n}+\ldots+Kp^{(n-1)/n}$, *can be converted into a continued fraction, although not, of course, into a terminating continued fraction.*

The process consists in finding the greatest integer in a series of surd numbers, and in rationalising the denominator of the reciprocal of the residue. Methods for effecting both these steps are known (see chap. x.), but both, in any but the simplest cases, are very laborious. It will be sufficient to give two simple examples, in each of which the result happens to be a periodic continued fraction.

Example 1.

To convert $\sqrt{13}$ into a continued fraction.

We have, 3 being the greatest integer $<\sqrt{13}$,

$$\sqrt{13}=3+(\sqrt{13}-3)=3+\frac{1}{1/(\sqrt{13}-3)},$$

$$=3+\frac{1}{(\sqrt{13}+3)/4} \qquad (1).$$

Again, since the greatest integer in $(\sqrt{13}+3)/4$ is 1, we have

$$\frac{\sqrt{13}+3}{4}=1+\frac{\sqrt{13}-1}{4}=1+\frac{1}{4/(\sqrt{13}-1)},$$

$$=1+\frac{1}{(\sqrt{13}+1)/3} \qquad (2).$$

Similarly, we have

$$\frac{\sqrt{13}+1}{3}=1+\frac{\sqrt{13}-2}{3}=1+\frac{1}{3/(\sqrt{13}-2)},$$

$$=1+\frac{1}{(\sqrt{13}+2)/3} \qquad (3);$$

$$\frac{\sqrt{13}+2}{3}=1+\frac{\sqrt{13}-1}{3}=1+\frac{1}{3/(\sqrt{13}-1)},$$

$$=1+\frac{1}{(\sqrt{13}+1)/4} \quad (4);$$

$$\frac{\sqrt{13}+1}{4}=1+\frac{\sqrt{13}-3}{4}=1+\frac{1}{4/(\sqrt{13}-3)},$$

$$=1+\frac{1}{\sqrt{13}+3} \quad (5);$$

$$\sqrt{13}+3=6+\sqrt{13}-3=6+\frac{1}{1/(\sqrt{13}-3)},$$

$$=6+\frac{1}{(\sqrt{13}+3)/4} \quad (6);$$

after which the process repeats itself.

From the equations (1) . . . (6) we derive

$$\sqrt{13}=3+\frac{1}{\overset{*}{1}+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{\overset{*}{6}+}\frac{1}{1+}\cdots,$$

where the * * indicate the beginning and end of the cycle of partial quotients.

Example 2.

To convert $\frac{\sqrt{3}-1}{2}$ into a continued fraction.

We have

$$\frac{\sqrt{3}-1}{2}=0+\frac{1}{2/(\sqrt{3}-1)},$$

$$=0+\frac{1}{\sqrt{3}+1};$$

$$\sqrt{3}+1=2+\sqrt{3}-1=2+\frac{1}{1/(\sqrt{3}-1)},$$

$$=2+\frac{1}{(\sqrt{3}+1)/2};$$

$$\frac{\sqrt{3}+1}{2}=1+\frac{\sqrt{3}-1}{2}=1+\frac{1}{2/(\sqrt{3}-1)},$$

$$=1+\frac{1}{\sqrt{3}+1};$$

after which the quotients recur. We have, therefore,

$$\frac{\sqrt{3}-1}{2}=0+\frac{1}{\overset{*}{2}+}\frac{1}{\overset{*}{1}+}\cdots.$$

It will be proved in chap. xxxiii. that every positive number of the form $(\sqrt{P}+Q)/R$, where P is a positive integer which is not a perfect square, and Q and R are positive or negative integers, can be converted into a periodic continued fraction; and that every periodic continued fraction represents an irrational number of this form.

EXERCISES XXVIII.

Express the following as simple continued fractions, terminating or periodic as the case may be :—

(1.) $\frac{15}{73}$. (2.) $\frac{532}{1193}$. (3.) $\frac{39293}{36932}$. (4.) $31\frac{76}{123}$.

(5.) 2·718281. (6.) ·0079. (7.) $\sqrt{2}$. (8.) $\sqrt{5}$. (9.) $\sqrt{(11)}$.

(10.) $\sqrt{(10)}$. (11.) $\sqrt{(12)}$. (12.) $\sqrt{\frac{5}{3}}$. (13.) $\sqrt{3}+1$.

(14.) $\frac{1+3\sqrt{5}}{2}$.

(15.) Show that $1+\frac{1}{\sqrt{6}}=1+\underset{*}{\frac{1}{2+}}\,\underset{*}{\frac{1}{2+}}\,\underset{*}{\frac{1}{4+}}\ \ldots\ .$

(16.) A line AB is divided in C, so that $AB.AC=BC^2$. Express the ratios AC/AB, BC/AB as simple continued fractions.

(17.) Express $\sqrt{(a^2+a)}$ and $\sqrt{(a^2-a)}$ as simple continued fractions, a being a positive integer.

(18.) If a be a positive integer, show that

$$2\sqrt{(1+a^2)}=2a+\underset{*}{\frac{1}{a+}}\,\underset{*}{\frac{1}{4a+}}\ \ldots\ .$$

(19.) If a be a positive integer >1, show that

$$\tfrac{1}{2}\sqrt{(9a^2+12)}=\frac{3a}{2}+\underset{*}{\frac{1}{a+}}\,\underset{*}{\frac{1}{3a+}}\ \ldots\ .$$

(20.) Show that

$$\sqrt[3]{(11)}=2+\underset{*}{\frac{1}{4+}}\,\underset{*}{\frac{1}{2+}}\,\underset{*}{\frac{1}{6+}}\ \ldots\ .$$

(21.) Show that every rational algebraical function of x can be expanded, and that in one way only, as a terminating continued fraction of the form

$$Q_1+\underset{*}{\frac{1}{Q_2+}}\,\underset{*}{\frac{1}{Q_3+}}\ \ldots\ \underset{*}{\frac{1}{Q_n}},$$

where $Q_1, Q_2, \ldots, Q_n$ are rational integral functions of x.

Exemplify with $(x^3+x^2+x+1)/(x^4+3x^3+2x^2+x+1)$.

(22.) If

$$x=\underset{*}{\frac{a}{1+}}\,\underset{*}{\frac{b}{1+}}\ \ldots,$$

and

$$y=\underset{*}{\frac{b}{1+}}\,\underset{*}{\frac{a}{1+}}\ \ldots,$$

show that

$$x-y=a-b.$$

PROPERTIES OF THE CONVERGENTS TO A CONTINUED FRACTION.

§ 6.] Let us denote the complete continued fraction by x_1, so that

$$x_1 = a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots\frac{1}{a_s} \qquad (1);$$

and let

$$x_2 = a_2 + \frac{1}{a_3+}\,\frac{1}{a_4+}\cdots\frac{1}{a_s} \qquad (2);$$

$$x_3 = a_3 + \frac{1}{a_4+}\cdots\frac{1}{a_s} \qquad (3);$$

and so on.

Then x_2, x_3, . . . are called the *complete quotients corresponding to* a_2, a_3, . . ., or, simply, the *second, third, . . . complete quotients*. The fraction itself, or x_1, may be called the first complete quotient. It will be observed that a_1, a_2, a_3, . . . are the integral parts of x_1, x_2, x_3, ...

Let us consider, on the other hand, the fractions which we obtain by first retaining only the first partial quotient, second by retaining only the first and second, and so on; and let us denote the fractions thus obtained, when reduced (without simplification, as under) so that their numerators and denominators are integral numbers, by p_1/q_1, p_2/q_2, . . . Then we have

$$a_1 = \frac{a_1}{1} = \frac{p_1}{q_1} \qquad (\alpha),$$

$$a_1 + \frac{1}{a_2} = \frac{a_1a_2+1}{a_2} = \frac{p_2}{q_2} \qquad (\beta),$$

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3} = \frac{a_1a_2a_3+a_1+a_3}{a_2a_3+1} = \frac{p_3}{q_3} \qquad (\gamma),$$

. . . . . .

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n} = \text{\&c.} = \frac{p_n}{q_n} \qquad (\delta),$$

and so on,

where

$$p_1 = a_1, \qquad q_1 = 1 \qquad (\alpha'),$$

$$p_2 = a_1a_2+1, \qquad q_2 = a_2 \qquad (\beta'),$$

$$p_3 = a_1a_2a_3+a_1+a_3, \quad q_3 = a_2a_3+1 \qquad (\gamma'),$$

and so on.

The fractions p_1/q_1, p_2/q_2, . . . are called the *first, second,* . . . *convergents* to the continued fraction.

Cor. *If the continued fraction terminates, the last convergent is, by its definition, the continued fraction itself.*

§ 7.] It will be seen, from the expressions for p_1, p_2, p_3 and q_1, q_2, q_3 in § 6 (α'), (β'), (γ'), that we have

$$p_3 = a_3 p_2 + p_1 \qquad (1);$$
$$q_3 = a_3 q_2 + q_1 \qquad (2).$$

This suggests the following *general formulæ for calculating the numerator and denominator of any convergent when the numerators and denominators of the two preceding convergents are known, namely,*

$$p_n = a_n p_{n-1} + p_{n-2} \qquad (3);$$
$$q_n = a_n q_{n-1} + q_{n-2} \qquad (4).$$

Let us suppose that this formula is true for the nth convergent. We observe, from the definitions (α), (β), . . ., (δ) of § 6, that the $\overline{n+1}$th convergent, p_{n+1}/q_{n+1}, is derived from the nth if we replace a_n by $a_n + 1/a_{n+1}$. Hence, since p_{n-1}, q_{n-1}, p_{n-2}, q_{n-2} do not contain a_n, and since, by hypothesis,

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}},$$

it follows that

$$\frac{p_{n+1}}{q_{n+1}} = \frac{(a_n + 1/a_{n+1}) p_{n-1} + p_{n-2}}{(a_n + 1/a_{n+1}) q_{n-1} + q_{n-2}},$$

or, after reduction,

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}},$$

$$= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}} \qquad \text{by (3) and (4).}$$

Hence it is sufficient if we take

$$p_{n+1} = a_{n+1} p_n + p_{n-1};$$
$$q_{n+1} = a_{n+1} q_n + q_{n-1}.$$

In other words, if the rule hold for the nth convergent, it holds for the $\overline{n+1}$th. Now, by (1) and (2), it holds for the third; hence, by what has just been proved, it holds for the fourth; hence for the fifth; and so on. That is to say, the rule is general.

Cor. 1. Since a_n is a positive integral number, it follows from (3) and (4) that *the numerators of the successive convergents form an increasing series of integral numbers, and that the same is true of the denominators.*

Cor. 2. *From* (3) *and* (4) *it follows that*

$$\frac{p_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1}+} \frac{1}{a_{n-2}+} \cdots \frac{1}{a_1} \qquad (5);$$

and

$$\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1}+} \frac{1}{a_{n-2}+} \cdots \frac{1}{a_2} \qquad (6).$$

For, dividing (3) by p_{n-1}, and writing successively $n-1$, $n-2$, . . ., 3 in place of n, we have

$$p_n/p_{n-1} = a_n + \frac{1}{p_{n-1}/p_{n-2}};$$

$$p_{n-1}/p_{n-2} = a_{n-1} + \frac{1}{p_{n-2}/p_{n-3}};$$

. . . . . .

$$p_3/p_2 = a_3 + p_1/p_2;$$

$$= a_3 + \frac{1}{a_2+} \frac{1}{a_1}.$$

From these equations, by successive substitution, we derive (5): and (6) may be proved in like manner.

Example 1.

The continued fraction which represents the ratio of the circumference of a circle to the diameter is $3 + \frac{1}{7+} \frac{1}{15+} \frac{1}{1+} \frac{1}{292+} \frac{1}{1+} \frac{1}{1+} \cdots$. It is required to calculate the successive convergents.

The first two convergents are 3 and $3 + \frac{1}{7}$, that is, $\frac{3}{1}$, $\frac{22}{7}$.

Hence, using the formulæ (3) and (4), we have the following table:—

n	a	p	q
1	3	3	1
2	7	22	7
3	15	333	106
4	1	355	113
5	292	103993	33102
6	1	104348	33215
7	1	208341	66317

where $p_4 = 355$, for example, is obtained by multiplying the number over it, namely 333, by 1, and adding to the product the number one place higher still, namely 22.

The successive convergents are therefore

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \ldots .$$

Example 2.

If p_1/q_1, p_2/q_2, . . . be the convergents to $1+\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\cdots\frac{1}{n+}\cdots$ ad ∞, show that

$$p_n=(n-1)p_{n-1}+(n-1)p_{n-2}+(n-2)p_{n-3}+\ldots+3p_2+2p_1+2.$$

By the recurrence-formula we have

$$\begin{aligned} p_n &= np_{n-1}+p_{n-2}; \\ p_{n-1} &= (n-1)p_{n-2}+p_{n-3}; \\ p_{n-2} &= (n-2)p_{n-3}+p_{n-4}; \\ &\cdots\cdots\cdots \\ p_3 &= 3p_2+p_1; \end{aligned}$$

and (since $p_1=1$, $p_2=3$)

$$p_2=2p_1+1.$$

Adding all these equations, and observing that $p_{n-2}, p_{n-3}, \ldots, p_2$ each occur three times, once on the left multiplied by 1, once on the right multiplied by 1, and again on the right multiplied by $n-1$, $n-2$, . . ., 3 respectively, we have

$$p_n=(n-1)p_{n-1}+(n-1)p_{n-2}+(n-2)p_{n-3}+\ldots+3p_2+2p_1+(p_1+1),$$

which gives the required result since $p_1=1$.

Example 3.

In the case of the continued fraction $a_1+\frac{1}{a_2+}\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_1+}\cdots$ prove that $p_{2n}=q_{2n+1}$, $p_{2n-1}=a_1q_{2n}/a_2$.

By the definition of a convergent, we have

$$\frac{p_{2n+1}}{q_{2n+1}}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_1} \qquad (\alpha),$$

since every odd partial quotient is a_1.

Again, by Cor. 2 above,

$$\frac{p_{2n+1}}{p_{2n}}=a_1+\frac{1}{a_2+}\cdots\frac{1}{a_1} \qquad (\beta).$$

Hence

$$\frac{p_{2n+1}}{q_{2n+1}}=\frac{p_{2n+1}}{p_{2n}},$$

which gives

$$p_{2n}=q_{2n+1} \qquad (\gamma).$$

Also, since

$$\begin{aligned} p_{2n} &= a_2p_{2n-1}+p_{2n-2}, \\ q_{2n+1} &= a_1q_{2n}+q_{2n-1}, \end{aligned}$$

(γ) leads to

$$a_2p_{2n-1}+p_{2n-2}=a_1q_{2n}+q_{2n-1} \qquad (\delta).$$

Now, if we write $n-1$ for n in (γ), we have $p_{2n-2}=q_{2n-1}$; hence (δ) gives

$$a_2p_{2n-1}=a_1q_{2n}.$$

Therefore

$$p_{2n-1}=\frac{a_1}{a_2}q_{2n} \qquad (\epsilon).$$

§ 8.] From equations (3) and (4) of last section we can prove the *following important property of any two consecutive convergents*:—

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n \qquad (1).$$

For, by § 7 (3) and (4),

$$\begin{aligned} p_{n+1} q_n - p_n q_{n+1} &= (a_{n+1} p_n + p_{n-1}) q_n - p_n (a_{n+1} q_n + q_{n-1}), \\ &= -(p_n q_{n-1} - p_{n-1} q_n). \end{aligned}$$

Hence, if (1) hold, we have

$$\begin{aligned} p_{n+1} q_n - p_n q_{n+1} &= -(-1)^n, \\ &= (-1)^{n+1}. \end{aligned}$$

In other words, if the property be true for any integer n, it holds for the next integer $n+1$. Now

$$\begin{aligned} p_2 q_1 - p_1 q_2 &= (a_1 a_2 + 1) 1 - a_1 a_2, \\ &= 1, \\ &= (-1)^2; \end{aligned}$$

that is to say, the property in question holds for $n = 2$, hence it holds for $n = 3$; hence for $n = 4$; and so on.

Cor. 1. *The convergents, as calculated by the rule of* § 7, *are fractions at their lowest terms.*

For, if p_n and q_n, for example, had any common factor, that factor would, by § 8 (1), divide $(-1)^n$ exactly. Hence p_n is prime to q_n; and p_n/q_n is at its lowest terms.

Cor. 2.

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}} \qquad (2).$$

Cor. 3.

$$\begin{aligned} \frac{p_n}{q_n} &= \frac{p_1}{q_1} + \left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right) + \left(\frac{p_3}{q_3} - \frac{p_2}{q_2}\right) + \ldots + \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right), \\ &= a_1 + \frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \ldots + \frac{(-1)^n}{q_{n-1} q_n} \qquad (3). \end{aligned}$$

Cor. 4.

$$p_n q_{n-2} - p_{n-2} q_n = (-)^{n-1} a_n \qquad (4).$$

For

$$\begin{aligned} p_n q_{n-2} - p_{n-2} q_n &= (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2} (a_n q_{n-1} + q_{n-2}), \\ &= (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) a_n, \\ &= (-)^{n-1} a_n, \text{ by Cor. 1.} \end{aligned}$$

Cor. 5.

$$p_n/q_n - p_{n-s}/q_{n-s} = (-)^{n-1}a_n/q_n q_{n-s} \qquad (5).$$

Cor. 6. *The odd convergents continually increase in value, the even convergents continually decrease; every even convergent is greater than every odd convergent; and every odd convergent is less than and every even convergent greater than any following convergent.*

These conclusions follow at once from the equations (2) and (5).

Cor. 7. *Given two integers p and q which are prime to each other, we can always find two positive integers p' and q' such that $pq' - p'q = +1$ or $= -1$, as we please.*

For, by § 4, Cor., we can always convert p/q into a continued fraction having an even or an odd number of partial quotients, as we please. If p'/q' be the penultimate convergent to this continued fraction, we have in the former case $pq' - p'q = +1$, in the latter $pq' - p'q = -1$.

Example. If p_n/q_n be the nth convergent to $a_1 + \dfrac{1}{a_2+}\ \dfrac{1}{a_3+}\ \cdots\ \dfrac{1}{a_p}$, and ${}_sP_n/{}_sQ_n$ the convergent to $a_s + \dfrac{1}{a_{s+1}+}\ \cdots\ \dfrac{1}{a_p}$ which corresponds to the partial quotient a_n, show that

$$p_n q_{n-r} - p_{n-r}q_n = (-1)^{n-r+1}{}_{n-r+1}Q_n.$$

We have, by our data,

$$\frac{p_n}{q_n} = a_1 + \frac{1}{a_2+}\ \cdots\ \frac{1}{a_n} \qquad (\alpha),$$

$$\frac{p_{n-r}}{q_{n-r}} = a_1 + \frac{1}{a_2+}\ \cdots\ \frac{1}{a_{n-r}} \qquad (\beta);$$

hence

$$\frac{p_n}{q_n} = a_1 + \frac{1}{a_2+}\ \cdots\ \frac{1}{a_{n-r}+}\ \frac{1}{{}_{n-r+1}P_n/{}_{n-r+1}Q_n} \qquad (\gamma).$$

Now

$$\frac{p_{n-r}}{q_{n-r}} = \frac{a_{n-r}p_{n-r-1} + p_{n-r-2}}{a_{n-r}q_{n-r-1} + q_{n-r-2}}.$$

Hence, by (α) and (γ),

$$\frac{p_n}{q_n} = \frac{(a_{n-r} + {}_{n-r+1}Q_n/{}_{n-r+1}P_n)p_{n-r-1} + p_{n-r-2}}{(a_{n-r} + {}_{n-r+1}Q_n/{}_{n-r+1}P_n)q_{n-r-1} + q_{n-r-2}},$$

$$= \frac{p_{n-r} + {}_{n-r+1}Q_n p_{n-r-1}/{}_{n-r+1}P_n}{q_{n-r} + {}_{n-r+1}Q_n q_{n-r-1}/{}_{n-r+1}P_n},$$

$$= \frac{{}_{n-r+1}P_n p_{n-r} + {}_{n-r+1}Q_n p_{n-r-1}}{{}_{n-r+1}P_n q_{n-r} + {}_{n-r+1}Q_n q_{n-r-1}} \qquad (\delta).$$

Now it is easy to see that the numerator and denominator of the fraction last written are mutually prime; therefore

$$\left.\begin{aligned} p_n &= {}_{n-r+1}P_n p_{n-r} + {}_{n-r+1}Q_n p_{n-r-1}, \\ q_n &= {}_{n-r+1}P_n q_{n-r} + {}_{n-r+1}Q_n q_{n-r-1}, \end{aligned}\right\} \qquad (\epsilon).$$

From (ϵ) we derive

$$\begin{aligned} p_n q_{n-r} - p_{n-r} q_n &= -(p_{n-r} q_{n-r-1} - p_{n-r-1} q_{n-r})_{n-r+1} Q_n, \\ &= (-1)(-1)^{n-r}{}_{n-r+1} Q_n, \\ &\qquad \text{by (1) above,} \\ &= (-1)^{n-r+1}{}_{n-r+1} Q_n ; \end{aligned}$$

as was to be shown.

§ 9.] *The convergents of odd order are each less than the whole continued fraction, and the convergents of even order are each greater; and each convergent is nearer in value to the whole continued fraction than the preceding.*

We have, by § 7,

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}};$$

and the whole continued fraction x_1 is derived from p_{n+1}/q_{n+1} by replacing the partial quotient a_{n+1} by the complete quotient x_{n+1}. Hence

$$x_1 = \frac{x_{n+1} p_n + p_{n-1}}{x_{n+1} q_n + q_{n-1}}.$$

From this value of x_1 we obtain

$$\begin{aligned} x_1 - \frac{p_n}{q_n} &= \frac{x_{n+1} p_n + p_{n-1}}{x_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n}, \\ &= \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n (x_{n+1} q_n + q_{n-1})} \qquad (1). \end{aligned}$$

Similarly

$$x_1 - \frac{p_{n-1}}{q_{n-1}} = \frac{x_{n+1}(p_n q_{n-1} - p_{n-1} q_n)}{q_{n-1}(x_{n+1} q_n + q_{n-1})} \qquad (2).$$

From (1) and (2) we deduce

$$\frac{x_1 - \dfrac{p_n}{q_n}}{x_1 - \dfrac{p_{n-1}}{q_{n-1}}} = -\frac{q_{n-1}}{q_n x_{n+1}} \qquad (3).$$

Now q_{n-1}, q_n are positive integers; $x_{n+1} \not< 1$; and, by § 7, Cor. 1, $q_{n-1} < q_n$. It follows, therefore, from (3) that $x_1 - p_n/q_n$ is opposite in sign to, and numerically less than, $x_1 - p_{n-1}/q_{n-1}$. In other words, p_n/q_n differs from x_1 by less than p_{n-1}/q_{n-1} does; and if the one be less than x_1, the other is

greater, and *vice versâ*. Now the first convergent is obviously less than x_1, hence the second is greater, the third less, and so on ; and the difference between x_1 and the successive convergents continually decreases.

Cor. 1. *The difference between the continued fraction and the nth convergent is less than* $1/q_nq_{n+1}$, *and greater than* a_{n+2}/q_nq_{n+2}.

For, by what has just been proved,

$$\frac{p_n}{q_n}, \quad \frac{p_{n+2}}{q_{n+2}}, \quad x_1, \quad \frac{p_{n+1}}{q_{n+1}}$$

are, in order of magnitude, either ascending or descending. Hence

$$\frac{p_n}{q_n} \sim x_1 < \frac{p_n}{q_n} \sim \frac{p_{n+1}}{q_{n+1}},$$

$$< \frac{1}{q_nq_{n+1}}, \text{ by § 8 (2).}$$

Again,

$$\frac{p_n}{q_n} \sim x_1 > \frac{p_n}{q_n} \sim \frac{p_{n+2}}{q_{n+2}},$$

$$> \frac{a_{n+2}}{q_nq_{n+2}}, \text{ by § 8 (5).}$$

Since $q_{n+1} > q_n$, and since $q_{n+2}/a_{n+2} = (a_{n+2}q_{n+1} + q_n)/a_{n+2}$ $= q_{n+1} + q_n/a_{n+2} < q_{n+1} + q_n$ (a_{n+2} being $\not< 1$), it follows that the upper and lower limits of the error committed by taking the nth convergent instead of the whole continued fraction may be taken to be $1/q_n^2$ and $1/q_n(q_n + q_{n+1})$. These, of course, are not so close as those given above, but they are simpler, and in many cases they will be found sufficient.

Cor. 2. *In order to obtain a good approximation to a continued fraction, it is advisable to take that convergent whose corresponding partial quotient immediately precedes a very much larger partial quotient.*

For, if the next quotient be large. there is a sudden increase in q_{n+1}, so that $1/q_nq_{n+1}$ is a very small fraction.

The same thing appears from the consideration that, in taking p_n/q_n instead of the whole fraction, we take a_n instead of

$a_n + \dfrac{1}{a_{n+1}+} \ldots$, that is, we neglect the part $\dfrac{1}{a_{n+1}+} \ldots$ of the complete quotient. Now, if a_{n+1} be very large, this neglected part will of course be very small.

Cor. 3. *The odd convergents form an increasing series of rational fractions continually approaching to the value of the whole continued fraction; and the even convergents form a decreasing series having the same property.**

Cor. 4. *If* $p_n/q_n - x_1 < 1/q_n(q_n + q_{n-1})$, *where* q_{n-1} *is the denominator of the penultimate convergent to* p_n/q_n *when converted into a simple continued fraction having an even number of quotients, then* p_n/q_n *is one of the convergents to the simple continued fraction which represents* x_1; *and the like holds if* $x_1 - p_n/q_n < 1/q_n(q_n + q_{n-1})$, *where* q_{n-1} *is the denominator of the penultimate convergent to* p_n/q_n *when converted into a simple continued fraction having an odd number of quotients.*

Let $a_1, a_2, \ldots, a_n$ be the n partial quotients of p_n/q_n when converted into a simple continued fraction having an even number of quotients, and let p_{n-1}/q_{n-1} be the penultimate convergent. Then $p_n q_{n-1} - p_{n-1} q_n = 1$.

Let x_{n+1} be determined by the equation

$$x_1 = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n + \dfrac{1}{x_{n+1}}}.$$

Then we have

$$x_1 = (x_{n+1} p_n + p_{n-1})/(x_{n+1} q_n + q_{n-1}),$$

whence

$$x_{n+1} = (x_1 q_{n-1} - p_{n-1})/(p_n - x_1 q_n),$$

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* The value of every simple continued fraction lies, of course, between 0 and ∞; and we may, in fact, regard these as the first and second convergents respectively to every continued fraction. If we write $0 = \frac{1}{0}$, and $\infty = \frac{1}{0}$, and denote these by $\dfrac{p_{-1}}{q_{-1}}$ and $\dfrac{p_0}{q_0}$, so that we understand p_{-1} to be 0, p_0 to be 1, q_{-1} to be 1, and q_0 to be 0, then p_{-1} and p_0 will be found to fall into the series p_1, p_2, p_3, &c., and q_{-1} and q_0 into the series q_1, q_2, q_3, &c. It will be found, for example, that $p_1 = a_1 p_0 + p_{-1}$, $q_1 = a_1 q_0 + q_{-1}$, $p_0 q_{-1} - p_{-1} q_0 = (-1)^0 = 1$, and so on.

or, if we put $\xi = p_n/q_n - x_1$,

$$x_{n+1} = \{(p_n q_{n-1} - p_{n-1} q_n)/q_n - q_{n-1}\xi\}/q_n\xi,$$
$$= (1/q_n - q_{n-1}\xi)/q_n\xi.$$

Hence the necessary and sufficient condition that $x_{n+1} > 1$ is that

$$1/q_n - q_{n-1}\xi > q_n\xi,$$

that is,

$$\xi < 1/q_n(q_n + q_{n-1}),$$

which is fulfilled by the condition in the first of our two theorems.

Let now $b_1, b_2, \ldots, b_n$ be the first n partial quotients in the simple continued fraction that represents x_1. Then we have

$$x_1 = b_1 + \frac{1}{b_2 +} \cdots \frac{1}{b_n + \dfrac{1}{y_{n+1}}},$$

where $y_{n+1} > 1$.

Hence

$$a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_n + \dfrac{1}{x_{n+1}}} = b_1 + \frac{1}{b_2 +} \cdots \frac{1}{b_n + \dfrac{1}{y_{n+1}}}.$$

Therefore, by § 3, Cor., we must have

$$a_1 = b_1, \quad a_2 = b_2, \quad \ldots, \quad a_n = b_n, \quad x_{n+1} = y_{n+1}.$$

Hence $a_1 + \dfrac{1}{a_2 +} \cdots + \dfrac{1}{a_n}$, that is, $\dfrac{p_n}{q_n}$ is the nth convergent to x_1.

The second theorem is proved in precisely the same way.

Since $q_{n-1} < q_n$, the conditions above are *a fortiori* fulfilled if $x_1 \sim p_n/q_n < 1/2q_n^2$.

§ 10.] The propositions and corollaries of last section show that the method of continued fractions possesses the two most important advantages that any system of numerical calculation can have, namely, 1st, it furnishes a regular series of rational approximations to the quantity to be evaluated, which increase step by step in complexity, but also in exactness; 2nd, the error committed by arresting the approximation at any step can at once be estimated. The student should compare it in these respects with the decimal system of notation.

§ 11.] It should be observed that the formation of the successive convergents virtually determines the meaning we attach to the chain of operations in a continued fraction.

If the continued fraction terminate, we might of course proceed to reduce it by beginning at the lower end and taking in the partial quotients one by one in the reverse order. The reader may, as an exercise, work out this treatment of finite continued fractions, and he will find that, from the arithmetical point of view, it presents few or none of the advantages of the ordinary plan developed above.

In the case of non-terminating continued fractions, no such alternative course is, strictly speaking, open to us. Indeed, the further difficulty arises that, *a priori*, we have no certainty that such a continued fraction has any definite meaning at all. The point of view to be taken is the following:—If we arrest the continued fraction at any partial quotient, say the sth, then, in the case of a simple continued fraction, however great s may be, we have seen that the two convergents, p_{2n-1}/q_{2n-1}, p_{2n}/q_{2n}, include the fraction p_s/q_s between them. Hence, if we can show that p_{2n-1}/q_{2n-1} and p_{2n}/q_{2n} each approach the same finite value when n is increased without limit, it will follow that as s is increased without limit, that is, as more and more of the partial quotients of the continued fraction are taken into account, p_s/q_s approaches a certain definite value, which we may call the value of the whole continued fraction. Now, by § 8, Cor. 5, p_{2n-1}/q_{2n-1} continually increases with n, and p_{2n}/q_{2n} continually decreases, and $p_{2n}/q_{2n} > p_{2n-1}/q_{2n-1}$. Hence, since both are positive, each of the two must approach a certain finite limit. Also the two limits must be the same; for by § 8, Cor. 2, $p_{2n}/q_{2n} - p_{2n-1}/q_{2n-1} = 1/q_{2n}q_{2n-1}$, and by the recurrence formula for q_n it follows that q_{2n} and q_{2n-1} increase without limit with n; therefore $p_{2n}/q_{2n} - p_{2n-1}/q_{2n-1}$ may be made as small as we please by sufficiently increasing n.

It appears, therefore, that *every simple continued fraction has a definite finite value.*

Example.

To obtain a good commensurable approximation to the ratio of the circum-

ference of a circle to the diameter. Referring to Example 1, § 7, we have the following approximations in defect:—

$$\frac{3}{1}, \quad \frac{333}{106}, \quad \frac{103993}{33102}, \text{ \&c.;}$$

and the following in excess:—

$$\frac{22}{7}, \quad \frac{355}{113}, \quad \frac{104348}{33215}, \text{ \&c.}$$

Two of these,* namely, 22/7 and 355/113, are distinguished beyond the others by preceding large partial quotients, namely, 15 and 292.

The last of these is exceedingly accurate, for in this case $1/q_n q_{n+1} = 1/113 \times 33102 = \cdot 0000002673$, and $a_{n+2}/q_n q_{n+2} = 1/113 \times 33215 = \cdot 0000002665$. The error therefore lies between ·000000266 and ·000000267; that is to say, 355/113 is accurate to the 6th decimal place. In point of fact, we have

$$\pi = 3 \cdot 14159265358 \ldots$$
$$355/113 = 3 \cdot 14159292035 \ldots$$
$$\text{Difference} = \quad \cdot 00000026677 \ldots$$

Exercises XXIX.

(1.) Calculate the various convergents to $\frac{769}{1114}$, and estimate the errors committed by taking the first, second, third, &c., instead of the fraction.

(2.) Find a convergent to the infinite continued fraction $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\ldots$ which shall represent its value within a millionth.

(3.) Find a commensurable approximation to $\sqrt{(17)}$ which shall be accurate within 1/100000, and such that no nearer fraction can be found not having a greater denominator.

(4.) The sidereal period of Venus is 224·7 days, that of the earth 365·25 days; calculate the various cycles in which transits of Venus may be expected to occur. Calculate the number of degrees in each case by which Venus is displaced from the node, when the earth is there, at the end of the first cycle after a former central transit.

(5.) Work out the same problem for Mercury, whose sidereal period is 87·97 days.

(6.) According to the Northampton table of mortality, out of 3635 persons who reach the age of 40, 3559 reach the age of 41. Show that this is expressed very accurately by saying that 47 out of 48 survive.

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* The first of them, 22/7, was given by Archimedes (212 B.C.) The second, 355/113, was given by Adrian Metius (published by his son, 1640 A.D.): it is in great favour, not only on account of its accuracy, but because it can be easily remembered as consisting of the first three odd numbers each repeated twice in a certain succession.

(7.) Find a good rational approximation to $\sqrt{(19)}$ which shall differ from it by less than 1/100000; and compare this with the rational approximation obtained by expressing $\sqrt{(19)}$ as a decimal fraction correct to the 6th place.

(8.) If α be any incommensurable quantity whatever, show that two integers, m and n, can always be found, so that $0 < \alpha n - m < \kappa$, however small κ may be.

(9.) Show that the numerators and also the denominators of any two consecutive convergents to a simple continued fraction are prime to each other; also that if p_n and p_{n-2} have any common factor it must divide a_n exactly.

(10.) Show that the difference between any two consecutive odd convergents to $\sqrt{(a^2+1)}$ is a fraction whose numerator, when at its lowest terms, is $2a$.

(11.) Prove directly, from the recursive relation connecting the numerators and denominators, that every convergent to a simple continued fraction is intermediate in value to the two preceding.

(12.) Prove that

$$q_n x_1 - p_n = (-1)^{n+1}/x_2 x_3 \ldots x_{n+1}.$$

Show that p_n/q_n differs from x_1 by less than $1/a_2 a_3 \ldots a_{n+1} q_n$. Is this a better estimate of the error than $1/q_n q_{n+1}$?

(13.) If the integers x and y be prime to each other, show that an integer u can always be found such that

$$(x^2+y^2)u = z^2+1,$$

where z is an integer.

(14.) Prove that

$$(p_n^2 - q_n^2)(p_{n-1}^2 - q_{n-1}^2) = (p_n p_{n-1} + q_n q_{n-1})^2 - 1;$$

$$\frac{p_n^2+q_n^2}{p_{n-2}^2+q_{n-2}^2} = \frac{(p_n p_{n-1}+q_n q_{n-1})^2+1}{(p_{n-1}p_{n-2}+q_{n-1}q_{n-2})^2+1}.$$

(15.) Prove that $p_{n-1}p_n - q_{n-1}q_n x_1^2$ is positive or negative according as n is even or odd.

(16.) If P/Q, P′/Q′, P″/Q″ be the nth, $\overline{n-1}$th, $\overline{n-2}$th convergents of

$$\frac{1}{a_1+}\ \frac{1}{a_2+}\ \frac{1}{a_3+}\ \frac{1}{a_4+}\ \ldots,$$

$$\frac{1}{a_2+}\ \frac{1}{a_3+}\ \frac{1}{a_4+}\ \ldots,$$

$$\frac{1}{a_3+}\ \frac{1}{a_4+}\ \ldots$$

respectively, show that

$$P = a_2 P' + P'', \quad Q = (a_1 a_2 + 1)P' + a_1 P''.$$

(17.) If the partial quotients of $x_1 = p_n/q_n$ form a reciprocal series (that is, a series in which the first and last terms are equal, the second and second last equal, and so on), then $p_{n-1} = q_n$, and $(q_n^2 \pm 1)/p_n$ is an integer; and, conversely, if these conditions be satisfied, the quotients will form a reciprocal series.

(18.) Show, from last exercise, that every integer which divides the sum of two integral squares that are prime to each other is itself the sum of two squares. (See Serret, *Alg. Sup.*, 4me éd., t. i., p. 29.)

(19.) Show that

$$\frac{a_1+\cfrac{1}{a_2+}\ \cdots\ \cfrac{1}{a_n}}{a_1+\cfrac{1}{a_2+}\ \cdots\ \cfrac{1}{a_{n-1}}} = \frac{a_n+\cfrac{1}{a_{n-1}+}\ \cdots\ \cfrac{1}{a_1}}{a_n+\cfrac{1}{a_{n-1}+}\ \cdots\ \cfrac{1}{a_2}}.$$

(20.) If $x_1=\cfrac{1}{a+}\ \cfrac{1}{a+}\ \cfrac{1}{a+}\ \ldots$, show that $p_n=q_{n-1}$.

(21.) The successive convergents of $2a+\cfrac{1}{a+}\ \cfrac{1}{4a+}\ \cfrac{1}{a+}\ \cfrac{1}{4a+}\ \ldots$ are always double those of $a+\cfrac{1}{2a+}\ \cfrac{1}{2a+}\ \ldots$.

(22.) If the reduced form of the nth complete quotient, x_n, in $a_1+\cfrac{1}{a_2+}\ \cfrac{1}{a_3+}\ \ldots$ be ζ_n/η_n, show that

$$\zeta_n=a_n\zeta_{n+1}+\zeta_{n+2},$$
$$\eta_n=\zeta_{n+1}.$$

(23.) Find the numerically least value of $ax-by$ for positive integral values of x and y, a and b being positive integers, which may or may not be prime to each other.

CLOSEST COMMENSURABLE APPROXIMATIONS OF GIVEN COMPLEXITY.

§ 12.] One commensurable approximation to a number (commensurable or incommensurable) is said to be more complex than another when the denominator of the representative fraction is greater in the one case than in the other. The problem which we put before ourselves here is *to find the fraction, whose denominator does not exceed a given integer* D, *which shall most closely approximate (by excess or by defect, as may be assigned) to a given number commensurable or incommensurable.* The solution of this problem is one of the most important uses of continued fractions. It depends on a principle of great interest in the theory of numbers, which we proceed to prove.

Lemma.—*If p/q and p'/q' be two fractions such that $pq'-p'q=1$, then no fraction can lie between them unless its denominator is greater than the denominator of either of them.*

Proof.—Let a/b be a fraction intermediate in magnitude to p/q and p'/q'. Then

$$\frac{p}{q}-\frac{a}{b}<\frac{p}{q}-\frac{p'}{q'} \qquad (1);$$

$$\frac{a}{b}-\frac{p'}{q'}<\frac{p}{q}-\frac{p'}{q'} \qquad (2).$$

From (1), $$\frac{pb-qa}{qb} < \frac{pq'-p'q}{qq'};$$

$$\frac{pb-qa}{qb} < \frac{1}{qq'}.$$

Hence $$qb > qq'(pb-qa);$$

and $$b > (pb-qa)q'.$$

Now $p/q - a/b$ is positive, hence $pb-qa$ is a positive integer. It follows, therefore, that $b > q'$.

Similarly it follows from (2) that $b > q$.

Hence no fraction can lie between p/q and p'/q' unless its denominator is greater than both q and q'. In other words, if $pq'-p'q=1$, no commensurable number can lie between p/q and p'/q' which is not more complex than either of them.

§ 13.] *The nth convergent to a continued fraction is a nearer approximation to the value of the complete fraction than any fraction whose denominator is not greater than that of the convergent.* For any fraction a/b which is nearer in value to the continued fraction than p_n/q_n must, *a fortiori*, be nearer than p_{n-1}/q_{n-1}. Hence, since p_n/q_n and p_{n-1}/q_{n-1} include the value of the continued fraction between them, it follows that a/b must lie between these two fractions. Now we have, by § 8, either $p_nq_{n-1}-p_{n-1}q_n=1$, or $p_{n-1}q_n-p_nq_{n-1}=1$. Hence, by § 12, b must be greater than q_n, which proves our proposition.

Example.

Consider the continued fraction $x_1 = 3+\cfrac{1}{1+}\ \cfrac{1}{3+}\ \cfrac{1}{4+}\ \cfrac{1}{2+}\ \cfrac{1}{5}$.

The successive convergents are $\frac{3}{1}, \frac{4}{1}, \frac{15}{4}, \frac{64}{17}, \frac{143}{38}, \frac{779}{207}$. If we take any one of these, say 64/17, the statement is, that no fraction whose denominator does not exceed 17 can be nearer in value to x_1 than 64/17.

§ 14.] The result of last section is a step towards the solution of the general problem of § 12; but something more is required.

Consider, for example, the successive convergents p_{n-2}/q_{n-2}, p_{n-1}/q_{n-1}, p_n/q_n to x_1, and let n be odd, say. Then

$$\frac{p_{n-2}}{q_{n-2}}, \quad \frac{p_n}{q_n}, \quad x_1, \quad \frac{p_{n-1}}{q_{n-1}}$$

are in increasing order of magnitude. We know, by last

section, that no fraction whose denominator is less than q_{n-1} can lie in the interval $\overline{p_{n-2}/q_{n-2},\ p_{n-1}/q_{n-1}}$, and also that no fraction whose denominator is less than q_n can lie in the interval $\overline{p_n/q_n,\ p_{n-1}/q_{n-1}}$; but we have no assurance that a fraction whose denominator is less than q_n may not lie in the interval $\overline{p_{n-2}/q_{n-2},\ p_n/q_n}$, for $p_nq_{n-2} - p_{n-2}q_n = a_n$, where a_n may be > 1.

This lacuna is filled by the following proposition :—

1°. *The series of fractions*

$$\frac{p_{n-2}}{q_{n-2}},\quad \frac{p_{n-2}+p_{n-1}}{q_{n-2}+q_{n-1}},\quad \frac{p_{n-2}+2p_{n-1}}{q_{n-2}+2q_{n-1}},\quad \ldots,$$
$$\frac{p_{n-2}+\overline{a_n-1}\,p_{n-1}}{q_{n-2}+\overline{a_n-1}\,q_{n-1}},\quad \frac{p_{n-2}+a_np_{n-1}}{q_{n-2}+a_nq_{n-1}}\left(=\frac{p_n}{q_n}\right) \qquad (1),$$

form (according as n is odd or even) an increasing or a decreasing series.

2°. *Each of them is at its lowest terms; and each consecutive pair, say* P/Q, P′/Q′, *satisfies the condition* PQ′ − P′Q = ± 1 ; *so that no commensurable quantity less complex than the more complex of the two can be inserted between them.*

The first and last of these fractions (formerly called *Convergents* merely) we now call, for the sake of distinction, *Principal Convergents*; the others are called *Intermediate Convergents* to the continued fraction. To prove the above properties, let us consider any two consecutive fractions of the series (1), say P/Q, P′/Q′; then

$$\frac{P}{Q}-\frac{P'}{Q'}=\frac{p_{n-2}+rp_{n-1}}{q_{n-2}+rq_{n-1}}-\frac{p_{n-2}+\overline{r+1}\,p_{n-1}}{q_{n-2}+\overline{r+1}\,q_{n-1}}$$

(where $r = 0$, or 1, or 2, . . ., or $a_n - 1$),

$$=\frac{-(p_{n-1}q_{n-2}-p_{n-2}q_{n-1})}{(q_{n-2}+rq_{n-1})(q_{n-2}+\overline{r+1}\,q_{n-1})},$$

$$=\frac{\mp 1}{(q_{n-2}+rq_{n-1})(q_{n-2}+\overline{r+1}\,q_{n-1})},$$

$$\left.\begin{aligned}&=\frac{-1}{QQ'}\text{ if } n \text{ be odd},\\ &=\frac{+1}{QQ'}\text{ if } n \text{ be even}.\end{aligned}\right\} \qquad (2).$$

Hence

$$\left.\begin{aligned} PQ' - P'Q &= -1 \text{ if } n \text{ be odd,} \\ &= +1 \text{ if } n \text{ be even.} \end{aligned}\right\} \quad (3),$$

(2) and (3) are sufficient to establish 1° and 2°.

3°. Since $P/Q - p_{n-1}/q_{n-1} = \pm 1/q_{n-1}(q_{n-2} + rq_{n-1})$, and since x_1 obviously lies between P/Q and p_{n-1}/q_{n-1}, it follows that *the intermediate convergent* P/Q *differs from the continued fraction by less than* $1/q_{n-1}Q$, a fortiori *by less than* $1/q_{n-1}^2$.

§ 15.] *If we take all the principal convergents of odd order with their intermediates wherever the partial quotients differ from unity, and form the series*

$$\frac{0}{1}, \ \ldots, \ \frac{p_1}{q_1}, \ \ldots, \ \frac{p_3}{q_3}, \ \ldots, \ \frac{p_{n-2}}{q_{n-2}}, \ \ldots, \ \frac{p_n}{q_n}, \ \ldots \quad (A),$$

and likewise all the principal convergents of even order with their intermediates, and form the series

$$\frac{1}{0}, \ \ldots, \ \frac{p_2}{q_2}, \ \ldots, \ \frac{p_4}{q_4}, \ \ldots, \ \frac{p_{n-2}}{q_{n-2}}, \ \ldots, \ \frac{p_{n-1}}{q_{n-1}}, \ \ldots \quad (B),$$

then (A) *is a series of commensurable quantities, increasing in complexity and increasing in magnitude, which continually approach the continued fraction; and* (B) *is a series of commensurable quantities, increasing in complexity and decreasing in magnitude, which continually approach the same; and it is impossible between any consecutive pair of either series to insert a commensurable quantity which shall be less complex than the more complex of the two.*

If the continued fraction be non-terminating, each of the two series (A) and (B) is non-terminating.

If the continued fraction terminates, one of the series will terminate, since the last member of one of them will be the last convergent to x_1; that is to say, x_1 itself. The other series may, however, be prolonged as far as we please; for, *if* p_{n-1}/q_{n-1} *and* p_n/q_n *be the last two convergents, the series of fractions*

$$\frac{p_{n-1}}{q_{n-1}}, \ \frac{p_{n-1}+p_n}{q_{n-1}+q_n}, \ \frac{p_{n-1}+2p_n}{q_{n-1}+2q_n}, \ \ldots$$

forms either a continually increasing or a continually decreasing series,

*in which no principal convergent occurs, but whose terms approach more and more nearly the value p_n/q_n, that is, x_1.**

§ 16.] We are now in a position to solve the general problem of § 12.† Suppose, for example, that we are required to find the fraction, whose denominator does not exceed D, which shall approximate most closely by defect to the quantity x_1. *What we have to do is to convert x_1 into a simple continued fraction, form the series* (A) *of last section, and select that fraction from it whose denominator is either* D, *or, failing that, less than but nearest to* D, *say* P/Q. For, if there were any fraction nearer to x_1 than P/Q, it would lie to the right of P/Q in the series; that is to say, would fall between P/Q and the next fraction P′/Q′ of the series, or between two fractions still more complex. Hence the denominator of the supposed fraction would be greater than Q′, and hence greater than D.

Similarly, *the fraction which most nearly approximates to x_1 by excess, and whose denominator does not exceed* D, *is obtained by taking that fraction in series* (B) *of last section whose denominator most nearly equals without exceeding* D.

N.B.—If the denominator in the (A) series which most

* This may also be seen from the fact that the continued fraction $a_1+\frac{1}{a_2+}\cdots\frac{1}{a_n}$ may also be written $a_1+\frac{1}{a_2+}\cdots\frac{1}{a_n+}\frac{1}{\infty}$; that is to say we may consider the last quotient to be ∞, and the last convergent $(p_{n-1}+\infty p_n)/(q_{n-1}+\infty q_n)$.

† The first general solution of this problem was given by Wallis (see his *Algebra* (1685), chap. x.); Huyghens also was led to discuss it when designing the toothed wheels of his Planetarium (see his *Descriptio Automati Planetarii*, 1682). One of the earlier appearances of continued fractions in mathematics was the value of $4/\pi$ given by Lord Brouncker (about 1655). While discussing Brouncker's Fraction in his *Arithmetica Infinitorum* (1656), Wallis gives a good many of the elementary properties of the convergents to a general continued fraction, including the rule for their formation. Saunderson, Euler, and Lambert all helped in developing the theory of the subject. See two interesting bibliographical papers by Günther and Favaro, *Bulletino di Bibliographia e di Storia delle Scienze Mathematiche e Fisiche*, t. vii. In this chapter we have mainly followed Lagrange, who gave the first full exposition of it in his additions to the French edition of Euler's *Algebra* (1795). We may here direct the attention of the reader to a series of comprehensive articles on continued fractions by Stern, *Crelle's Jour.*, x., xi. xviii.

nearly equals without exceeding D be the denominator of an intermediate convergent, the denominator in the (B) series which most nearly equals without exceeding D will be the denominator of a principal convergent.

Example 1.

To find the fraction, whose denominator does not exceed 60, which approximates most closely to $\frac{779}{207}$.

We have $$\frac{779}{207}=3+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{5}.$$

The odd convergents are $\frac{0}{1}, \frac{3}{1}, \frac{15}{4}, \frac{143}{38}$;

the even convergents $\frac{1}{0}, \frac{4}{1}, \frac{64}{17}, \frac{779}{207}$.

The two series are

$$\frac{0}{1}\ \frac{1}{1}\ \frac{2}{1}\ \frac{3}{1}\ \frac{7}{2}\ \frac{11}{3}\ \frac{15}{4}\ \frac{79}{21}\ \frac{143}{38}\ \frac{922}{245}\ \frac{1701}{452}\ \frac{2480}{659}\ \ldots \text{ (A)},$$

$$\frac{1}{0}\ \frac{4}{1}\ \frac{19}{5}\ \frac{34}{9}\ \frac{49}{13}\ \frac{64}{17}\ \frac{207}{55}\ \frac{350}{93}\ \frac{493}{131}\ \frac{636}{169}\ \frac{779}{207} \text{ (B)}.$$

Hence, of the fractions whose denominators do not exceed 60, 143/38 is the closest by defect and 207/55 the closest by excess to 779/207.

Of these two it happens that 143/38 is the closer, although its denominator is less than that of 207/55; for we have $143/38 = 3{\cdot}76315\ldots$, $207/55 = 3{\cdot}76363\ldots$, and $779/207 = 3{\cdot}76328\ldots$ For a rule enabling us in most cases to save calculation in deciding between the closeness of the (A) and (B) approximations, see Exercises XXX., 10.

Example 2.

Adopting La Caille's determination of the length of the tropical year as $365^{d}\ 5^{h}\ 48'\ 49''$, so that it exceeds the civil year by $5^{h}\ 48'\ 49''$, we are required to find the various ways of rectifying the calendar by intercalating an integral number of days at equal intervals of an integral number of years. (Lagrange.)

The intercalation must be at the rate of $\frac{20929^{d}}{86400}$ per year; that is to say, at the rate of 20929 days in 86400 years. If, therefore, we were to intercalate 20929 days at the end of every 864 centuries we should exactly represent La Caille's determination. Such a method of rectifying the calendar is open to very obvious objections, and consequently we seek to obtain an approximate rectification by intercalating a smaller number of days at shorter intervals. If we turn 86400/20929 into a continued fraction and form the (A) and (B) series of convergents, we have (omitting the earlier terms)

$$\frac{4}{1}\ \frac{33}{8}\ \frac{161}{39}\ \frac{2865}{694}\ \frac{8434}{2043}\ \frac{14003}{3392}, \text{ \&c.} \quad \text{(A)},$$

$$\frac{5}{1}\ \frac{9}{2}\ \frac{13}{3}\ \frac{17}{4}\ \frac{21}{5}\ \frac{25}{6}\ \frac{29}{7}\ \frac{62}{15}\ \frac{95}{23}\ \frac{128}{31}\ \frac{289}{70}\ \frac{450}{109}\ \frac{611}{148}$$

$$\frac{772}{187}\ \frac{933}{226}\ \frac{1094}{265}, \text{ \&c.} \quad \text{(B)}.$$

Hence, if we take approximations which err by excess, we may with increasing accuracy intercalate 1 day every 4 years, 8 every 33, 39 every 161, and so on;* and be assured that each of these gives us the greatest accuracy obtainable by taking an integral number of days less than that indicated in the next of the series.

The (B) series may be used in a similar manner.†

Example 3.

An eclipse of the sun will happen if at the time of new moon the earth be within about 13° of the line of nodes of the orbits of earth and moon. The period between two new moons is on the average 29·5306 days, and the mean synodic period of the earth and moon is 346·6196 days. It is required to calculate the simpler periods for the recurring of eclipses.

Suppose that after any the same time from a new moon the moon and earth have made respectively the multiples x and y of a revolution, then $x \times 29{\cdot}5306 = y \times 346{\cdot}6196$. Hence $y/x = 295306/3466196 = 0 + \cfrac{1}{11+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{3+}}}}}} \cdots$

The successive convergents to this fraction are 1/11, 1/12, 3/35, 4/47, 19/223, 61/716.

Suppose we take the convergent 4/47, the error incurred thereby will be $< 1/47 \times 223$ in excess, and we may write on the most unfavourable supposition

$$\frac{y}{x} = \frac{4}{47} - \frac{1}{47 \times 223}.$$

Hence, if $x = 47$, $y = 4 - 1/223$. But $360°/223 = 1°{\cdot}61$. Hence 47 lunations after total eclipse new moon will happen when the earth is less than $1°{\cdot}61$ from the line of nodes, 47 lunations after that again when the earth is less than $3°{\cdot}2$ from the line of nodes, and so on. Hence, since 47 lunations = 1388 days, eclipses will recur after a total eclipse for a considerable number of periods of 1388 days.

If we take the next convergent we find for the period of recurrence 223 lunations, which amounts to 18 years and 10 or 11 days, according as five or four leap years occur in the interval. The displacement from the node in this case is certainly less than $360°/716$, that is, less than half a degree, so that

* The fraction 4/1 corresponds to the Julian intercalation, introduced by Julius Cæsar (45 B.C.). 33/8 gives the so-called Persian intercalation, said to be due to the mathematician Omar Alkhayami (1079 A.D.). The method in present use among most European nations is the Gregorian, which corrects the Julian intercalation by omitting 3 days every 4 centuries. This corresponds to the fraction 400/97, which is not one in the above series; in fact, 70 days every 289 years would be more accurate. The Gregorian method has, however, the advantage of proceeding by multiples of a century. The Greeks and Russians still use the Julian intercalation, and in consequence there is a difference of 12 days between their calendar and ours. See art. "Calendar," *Encyclopædia Britannica*, 9th ed.

† See Lagrange's additions to the French edition of Euler's *Algebra* (Paris, 1807), t. ii., p. 312.

this is a far more certain cycle than the last; in fact, it is the famous "saros" of antiquity which was known to the Chaldean astronomers.

Still more accurate results may of course be obtained by taking higher convergents.

EXERCISES XXX.

(1.) Find the first eight convergents to $1+\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{1+}\ldots$, and find the fraction nearest to it whose denominator does not exceed 600.

(2.) Work out the problem of Exercise XXIX., 4, using intermediate as well as principal convergents.

(3.) Work out all the convergents to 2π whose denominators do not exceed 1000.

(4.) Solve the same problem for the base of the Napierian system of logarithms $e=2{\cdot}71828183\ldots$.

(5.) Two scales, such that 1873 parts of the one is equal to 1860 parts of the other, are superposed so that the zeros coincide: find where approximate coincidences occur and estimate the divergence in each case.

(6.) Two pendulums are hung up, one in front of the other. The first beats seconds exactly; the second loses 5 min. 37 sec. in 24 hours. They pass the vertical together at 12 o'clock noon. Find the times during the day at which the first passes the vertical, and the second does so approximately at the same time.

(7.) Along the side AB and diagonal AC of a square field round posts are erected at equal intervals, the interval in the two cases being the same. A person looking from a distance in a direction perpendicular to AB sees in the perspective of the two rows of posts places where the posts seem very close together ("ghosts"), and places where the intervals are clear owing to approximate coincidences. Calculate the distances of the centres of the ghosts from A, and show that they grow broader and sparser as they recede from A.

(8.) Show that between two given fractions p/q and p'/q', such that $pq'-p'q=1$, an infinite number of fractions in order of magnitude can be inserted such that between any consecutive two of the series no fraction can be found less complex than either of them.

(9.) In the series of fractions whose denominators are 1, 2, 3, . . ., n there is at least one whose denominator is ν, say, such that it differs from a given irrational quantity x by less than $1/n\nu$. (For a proof of this theorem, due to Dirichlet, not depending on the theory of continued fractions, see Serret, *Alg. Sup.*, 4me éd., t. i., p. 27.)

(10.) If the nearest rational approximation in excess or defect (see § 16) be an intermediate convergent P/Q, where $Q=\lambda q_{n-1}+q_{n-2}$, show that the approximation in defect or excess will be nearer unless $Q>\frac{1}{2}q_n+q_{n-1}/2\alpha_{n+1}$.

(11.) If zero partial quotients be (contrary to the usual understanding) admitted, show that every continued fraction may be written in the form $0+\frac{1}{\alpha_1+}\frac{1}{\alpha_2+}\frac{1}{\alpha_3+}\ldots$, where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are each either 0 or 1. Show the bearing of this on the theory of the so-called intermediate convergents.

CHAPTER XXXIII.

On Recurring Continued Fractions.

EVERY SIMPLE QUADRATIC SURD NUMBER IS EQUAL TO A RECURRING CONTINUED FRACTION.

§ 1.] We have already seen in two particular instances (chap. xxxii., § 5) that a simple surd number can be expressed as a recurring continued fraction. We proceed in the present chapter to discuss this matter more closely.*

Let us consider the simple surd number $(P_1 + \sqrt{R})/Q_1$. We suppose that its value is positive; and we arrange, as we always may, that P_1, Q_1, R shall be integers, and that $\sqrt{R}$ shall have the positive sign as indicated. R will of course always be positive; but P_1 and Q_1 may be either positive or negative. It is further supposed that $R - P_1^2$ is exactly divisible by Q_1. This is allowable, for, if $R - P_1^2$ were, say, prime to Q_1, then we might write $(P_1 + \sqrt{R})/Q_1 = (P_1Q_1 + \sqrt{Q_1^2R})/Q_1^2 = (P_1' + \sqrt{R'})/Q_1'$, where $R' - P_1'^2 \{= Q_1^2(R - P_1^2) = (R - P_1^2)Q_1'^2\}$ is exactly divisible by Q_1'.

For example, to put $\frac{1}{4}\left(2 - \sqrt{\frac{3}{2}}\right)$ into the standard form contemplated, we must write

$$\frac{1}{4}\left(2 - \sqrt{\frac{3}{2}}\right) = \frac{1}{4}\left(2 - \frac{\sqrt{6}}{2}\right) = \frac{-4+\sqrt{6}}{-8} = \frac{-16+\sqrt{96}}{-32};$$

so that in this case $P_1 = -16$, $Q_1 = -32$, $R = 96$; $R - P_1^2 = 96 - 256 = -160$, which is exactly divisible by $Q_1 = -32$.

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* The following theory is due in the main to Lagrange. For the details of its exposition we are considerably indebted to Serret, *Alg. Sup.*, chap. ii.

§ 2.] If we adopt the process and notation of chap. xxxii., §§ 3 and 5, the calculation of the partial and complete quotients of the continued fraction which represents $(P_1 + \sqrt{R})/Q_1$ proceeds as follows:—

$$\left.\begin{aligned} x_1 &= \frac{P_1 + \sqrt{R}}{Q_1} = a_1 + \frac{1}{x_2}; \\ x_2 &= \frac{P_2 + \sqrt{R}}{Q_2} = a_2 + \frac{1}{x_3}; \\ &\cdots\cdots\cdots \\ x_n &= \frac{P_n + \sqrt{R}}{Q_n} = a_n + \frac{1}{x_{n+1}}; \\ &\cdots\cdots\cdots \end{aligned}\right\} \quad (1),$$

where it will be remembered that a_1, a_2, . . . are the greatest integers which do not exceed x_1, x_2, . . . respectively; and x_2, x_3, . . . are each positive, and not less than unity.

It should be noticed, however, that since we keep the radical $\sqrt{R}$ unaltered in our arrangement of the complete quotients, it by no means follows that P_2, Q_2, P_3, Q_3, &c., are integers, much less that they are positive integers.

The connection between any two consecutive pairs, say P_n, Q_n and P_{n+1}, Q_{n+1}, follows from the equation

$$\frac{P_n + \sqrt{R}}{Q_n} = a_n + \frac{1}{(P_{n+1} + \sqrt{R})/Q_{n+1}} \quad (2),$$

or

$$\{(P_n - a_nQ_n)P_{n+1} - Q_nQ_{n+1} + R\} + \{P_n - a_nQ_n + P_{n+1}\}\sqrt{R} = 0 \quad (3).$$

It follows from (3), by chap. xi., § 8, that

$$(P_n - a_nQ_n)P_{n+1} - Q_nQ_{n+1} + R = 0,$$

$$P_n - a_nQ_n + P_{n+1} = 0;$$

whence

$$P_{n+1} = a_nQ_n - P_n \quad (4),$$

$$P_{n+1}^2 + Q_nQ_{n+1} = R \quad (5).$$

If we write $n - 1$ for n in (5), we have

$$P_n^2 + Q_{n-1}Q_n = R \quad (6).$$

From (5), by means of (4) and (6), we have

$$Q_nQ_{n+1} = R - (a_nQ_n - P_n)^2,$$
$$= P_n^2 + Q_{n-1}Q_n - (a_nQ_n - P_n)^2,$$

so that

$$Q_{n+1} = Q_{n-1} + 2a_nP_n - a_n^2Q_n,$$
$$= Q_{n-1} + a_n(P_n - P_{n+1}) \qquad (7).$$

The formulæ (4) and (7) give a convenient means of calculating P_2, P_3, Q_2, P_4, Q_4, &c., and hence the successive complete quotients x_2, x_3, . . . Q_2 is given by the equation

$$P_2^2 + Q_2Q_1 = R,$$

namely,
$$Q_2 = \frac{R - (a_1Q_1 - P_1)^2}{Q_1},$$
$$= \frac{R - P_1^2}{Q_1} + 2a_1P_1 - a_1^2Q_1.$$

From this last equation it follows, since by hypothesis $(R - P_1^2)/Q_1$ is an integer, that Q_2 is an integer. Hence, since P_1, Q_1 are integers, it follows, by (4) and (7), that P_2, P_3, . . ., P_n, Q_2, . . ., Q_n are also all integers.

§ 3.] We shall now investigate formulæ connecting P_n and Q_n with the numerators and denominators of the convergents to the continued fraction which represents $(P_1 + \sqrt{R})/Q_1$.

We have (chap. xxxii., § 9)

$$\frac{P_1 + \sqrt{R}}{Q_1} = \frac{p_{n-1}x_n + p_{n-2}}{q_{n-1}x_n + q_{n-2}} \qquad (A),$$
$$= \frac{p_{n-1}P_n + p_{n-2}Q_n + p_{n-1}\sqrt{R}}{q_{n-1}P_n + q_{n-2}Q_n + q_{n-1}\sqrt{R}}.$$

Hence

$$(P_1 + \sqrt{R})(q_{n-1}P_n + q_{n-2}Q_n + q_{n-1}\sqrt{R})$$
$$= Q_1(p_{n-1}P_n + p_{n-2}Q_n + p_{n-1}\sqrt{R}) \qquad (1).$$

From (1) we derive

$$q_{n-1}P_n + q_{n-2}Q_n = Q_1p_{n-1} - P_1q_{n-1} \qquad (2);$$

$$p_{n-1}P_n + p_{n-2}Q_n = P_1p_{n-1} + \frac{R - P_1^2}{Q_1}q_{n-1} \qquad (3).$$

From (2) and (3) we obtain, since $p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = (-1)^{n-1}$,

$$(-1)^{n-1}P_n = P_1(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}) + \frac{R - P_1^2}{Q_1} q_{n-1}q_{n-2} - Q_1 p_{n-1}p_{n-2} \quad (4);$$

$$(-1)^{n-1}Q_n = -2p_{n-1}q_{n-1}P_1 - \frac{R - P_1^2}{Q_1} q_{n-1}^2 + Q_1 p_{n-1}^2 \quad (5).$$

The formulæ (4) and (5) give us the required expressions, and furnish another proof that $P_2, P_3, \ldots, P_n, Q_2, Q_3, \ldots, Q_n$ are all integral.

§ 4.] If in equation (2) of last paragraph we replace P_1 by its value $Q_1(p_{n-1}x_n + p_{n-2})/(q_{n-1}x_n + q_{n-2}) - \sqrt{R}$, derived from equation (A), we have

$$q_{n-1}P_n + q_{n-2}Q_n = \frac{(-1)^{n-1}Q_1}{q_{n-1}x_n + q_{n-2}} + q_{n-1}\sqrt{R} \quad (1).$$

Also, since $x_n = (P_n + \sqrt{R})/Q_n$, we have

$$P_n - x_n Q_n = -\sqrt{R} \quad (2).$$

From equations (1) and (2) we derive, by direct calculation, the following four :—

$$P_n = \frac{x_n}{(q_{n-1}x_n + q_{n-2})^2}\left\{\left(q_{n-1}x_n - q_{n-2}\right)\left(q_{n-1} + \frac{q_{n-2}}{x_n}\right)\sqrt{R} + (-1)^{n-1}Q_1\right\} \quad (3);$$

$$Q_n = \frac{1}{(q_{n-1}x_n + q_{n-2})^2}\{q_{n-1}(q_{n-1}x_n + q_{n-2})2\sqrt{R} + (-1)^{n-1}Q_1\} \quad (4);$$

$$\sqrt{R} - P_n = \frac{x_n}{(q_{n-1}x_n + q_{n-2})^2}\left\{2q_{n-2}\left(q_{n-1} + \frac{q_{n-2}}{x_n}\right)\sqrt{R} - (-1)^{n-1}Q_1\right\} \quad (5);$$

$$2\sqrt{R} - Q_n = \frac{1}{(q_{n-1}x_n + q_{n-2})^2}\{(x_n - 1)q_{n-1} + q_{n-2})(q_{n-1}x_n + q_{n-2})2\sqrt{R} - (-1)^{n-1}Q_1\} \quad (6).$$

The coefficients of $\sqrt{R}$ and $2\sqrt{R}$ in these four formulæ are positive, and increase without limit when n is increased without limit. Hence, since Q_1 is a fixed quantity, it follows that for

some value of n, say $n = \nu$, and for all greater values, P_n, Q_n, $\sqrt{R} - P_n$, $2\sqrt{R} - Q_n$ will all be positive. *In other words, on and after a certain value of n, $n = \nu$ say, P_n and Q_n will be positive; and $P_n < \sqrt{R}$, and $Q_n < 2\sqrt{R}$.*

Cor. 1. Since P_n and Q_n are integers, it follows that *after $n = \nu$ P_n cannot have more than $\sqrt{R}$ different values, and Q_n cannot have more than $2\sqrt{R}$ different values; so that $x_n = (P_n + \sqrt{R})/Q_n$ cannot have more than $\sqrt{R} \times 2\sqrt{R} = 2R$ different values.* In other words, after the νth complete quotient, the complete quotients must recur within 2R steps at most.

Hence the continued fraction which represents $(P_1 + \sqrt{R})/Q_1$ must recur in a cycle of 2R steps at most.

Since ever after $n = \nu$ P_n and Q_n remain positive, it is clear that *in the cycle of complete quotients there cannot occur any one in which P_n and Q_n are not both positive.*

It should be noticed that it is merely the fact that P_n and Q_n ultimately become positive that causes the recurrence.

If we knew that on and after $n = \nu$ P_n remains positive, then it would follow, from § 2 (4), that Q_ν and all following remain positive; and it would follow, from § 2 (5), that $P_{\nu+1}$ and all following are each $< \sqrt{R}$; and hence, from (4), that Q_ν and all following are each $< 2\sqrt{R}$; and we should thus establish the recurrence of the continued fraction by a somewhat different process of reasoning.

Cor. 2. Since a_n is the greatest integer in $(P_n + \sqrt{R})/Q_n$, and since, if $n > \nu$, P_n and Q_n are both positive, and $P_n < \sqrt{R}$, and $Q_n > 1$, it follows that, if $n > \nu$, $a_n < 2\sqrt{R}$.

It follows, therefore, that none of the partial quotients in the cycle can exceed the greatest integer in $2\sqrt{R}$.

Cor. 3. By means of (3) and (4), we can show *that ultimately*

$$P_n + Q_n > \sqrt{R} \qquad (7).$$

Cor. 4. From § 2 (5), we can also show *that ultimately*

$$P_n + Q_{n-1} > \sqrt{R} \qquad (8).$$

Cor. 5. Since $\sqrt{R} > P_m$, it follows from Cor. 3 and Cor. 4 *that ultimately*

$$P_m - P_n < Q_n, \quad < Q_{n-1} \qquad (9).$$

EVERY RECURRING CONTINUED FRACTION IS EQUAL TO A SIMPLE QUADRATIC SURD NUMBER.

§ 5.] We shall next prove the converse of the main proposition which has just been established, namely, we shall show that every recurring continued fraction, pure or mixed, is equal to a simple quadratic surd number.

First, let us consider the pure recurring continued fraction

$$x_1 = \underset{*}{a_1} + \frac{1}{a_2 +} \cdots \frac{1}{\underset{*}{a_r} +} \cdots \qquad (1).$$

Let the two last convergents to

$$a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r}$$

be p'/q' and p/q.

From (1) we have

$$x_1 = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r + \dfrac{1}{x_1}},$$

$$= \frac{px_1 + p'}{qx_1 + q'};$$

whence

$$qx_1^2 + (q' - p)x_1 - p' = 0 \qquad (2).$$

The quadratic equation (2) has two real roots; but one of them is negative and therefore not in question, hence the other must be the value of x_1 required.

We have, therefore,

$$x_1 = \frac{p - q' + \sqrt{(p - q')^2 + 4p'q}}{2q} \qquad (3),$$

$$= \frac{L + \sqrt{N}}{M}, \text{ say};$$

which proves the proposition in the present case.

It should be noticed that, since $a_1 \neq 0$, $p/q > 1$; so that $p > q > q'$. Hence $p - q'$ cannot vanish, and a pure recurring fraction can never represent a surd number of the form $\sqrt{N}/M$.

Next, consider the general case of a mixed recurring continued fraction.

Let

$$x_1 = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r +} \underset{*}{\frac{1}{a_1 +}} \frac{1}{a_2 +} \cdots \underset{*}{\frac{1}{a_s +}} \cdots \quad (4).$$

Also let

$$y_1 = \underset{*}{a_1 +} \frac{1}{a_2 +} \cdots \underset{*}{\frac{1}{a_s +}} \cdots \quad (5).$$

Then, by (3),

$$y_1 = \frac{L + \sqrt{N}}{M}.$$

From (4) we have

$$x_1 = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r +} \frac{1}{y_1},$$

whence, if P'/Q' and P/Q be the two last convergents to $a_1 + \dfrac{1}{a_2 +} \cdots \dfrac{1}{a_r}$,

$$x_1 = \frac{Py_1 + P'}{Qy_1 + Q'},$$

$$= \frac{PL + P'M + P\sqrt{N}}{QL + Q'M + Q\sqrt{N}} \quad (6).$$

Hence, rationalising the denominator, we deduce

$$x_1 = \frac{U + V\sqrt{N}}{W}.$$

Example 1.

Evaluate $x_1 = \underset{*}{1 +} \dfrac{1}{2 +} \dfrac{1}{\underset{*}{1 +}} \dfrac{1}{1 +} \cdots$.

The two last convergents to $1 + \dfrac{1}{2 +} \dfrac{1}{1}$ are $3/2$ and $4/3$; hence

$$x_1 = \frac{4x_1 + 3}{3x_1 + 2}.$$

We therefore have

$$3x_1^2 - 2x_1 - 3 = 0,$$

the positive root of which is

$$x_1 = \frac{1 + \sqrt{10}}{3}.$$

Example 2.

Evaluate $y_1 = 3 + \underset{*}{\frac{1}{4+}}\ \underset{*}{\frac{1}{1+}}\ \frac{1}{2+}\ \underset{*}{\frac{1}{1+}}\ \frac{1}{1+} \ldots$.

The two last convergents to $3+\frac{1}{4}$ are 3/1 and 13/4; and, by Example 1 above,

$$\underset{*}{1+}\ \frac{1}{2+}\ \underset{*}{\frac{1}{1+}} \ldots = \frac{1+\sqrt{10}}{3}.$$

We have, therefore,

$$\begin{aligned} y_1 &= 3 + \frac{1}{4+}\ \frac{1}{(1+\sqrt{10})/3}, \\ &= \frac{13(1+\sqrt{10})/3+3}{4(1+\sqrt{10})/3+1}, \\ &= \frac{22+13\sqrt{10}}{7+4\sqrt{10}}, \\ &= \frac{366-3\sqrt{10}}{111}, \\ &= \frac{122-\sqrt{10}}{37}. \end{aligned}$$

ON THE CONTINUED FRACTION WHICH REPRESENTS $\sqrt{(C/D)}$.

§ 6.] The square root of every positive rational number, say $\sqrt{(C/D)}$, where C and D are positive integers, and C/D is not the square of a commensurable number, can be put into the form $\sqrt{N}/M$, where $N = CD$ and $M = D$. Since $N/M = C$ is an integer, we know from what precedes that $\sqrt{N}/M$ can be developed, and that in one way only, as a continued fraction of the form

$$x_1 = a_1 + \frac{1}{a_2+} \ldots \frac{1}{a_r+}\ \underset{*}{\frac{1}{a_1+}}\ \frac{1}{a_2+} \ldots \underset{*}{\frac{1}{a_s+}} \ldots \quad (1).$$

We have, in fact, merely to put $P_1 = 0$, $R = N$, $Q_1 = M$ in our previous formulæ.

We suppose that $\sqrt{N}/M$ is greater than unity, so that $a_1 \neq 0$. If $\sqrt{N}/M$ were less than unity, then we have only to consider $M/\sqrt{N} = \sqrt{M^2N}/N$, which is greater than unity.

The acyclic part $a_1 + \frac{1}{a_2+} \ldots \frac{1}{a_r}$ must consist of one term at

least, for we saw, in § 5, that a pure recurring continued fraction cannot represent a surd number of the form $\sqrt{N}/M$. Let us suppose that there are at least *two* terms in this part of the fraction; and let P'/Q', P/Q be the two last convergents to $a_1 + \dfrac{1}{a_2 +} \cdots \dfrac{1}{a_r}$; and p'/q', p/q the two last convergents to $a_1 + \dfrac{1}{a_2 +} \cdots \dfrac{1}{a_r +} \dfrac{1}{a_1 +} \dfrac{1}{a_2 +} \cdots \dfrac{1}{a_s}$. Then, if

$$y_1 = \underset{*}{a_1} + \frac{1}{a_2 +} \cdots \frac{1}{\underset{*}{a_s} +} \cdots,$$

we have

$$x_1 = a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r +} \frac{1}{y_1},$$

$$= a_1 + \frac{1}{a_2 +} \cdots \frac{1}{a_r +} \frac{1}{a_1 +} \frac{1}{a_2 +} \cdots \frac{1}{a_s +} \frac{1}{y_1}.$$

Hence

$$x_1 = \frac{Py_1 + P'}{Qy_1 + Q'} = \frac{py_1 + p'}{qy_1 + q'} \qquad (2).$$

Eliminating y_1 from the equations (2), we have

$$(Qq' - Q'q)x_1^2 - (Qp' - Q'p + Pq' - P'q)x_1 + (Pp' - P'p) = 0 \qquad (3).$$

Now, if $x_1 = \sqrt{N}/M$, we must have

$$M^2x_1^2 - N = 0 \qquad (4).$$

In order that the equations (3) and (4) may agree, we must have

$$Qp' - Q'p + Pq' - P'q = 0 \qquad (5);$$

and

$$\frac{Pp' - P'p}{Qq' - Q'q} = -\frac{N}{M^2} \qquad (6).$$

It is easy to show that equation (6) cannot be satisfied. We have, in fact,

$$\frac{Pp' - P'p}{Qq' - Q'q} = \frac{P'p'}{Q'q'} \frac{P/P' - p/p'}{Q/Q' - q/q'} \qquad (7).$$

But, by chap. xxxii., § 7,

$$\frac{P}{P'} - \frac{p}{p'} = a_r + \frac{1}{a_{r-1} +} \cdots \frac{1}{a_1} - a_s - \frac{1}{a_{s-1} +} \cdots \frac{1}{a_1},$$

$$= a_r - a_s \pm f,$$

where f is a proper fraction.

Similarly

$$\frac{Q}{Q'} - \frac{q}{q'} = a_r + \frac{1}{a_{r-1}+} \cdots + \frac{1}{a_s} - a_s - \frac{1}{a_{s-1}+} \cdots \frac{1}{a_s},$$
$$= a_r - a_s \pm f',$$

where f' is a proper fraction.

Now $a_r - a_s$ cannot be zero, for, if that were so, we should have $a_r = a_s$, that is to say, the cycle of partial quotients would begin one place sooner, and would be $a_s, a_1, a_2, \ldots, a_{s-1}$, and not $a_1, a_2, \ldots, a_s$, as was supposed. It follows then that $a_r - a_s$ is a positive or negative *integral number*. Hence the signs of $P/P' - p/p'$ and $Q/Q' - q/q'$ are either both positive or both negative, and the sign of the quotient of the two is positive. Hence the left-hand side of (6) is positive, and the right-hand side negative.

There cannot, therefore, be more than one partial quotient in the acyclic part of (1).

Let us, then, write

$$x_1 = a + \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_s+} \frac{1}{a_1+} \cdots \tag{8},$$
$$= a + \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_s+} \frac{1}{1/(x_1 - a)}.$$

Hence

$$x_1 = \frac{p/(x_1 - a) + p'}{q/(x_1 - a) + q'},$$

which gives

$$q'x_1^2 - (p' + q'a - q)x_1 - (p - ap') = 0 \tag{9}.$$

From (9) we obtain

$$x_1 = \frac{p' + q'a - q}{2q'} + \frac{\sqrt{(p' + q'a - q)^2 + 4(p - ap')q'}}{2q'} \tag{10}.$$

In order that (10) may agree with $x_1 = \sqrt{N}/M$, we must have

$$p' + q'a - q = 0 \tag{11};$$

and

$$q'^2 N/M^2 = (p - ap')q' \tag{12}.$$

Cor. 1. By equation (11) we have

$$p'/q' + a = q/q'.$$

Hence, by chap. xxxii., § 7, Cor. 2,

$$2a + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\,\cdots\,\frac{1}{a_{s-1}} = a_s + \frac{1}{a_{s-1} +}\,\cdots\,\frac{1}{a_1}.$$

It follows, therefore, by chap. xxxii., § 3, that

$$a_s = 2a, \quad a_{s-1} = a_1, \quad a_{s-2} = a_2, \quad \ldots, \quad a_1 = a_{s-1}.$$

In other words, the last partial quotient of the cyclical part of the continued fraction which represents $\sqrt{N}/M$ *is double the unique partial quotient which forms the acyclical part; and the rest of the cycle is reciprocal, that is to say, the partial quotients equidistant from the two extremes are equal.*

In short, we may write

$$\frac{\sqrt{N}}{M} = a + \frac{1}{a_1 +}\,\frac{1}{a_2 +}\,\cdots\,\frac{1}{a_2 +}\,\frac{1}{a_1 +}\,\frac{1}{2a +}\,\frac{1}{a_1 +}\,\cdots \quad (13).$$

Cor. 2. *If we use the value of* $q'a$ *given by* (11), *we may throw* (12) *into the form*

$$q'^2N/M^2 = pq' - p'(q - p');$$

whence

$$q'^2N/M^2 - p'^2 = pq' - p'q, \\ = \pm 1 \quad (14),$$

the upper sign being taken if p/q *be an even convergent, the lower if it be an odd convergent.*

§ 7.] All the results already established for $(P_1 + \sqrt{R})/Q_1$ apply to $\sqrt{N}/M$. For convenience, we modify the notation as follows :—

$$\begin{aligned}
a_1 &= a, & x_1 &= (P_1 + \sqrt{R})/Q_1 = (0 + \sqrt{N})/M; \\
a_2 &= a_1, & x_2 &= (P_2 + \sqrt{R})/Q_2 = (L_1 + \sqrt{N})/M_1; \\
a_3 &= a_2, & x_3 &= (P_3 + \sqrt{R})/Q_3 = (L_2 + \sqrt{N})/M_2; \\
&\cdots & &\cdots \\
a_s &= a_{s-1}, & x_s &= (P_s + \sqrt{R})/Q_s = (L_{s-1} + \sqrt{N})/M_{s-1}; \\
a_{s+1} &= 2a, & &\cdots \\
a_{s+2} &= a_1, & &\cdots
\end{aligned}$$

From § 2 (4), we then have

$$L_n = a_{n-1}M_{n-1} - L_{n-1} \quad (1);$$

and, in particular, when $n = 1$,

$$L_1 = aM \quad (1').$$

From § 2 (5), we have

$$L_n^2 + M_{n-1}M_n = N \qquad (2);$$

and, in particular,

$$L_1^2 + MM_1 = N \qquad (2').$$

From § 3 (4) and (5), we have

$$(-1)^n L_n = (N/M)q_n q_{n-1} - Mp_n p_{n-1} \qquad (3);$$

$$(-)^n M_n = Mp_n^2 - (N/M)q_n^2 \qquad (4).$$

These formulæ are often useful in particular applications. It will be a good exercise for the student to establish them directly.

§ 8.] Let us call L_1, L_2, &c., the *Rational Dividends* and M, M_1, M_2, &c., the *Divisors* belonging to the development of $\sqrt{N}/M$.

Then, from the results of § 4, we see that

None of the rational dividends can exceed $\sqrt{N}$; *none of the partial quotients and none of the divisors can exceed* $2\sqrt{N}$.

All the rational dividends, and all the divisors, are positive.

It is, of course, obvious that *the rational dividends and the divisors form cycles collateral with the cycle of the partial and total quotients*; namely, just as we have

$$a_{s+1} = a_1, \quad a_{s+2} = a_2,$$

so we have

$$L_{s+1} = L_1, \quad L_{s+2} = L_2, \qquad (1),$$

and

$$M_{s+1} = M_1, \quad M_{s+2} = M_2, \qquad (2).$$

We can also show that the cycles of the rational dividends and of the divisors have a reciprocal property like the cycle of the partial quotients; namely, we have

$$\left.\begin{array}{llll} L_s = L_1, & M_s = M; \\ L_{s-1} = L_2, & M_{s-1} = M_1; \\ L_{s-2} = L_3, & M_{s-2} = M_2; \\ \cdots & \cdots \end{array}\right\} \qquad (3).$$

For, by § 7 (2),

$$L_{s+1}^2 + M_{s+1}M_s = L_1^2 + M_1 M;$$

but $L_{s+1} = L_1$ and $M_{s+1} = M_1$, hence

$$M_s = M \qquad (4).$$

Again, by § 7 (1),

$$L_{s+1} = a_s M_s - L_s ;$$

but $L_{s+1} = L_1$, $a_s = 2a$, $M_s = M$, hence we have

$$L_1 = 2aM - L_s.$$

Now, by § 7 (1'), $L_1 = aM$, hence

$$L_1 = 2L_1 - L_s,$$

therefore

$$L_s = L_1 \qquad (5).$$

Again, by § 7 (2),

$$L_s^2 + M_s M_{s-1} = L_1^2 + M_1 M,$$

whence, bearing in mind what we have already proved, we have

$$M_{s-1} = M_1 \qquad (6).$$

Once more, by § 7 (1),

$$L_s = a_{s-1} M_{s-1} - L_{s-1},$$
$$L_2 = a_1 M_1 - L_1.$$

Now $M_{s-1} = M_1$ and $a_{s-1} = a_1$, hence

$$L_s - L_2 = L_1 - L_{s-1}.$$

But $L_s = L_1$, hence

$$L_{s-1} = L_2.$$

Proceeding step by step, in this way, we establish all the equations (3).

It appears, then, that we may write the cycles of the rational dividends and of the divisors thus—

$$\begin{array}{l} L_1, \ L_2, \ L_3, \ \ldots, \qquad L_3, \ L_2, \ L_1 ; \\ M_1, \ M_2, \ M_3 \ \ldots, \ M_3, \ M_2, \ M_1, \ M. \end{array}$$

Since M precedes M_1, we may make the cycle of the divisors commence one step earlier, and we thus have for partial quotients, rational dividends, and divisors the following cycles :—

$$\begin{array}{rl} a_1, \ a_2, \ a_3, \ \ldots, \ a_3, \ a_2, \ a_1, & 2a \ ; \ a_1. \\ L_1, \ L_2, \ L_3, \ \ldots, \qquad L_3, \ L_2, \ L_1 \ ; & L_1. \\ M, \ M_1, \ M_2 \ M_3, \ \ldots, \qquad M_3, \ M_1 \ ; & M, \ M_1. \end{array}$$

That is to say, *the cycle of the rational dividends is collateral with the cycle of the partial quotients, and is completely reciprocal; the*

cycle of the divisors begins one step earlier (that is, from the very beginning), and is reciprocal after the first term.*

§ 9.] The following theorem forms, in a certain sense, a converse to the propositions just established regarding the cycles of the continued fraction which represents $\sqrt{N}/M$.

$$\begin{array}{llll} \textit{If} & L_m = L_{n+1}, & M_m = M_n, & a_m = a_n, \\ \textit{then} & L_{m-1} = L_{n+2}, & M_{m-1} = M_{n+1}, & a_{m-1} = a_{n+1} \end{array} \tag{1}$$

We have, by § 7 (2),

$$L_m^2 + M_m M_{m-1} = L_{n+1}^2 + M_{n+1} M_n,$$

whence, remembering our data, we deduce

$$M_{m-1} = M_{n+1} \tag{2}$$

Again, by § 7 (1),

$$L_m + L_{m-1} = a_{m-1} M_{m-1},$$
$$L_{n+2} + L_{n+1} = a_{n+1} M_{n+1},$$

whence, since $L_m = L_{n+1}$ by data,

$$\begin{aligned} L_{m-1} - L_{n+2} &= (a_{m-1} - a_{n+1}) M_{m-1}, \\ &= (a_{m-1} - a_{n+1}) M_{n+1} \end{aligned} \tag{3}$$

If $L_{m-1} > L_{n+2}$, we may write (3)

$$(L_{m-1} - L_{n+2})/M_{n+1} = a_{m-1} - a_{n+1} \tag{4};$$

if $L_{m-1} < L_{n+2}$ we may write

$$(L_{n+2} - L_{m-1})/M_{m-1} = a_{n+1} - a_{m-1} \tag{5}.$$

But, by § 4 (9), the left-hand sides of (4) and (5) (if they differ from 0) are each < 1, while the right-hand sides are each positive integers (if they differ from 0).

It follows, then, that each side of equation (3) must vanish, so that

$$L_{m-1} = L_{n+2} \tag{6},$$

$$a_{m-1} = a_{n+1} \tag{7},$$

which completes the proof.

---

* The fact that the cycle of the divisors begins one step earlier than the cycles of the partial quotients and rational dividends is true for the general recurring continued fraction. Several other propositions proved for the special case now under consideration have a more general application. The circumstances are left for the reader himself to discover.

Cor. 1. *Starting with the equations in the second line of* (1) *as data, we could in like manner prove that*

$$L_{m-s} = L_{n+s}, \quad M_{m-s} = M_{n+s}, \quad a_{m-s} = a_{n+s};$$

and so on, forwards and backwards.

Cor. 2. If we put $m = n$, the conditions in (1) become

$$L_n = L_{n+1}, \quad M_n = M_n, \quad a_n = a_n;$$

in other words, the conditions reduce to

$$L_n = L_{n+1};$$

and the conclusion becomes

$$L_{n-1} = L_{n+s}, \quad M_{n-1} = M_{n+1}, \quad a_{n-1} = a_{n+1}.$$

Hence, if two consecutive rational dividends be equal, they are the middle terms of the cycle of rational dividends, which must therefore be an even cycle; and the partial quotient and divisor corresponding to the first of the two rational dividends will be the middle terms of their respective cycles, which must therefore be odd cycles.

Cor. 3. If we put $m = n + 1$, the conditions in (1) reduce to

$$M_{n+1} = M_n, \quad a_{n+1} = a_n;$$

and the conclusion gives

$$L_n = L_{n+s}, \quad M_n = M_{n+1}, \quad a_n = a_{n+1}.$$

Using this conclusion as data in (1), we have as conclusion

$$L_{n-1} = L_{n+s}, \quad M_{n-1} = M_{n+s}, \quad a_{n-1} = a_{n+s};$$

and so on.

Hence, if two consecutive divisors (M_n, M_{n+1}) *be equal, and also the two corresponding partial quotients* (a_n, a_{n+1}) *be equal, these two pairs are the middle terms of their respective cycles, which are both even; and the rational dividend* (L_{n+1}) *corresponding to the second member of either pair is the middle term of its cycle, which is odd.*

These theorems enable us to save about half the labour of calculating the constituents of the continued fraction which represents $\sqrt{N}/M$. In certain cases they are useful also in reducing surds of the more general form $(L + \sqrt{N})/M$ to continued fractions.

Example 1.

Express $\sqrt{8463}/39$ as a simple continued fraction; and exhibit the cycles of the rational dividends and of the divisors.

We have

$$\frac{\sqrt{8463}}{39}=2+\frac{-78+\sqrt{8463}}{39}=2+\frac{1}{(78+\sqrt{8463})/61};$$

$$\frac{78+\sqrt{8463}}{61}=2+\frac{-44+\sqrt{8463}}{61}=2+\frac{1}{(44+\sqrt{8463})/107};$$

$$\frac{44+\sqrt{8463}}{107}=1+\frac{-63+\sqrt{8463}}{107}=1+\frac{1}{(63+\sqrt{8463})/42};$$

$$\frac{63+\sqrt{8463}}{42}=3+\frac{-63+\sqrt{8463}}{42}=3+\frac{1}{(63+\sqrt{8463})/107};$$

$$\frac{63+\sqrt{8463}}{107}=1+\text{ \&c.}$$

Since we have now two successive rational dividends each equal to 63, we know that the cycle of partial quotients has culminated in 3. Hence the cycles of partial quotients, rational dividends, and divisors are—

Partial quotients . .	2,	1,	3,	1,	2,	4;
Rational dividends .	78,	44,	63,	63,	44,	78;
Divisors . . .	39,	61,	107,	42,	107,	61;

and we have

$$\frac{\sqrt{8463}}{39}=2+\underset{*}{\frac{1}{2+}}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{2+}\ \underset{*}{\frac{1}{4+}}\cdots.$$

Example 2.

If c denote the number of partial quotients in the cycle of the continued fraction which represents $\sqrt{\mathrm{N}}/\mathrm{M}$, prove the following formulæ:—

If $c=2t$,

$$\frac{p_c}{q_c}=\frac{p_{t+1}q_t+p_tq_{t-1}}{q_t(q_{t+1}+q_{t-1})} \qquad \text{(I.)};$$

if $c=2t+1$,

$$\frac{p_c}{q_c}=\frac{p_{t+1}q_{t+1}+p_tq_t}{{q_{t+1}}^2+{q_t}^2} \qquad \text{(II.)};$$

if m be any positive integer,

$$\frac{p_{2mc}}{q_{2mc}}=\frac{{p_{mc}}^2+(\mathrm{N}/\mathrm{M}^2){q_{mc}}^2}{2p_{mc}q_{mc}} \qquad \text{(III.)}$$

For brevity we shall prove III. alone. The reader will find that I. and II. may be proved in a similar manner. For a different kind of demonstration, see chap. xxxiv., § 6.

We have

$$\frac{p_{2mc}}{q_{2mc}}=a+\frac{1}{a_1+}\cdots\frac{1}{a_1+}\ \frac{1}{2a+}\cdots\frac{1}{a_1}\ (2m \text{ cycles}),$$

$$=a+\frac{1}{a_1+}\cdots\frac{1}{a_1+}\ \frac{1}{2a+}\cdots\frac{1}{a_1+}\ \frac{1}{a+p_{mc}/p_{mc}}\ (m \text{ cycles}),$$

$$=\frac{(a+p_{mc}/q_{mc})p_{mc}+p_{mc-1}}{(a+p_{mc}/q_{mc})q_{mc}+q_{mc-1}},$$

$$=\frac{(ap_{mc}+p_{mc-1})q_{mc}+{p_{mc}}^2}{q_{mc}(aq_{mc}+q_{mc-1}+p_{mc})} \qquad (a)$$

Now the equations (2) and (3) of § 3 give us

$$\left.\begin{aligned} q_{mc}\,P_{mc+1}+q_{mc-1}\,Q_{mc+1}&=Mp_{mc}\\ p_{mc}\,P_{mc+1}+p_{mc-1}\,Q_{mc+1}&=(N/M)\,q_{mc}\end{aligned}\right\}\qquad(\beta).$$

In the present case,

$$\begin{aligned} P_{mc+1}=P_{c+1}=L_c=L_1&=aM,\\ Q_{mc+1}=Q_{c+1}=M_c\quad&=M.\end{aligned}$$

The equations (β) therefore give

$$\left.\begin{aligned} aq_{mc}+q_{mc-1}&=p_{mc}\\ ap_{mc}+p_{mc-1}&=(N/M^2)q_{mc}\end{aligned}\right\}\qquad(\gamma).$$

From (α) and (γ) (III.) follows at once.

The formulæ (I.), (II.), (III.) enable us, after a certain number of convergents to $\sqrt{N}/M$ have been calculated, to calculate high convergents without finding all the intermediate ones.

Consider, for example,

$$\frac{\sqrt{8463}}{39}=2+\underset{*}{\frac{1}{2+}}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\underset{*}{\frac{1}{4+}}\,.$$

Here $c=6$, $t=3$, and we have for the first four convergents 2/1, 5/2, 7/3, 26/11; hence

$$\frac{p_6}{q_6}=\frac{p_4q_3+p_3q_2}{q_3(q_4+q_2)},$$

$$=\frac{26\times3+7\times2}{3(11+2)}=\frac{92}{39}.$$

Also

$$\frac{p_{12}}{q_{12}}=\frac{p_6^2+(N/M^2)q_6^2}{2p_6q_6},$$

$$=\frac{92^2+(8463/39^2).39^2}{2\times92\times39}=\frac{16927}{7176};$$

$$\frac{p_{24}}{q_{24}}=\frac{p_{12}^2+(N/M^2)q_{12}^2}{2p_{12}q_{12}},$$

$$=\frac{16927^2\times39^2+8463\times7176^2}{2\times39^2\times16927\times7176}.$$

The rapidity and elegance of this method of forming rational approximations cannot fail to strike the reader.

Exercises XXXI.

Express the following surd numbers as simple continued fractions, and exhibit the cycles of the partial quotients, rational dividends, and divisors:—

(1.) $\sqrt{(101)}$. (2.) $\frac{1}{3}\sqrt{(63)}$. (3.) $\sqrt{(\frac{1}{11})}$.

(4.) $\dfrac{1}{\sqrt{(61)}}$. (5.) $\dfrac{2+\sqrt{(29)}}{5}$. (6.) $1+\sqrt{\frac{3}{4}}$.

(7.) Express the positive root of $x^2-x-4=0$ as a continued fraction, and find the 6th convergent to it.

(8.) Express both roots of $2x^2-6x-1=0$ as continued fractions, and point out the relations between the various cycles in the two fractions.

(9.) Show that

$$\sqrt{(a^2+b)} = a + \underset{*}{\frac{b}{2a+}} \cdots;$$

$$\sqrt{(a^2-b)} = a - \underset{*}{\frac{b}{2a-}} \cdots.$$

(10.) Express $\sqrt{(a^2+1)}$ as a simple continued fraction, and find an expression for the nth convergent.

Evaluate the following recurring continued fractions, and find, where you can, closed expressions for their nth convergents; also obtain recurring formulæ for simplifying the calculation of high convergents:—

(11.) $$a + \underset{*}{\frac{1}{a+}} \cdots.$$

(12.) $$\underset{*}{\frac{1}{a-}} \cdots.$$

(13.) $$\underset{*}{\frac{1}{a+}} \underset{*}{\frac{1}{b+}} \cdots.$$

Show, in this case, that

$$p_{2n+2} - 2p_{2n} + p_{2n-2} = abp_{2n}.$$

(14.) $$\underset{*}{\frac{1}{1+}} \underset{*}{\frac{1}{1+}} \cdots \underset{*}{\frac{1}{2+}} \cdots,$$

where the cycle consists of n units followed by 2.

(15.) Show that

$$\left(\underset{*}{\frac{1}{x+}} \underset{*}{\frac{1}{4x+}} \cdots\right)\left(\underset{*}{\frac{1}{2x+}} \cdots\right)$$

is independent of x.

(16.) Show that

$$\left(x + \underset{*}{\frac{1}{2x+}} \cdots\right)^2 - \left(x - \underset{*}{\frac{1}{2x-}} \cdots\right)^2 = 2.$$

(17.) If $$x = \underset{*}{a+} \frac{1}{b+} \frac{1}{c+} \cdots, \quad y = \underset{*}{b+} \frac{1}{c+} \frac{1}{a+} \cdots,$$

$$z = \underset{*}{c+} \frac{1}{a+} \frac{1}{b+} \cdots, \quad u = \underset{*}{a+b+c+} \frac{1}{a+b+c+} \cdots,$$

show that

$$\frac{2(x+y+z)-(a+b+c)}{2u-(a+b+c)-abc} = \frac{1}{bc+1} + \frac{1}{ca+1} + \frac{1}{ab+1}.$$

(18.) Show that

$$\left(\underset{*}{\frac{a}{b+}} \cdots\right)^2 = \underset{*}{\frac{a^2}{2a+b^2-}} \cdots.$$

(19.) If p be the numerator of any convergent to $\sqrt{2}$, then $2p^2 \pm 1$ will also be the numerator of a convergent, the upper or lower sign being taken according as p/q is an odd or an even convergent; also, if q, q' be two consecutive denominators, $q^2+q'^2$ will be a denominator.

(20.) Evaluate

$$\frac{1}{1+}\,\frac{1}{1+}\cdots\frac{1}{n+}\cdots,$$

where the cycle consists of $\overline{n-1}$ units followed by n.

(21.) In the case of $\dfrac{1}{1+}\,\dfrac{1}{4+}\cdots$, prove that

$$p_{2n} = q_{2n+1} = \{(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}\}/2\sqrt{2},$$
$$p_{2n-1} = \tfrac{1}{2}q_{2n} = \{(\sqrt{2}+1)^{2n} - (\sqrt{2}-1)^{2n}\}/4\sqrt{2}.$$

(22.) Convert the positive root of $ax^2+abx-b=0$ into a simple continued fraction; and show that p_n and q_n are the coefficients of x^n in $(x+bx^2-x^4)/(1-\overline{ab+2}.x^2+x^4)$ and $(ax+\overline{ab+1}.x^2+x^4)/(1-\overline{ab+2}.x^2+x^4)$ respectively.

Hence, or otherwise, show that if α, β be the roots of $1-(ab+2)z+z^2=0$, then

$$ap_{2n} = bq_{2n-1} = ab\frac{\alpha^n-\beta^n}{\alpha-\beta},$$
$$p_{2n+1} = q_{2n} = \frac{(\alpha^{n+1}-\beta^{n+1})-(\alpha^n-\beta^n)}{\alpha-\beta}.$$

(23.) If the number of quotients in the cycle of

$$\sqrt{\frac{N}{M}} = a+\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_2+}\,\frac{1}{a_1+}\,\frac{1}{2a+}\cdots \text{ be } c,$$

show that

$$a+\frac{1}{a_1+}\cdots\frac{1}{a_1+}\,\frac{1}{2a+}\,\frac{1}{a_1+}\cdots\frac{1}{a_1+}\,\frac{1}{a} \ (m \text{ cycles}) = \frac{Nq_{mc}}{M^2p_{mc}}.$$

(24.)* If c be the number of quotients in the cycle of $\sqrt{N/M}$, show that if $c=2t+1$,

$$\frac{p^2_{t-r-1}+p^2_{t+r}}{q^2_{t-r-1}+q^2_{t+r}} = \frac{N}{M^2},$$

$r=0,\ 1,\ \ldots,\ t-1$;

and if $c=2t$,

$$\frac{p_{t-r-2}p_{t-r-1}+p_{t+r-1}p_{t+r}}{q_{t-r-2}q_{t-r-1}+q_{t+r-1}q_{t+r}} = \frac{N}{M^2}.$$

(25.)† If $\sqrt{Z} = a+\dfrac{1}{a_1+}\,\dfrac{1}{a_2+}\cdots\dfrac{1}{a_2+}\,\dfrac{1}{a_1+}\,\dfrac{1}{2a+}\cdots$, and if the convergent

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* For solutions of Exercises 24 and 26-29 see Muir's valuable little tract on *The Expression of a Quadratic Surd as a Continued Fraction*, Glasgow (Maclehose), 1874.

† In connection with Exercises 25 and 30-32 see Serret's *Cours d'Algèbre Supérieure*, 3me éd., t. i., chaps. i. and ii.

obtained by taking 1, 2, . . ., i periods, ending in each case with a_1, be $Z_1, Z_2, \ldots, Z_i$, and if $Z_1 = P_1/Q_1, \ldots, Z_i = P_i/Q_i$, P_1 and Q_1 being integers prime to each other as usual, then

$$P_i - Q_i\sqrt{Z} = (P_{i-1} - Q_{i-1}\sqrt{Z})(P_1 - Q_1\sqrt{Z}),$$
$$= (P_1 - Q_1\sqrt{Z})^i;$$
$$\frac{Z_i + \sqrt{Z}}{Z_i - \sqrt{Z}} = \left(\frac{Z_1 + \sqrt{Z}}{Z_1 - \sqrt{Z}}\right)^i.$$

(26.) If N be an integer, and if a cyclical partial quotient occur in the development of $\sqrt{N}$ equal to the acyclic partial quotient a, that quotient will be the middle term of the reciprocal part of the cycle; and no cyclical partial quotient can occur lying between a and $2a$.

(27.) When N is a prime integer, the cycle of partial quotients is even, and the middle term of the reciprocal part of the cycle is a or $a-1$, according as a is odd or even.

(28.) If N be an integer, and the cycle of $\sqrt{N}$ be odd, then A is the sum of the squares of two integers which are prime to each other.

Exhibit 365, as the sum of two squares.

(29.) The general expression for every integer whose square root has a cycle of c terms, the reciprocal part of which has the terms $a_1, a_2, \ldots, a_2, a_1$, is

$$(\tfrac{1}{2}pm - (-1)^c p'q')^2 + p'm - (-1)^c q'^2,$$

where m is any positive integer, and p'/q', p/q are the two last convergents to

$$a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_2 + \cfrac{1}{a_1}}}.$$

Find an expression for all the integers that have 1, 2, 1 for the reciprocal part of the cycle of their square root.

(30.) If two positive irrational quantities, x and x', can be developed in continued fractions which are identical on and after a certain constituent, show that

$$x' = (ax + b)/(a'x + b'),$$

where a, b, a', b', are integers such that $ab' - a'b = \pm 1$; and that this condition is sufficient.

(31.) The equation of the 2nd degree with rational coefficients which is satisfied by a given recurring continued fraction has its roots of opposite signs if the fraction is purely recurring, and of the same sign if it is mixed and has more than one acyclic partial quotient.

(32.) Investigate the relation between the cycles of the partial and complete quotients of the two continued fractions which represent the numerical values of the two roots of an equation of the 2nd degree with rational coefficients.

Illustrate with $27x^2 - 97x + 77 = 0$.

APPLICATIONS TO THE SOLUTION OF DIOPHANTINE PROBLEMS.

§ 10.] When an equation or a system of equations is indeterminate, we may limit the solution by certain extraneous conditions, and then the indeterminateness may become less in degree or may cease, or it may even happen that there is no solution at all of the kind demanded.

Thus, for example, we may require (I.) that the solution be in rational numbers; (II.) that it be in integral numbers; or, still more particularly, (III.) that it be in positive integral numbers. Problems of this kind are called Diophantine Problems, in honour of the Alexandrine mathematician Diophantos, who, so far as we know, was the first to systematically discuss such problems, and who showed extraordinary skill in solving them.* We shall confine ourselves here mainly to the third class of Diophantine problems, where positive integral solutions are required, and shall consider the first and second classes merely as stepping-stones toward the solution of the third. We shall also treat the subject merely in so far as it illustrates the use of continued fractions: its complete development belongs to the higher arithmetic, on which it is beyond the purpose of the present work to enter.†

Equations of the 1st Degree in Two Variables.

§ 11.] Since we are ultimately concerned only with positive integral solutions, we need only consider equations of the form $ax \pm by = c$, where a, b, c are positive integers. We shall suppose that any factor common to the three coefficients has been

* See Heath's *Diophantos of Alexandria* (Camb. 1885).

† The reader who wishes to pursue the study of the higher arithmetic should first read the late Henry Smith's series of Reports on the Theory of Numbers, published in the Annual Reports of the British Association (1859-60-61-62); then Legendre, *Théorie des Nombres*; Dirichlet's *Vorlesungen über Zahlentheorie*, ed. by Dedekind; and finally Gauss's *Disquisitiones Arithmeticæ*. He will then be in a position to master the various special memoirs in which Jacobi, Hermite, Kummer, Henry Smith, and others have developed this great branch of pure mathematics.

removed. We may obviously confine ourselves to the cases where a is prime to b; for, if x and y be integers, any factor common to a and b must be a factor in c. In other words, if a be not prime to b, the equation $ax \pm by = c$ has no integral solution.

§ 12.] *To find all the integral solutions of* $ax - by = c$; *and to separate the positive integral solutions.*

We can always find a particular integral solution of

$$ax - by = c \tag{1}.$$

For, since a is prime to b, if we convert a/b into a continued fraction, its last convergent will be a/b. Let the penultimate convergent be p/q, then, by chap. xxxii., § 8,

$$aq - pb = \pm 1 \tag{2}.$$

Therefore

$$a(\pm cq) - b(\pm cp) = c \tag{3}.$$

Hence

$$x' = \pm cq, \quad y' = \pm cp \tag{4}$$

is a particular integral solution of (1).

Next, let (x, y) be any integral solution of (1) whatever. Then from (1) and (3) by subtraction we derive

$$a\{x - (\pm cq)\} - b\{y - (\pm cp)\} = 0.$$

Therefore

$$\{x - (\pm cq)\}/\{y - (\pm cp)\} = b/a \tag{5}.$$

Since a is prime to b, it follows from (5), by chap. iii., Exercises IV., 1, that

$$x - (\pm cq) = bt, \quad y - (\pm cp) = at,$$

where t is zero or some integer positive or negative. Hence every integral solution of (1) is included in

$$x = \pm cq + bt, \quad y = \pm cp + at \tag{6},$$

where the upper or lower sign must be taken according as the upper or lower sign is to be taken in (2).

Finally, let us discuss the number of possible integral solutions, and separate those which are positive.

1°. If $a/b > p/q$, then the upper sign must be taken in (2), and we have

$$x = cq + bt, \quad y = cp + at \tag{6'}.$$

There are obviously an infinity of integral solutions. To get positive values for x and y we must (since $cp/a < cq/b$) give to t values such that $-cp/a \not> t \not> +\infty$. There are, therefore, an infinite number of positive integral solutions.

2°. If $a/b < p/q$, so that $cp/a > cq/b$, we must write

$$x = -cq + bt, \quad y = -cp + at \qquad (6'').$$

All our conclusions remain as before, except that for positive solutions we must have $cp/a \not> t \not> +\infty$.

We see, therefore, *that $ax - by = c$ has in all cases an infinite number of positive integral solutions.*

§ 13.] *To find all the integral solutions of*

$$ax + by = c \qquad (7),$$

and to separate the positive integral solutions.

We can always find an integral solution of (7); for, if p and q have the same meaning as in last paragraph, we have

$$(\pm cq)a + (\mp cp)b = c \qquad (8),$$

that is, $x' = \pm cq$, $y' = \mp cp$ is a particular integral solution of (7).

By exactly the same reasoning as before, we show that all the integral solutions of (7) are given by

$$x = \pm cq - bt, \quad y = \mp cp + at \qquad (9);$$

so that there are in this case also an infinity of integral solutions.

To get the positive integral solutions:—

1°. Let us suppose that $a/b > p/q$, so that $cp/a < cq/b$. Then the general solution is

$$x = cq - bt, \quad y = -cp + at \qquad (9').$$

Hence for positive integral solutions we must have $cp/a \not> t \not> cq/b$.

2°. Let us suppose that $a/b < p/q$, so that $cp/a > cq/b$, then

$$x = -cq - bt, \quad y = cp + at \qquad (9'').$$

Hence for positive integral solutions we must have $-cp/a \not> t \not> -cq/b$.

In both these cases the number of positive integral solutions is limited. In fact, the number of such solutions cannot exceed $1 + \text{mod}\,(cq/b - cp/a)$; that is, since $\text{mod}\,(aq - pb) = 1$, *the number of positive integral solutions of the equation* $ax + by = c$ *cannot exceed* $1 + c/ab$.

Example 1. To find all the integral and all the positive integral solutions of $8x + 13y = 159$.

We have

$$\frac{8}{13} = \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}.$$

The penultimate convergent is $3/5$; and we have

$$8 \times 5 - 13 \times 3 = 1,$$
$$8(795) + 13(-477) = 159.$$

Hence a particular solution of the given equation is $x' = 795$, $y' = -477$; and the general solution is

$$x = 795 - 13t, \quad y = -477 + 8t.$$

For positive integral solutions we must have $795/13 < t < 477/8$, that is, $61\frac{2}{13} < t < 59\frac{5}{8}$. The only admissible values of t are therefore 60 and 61; these give $x = 15$, $y = 3$, and $x = 2$, $y = 11$, which are the only positive integral solutions.

Example 2. Find all the positive integral solutions of $3x + 2y + 3z = 8$.

We may write this equation in the form

$$3x + 2y = 8 - 3z,$$

rom which it appears that those solutions alone are admissible for which $z = 0$, 1, or 2.

The general integral solution of the given equation is obviously

$$x = 8 - 3z - 2t, \quad y = -8 + 3z + 3t.$$

In order to obtain positive values for x and y, we must give to t integral values lying between $+4 - \frac{3}{2}z$ and $+2\frac{2}{3} - z$. The admissible values of t are 3 and 4, when $z = 0$; 2, when $z = 1$; and 1, when $z = 2$. Hence the only positive integral solutions are

$$\begin{array}{llll} x = 2, & 0, & 1, & 0; \\ y = 1, & 4, & 1, & 1; \\ z = 0, & 0, & 1, & 2. \end{array}$$

In a similar way we may treat any single equation involving more than two variables.

§ 14.] Any system of equations in which the number of variables exceeds the number of equations may be treated by methods which depend ultimately on what has been already done.

Consider, for example, the system

$$ax + by + cz = d \quad (1),$$
$$a'x + b'y + c'z = d' \quad (2),$$

where a, b, c, d, a', &c. denote any integers positive or negative.

This system is equivalent to the following :—

$$-(ca')x + (bc')y = (dc') \quad (3),$$
$$ax + by + cz = d \quad (4),$$

where (ca') stands for $ca' - c'a$, &c.

Let δ be the G.C.M. of the integers (ac'), (bc'). Then, if δ be not a factor in (dc'), (3) has no integral solution, and consequently the system (1) and (2) has no integral solution.

If, however, δ be a factor in (dc'), then (3) will have integral solutions the general form of which is

$$x = x'' + (bc')t/\delta, \quad y = y'' + (ca')t/\delta \quad (5),$$

where (x'', y'') is any particular integral solution of (3), and t is any integer whatever.

If we use (5) in (4), we reduce (4) to

$$cz - c(ab')t/\delta = d - ax'' - by'' \quad (6),$$

where $c(ab')/\delta$ is obviously integral.

In order that the system (1), (2) may be soluble in integers, (6) must have an integral solution. Let any particular solution of (6) be $z = z'$, $t = t'$. Then

$$\frac{z - z'}{t - t'} = \frac{(ab')}{\delta}.$$

Hence, if ϵ be the G.C.M. of (ab') and δ, that is, the G.C.M. of (bc'), (ca'), (ab'), then

$$z = z' + (ab')u/\epsilon, \quad t = t' + \delta u/\epsilon \quad (7),$$

where u is any integer.

From (5) and (7) we now have

$$x = x' + (bc')u/\epsilon, \quad y = y' + (ca')u/\epsilon, \quad z = z' + (ab')u/\epsilon \quad (8),$$

where $x' = x'' + (bc')t'/\delta$, $y' = y'' + (ca')t'/\delta$.

If in (8) we put $u = 0$, we get $x = x'$, $y = y'$, $z = z'$; therefore (x', y', z') is a particular integral solution of the system (1), (2). A little consideration will show that we might replace (x', y', z') by *any* particular integral solution whatever. Hence (8) *gives all*

the integral solutions of (1), (2), (x', y', z') *being any particular integral solution,* ϵ *the G.C.M. of* (bc'), (ca'), (ab'), *and* u *any integer whatever.*

The positive integral solutions can be found by properly limiting u.

Example.

$$3x+4y+27z=34, \quad 3x+5y+21z=29.$$

Here $(bc')=-51$, $(ca')=18$, $(ab')=3$. Hence $\epsilon=3$; a particular integral solution is (1, 1, 1); and we have for the general integral solution

$$x=1-17u, \quad y=1+6u, \quad z=1+u.$$

The only positive integral solution is $x=1$, $y=1$, $z=1$.

Equations of the 2nd Degree in Two Variables.

§ 15.] It follows from § 7 (4) that, if p_n/q_n be the nth convergent and M_n the $(n+1)$th rational divisor belonging to the development of $\sqrt{(C/D)}$ as a simple periodic continued fraction, then

$$Dp_n^2 - Cq_n^2 = (-)^n M_n \qquad (1).$$

Hence the equation $Dx^2 - Cy^2 = +H$, *where* C, D, H *are positive integers, and* C/D *is not a perfect square, admits of an infinite number of integral solutions provided its right-hand side occurs among the quantities* $(-)^n M_n$ *belonging to the simple continued fraction which represents* $\sqrt{(C/D)}$; *and the same is true of the equation* $Dx^2 - Cy^2 = -H$.

The most important case of this proposition arises when we suppose D = 1. We thus get the following result:—

The equation $x^2 - Cy^2 = \pm H$, *where* C *and* H *are positive integers, and* C *is not a perfect square, admits of an infinite number of integral solutions provided its right-hand side occurs among the quantities* $(-)^n M_n$ *belonging to the development of* $\sqrt{C}$ *as a simple continued fraction.*

Cor. 1. *The equation* $x^2 - Cy^2 = 1$, *where* C *is positive and not a perfect square, always admits of an infinite number of solutions.**

* By what seems to be a historical misnomer, this equation is commonly spoken of as the Pellian Equation. It was originally proposed by Fermat as a challenge to the English mathematicians. Solutions were obtained by

For, if the number of quotients in the period of $\sqrt{C}$ be even, $= 2s$ say, then $(-)^{2s}M_{2s}$ will be $+1$ (since here $M = +1$). Therefore we have

$$p_{2ts}^2 - Cq_{2ts}^2 = +1,$$

where t is any positive integer; that is to say, we have the system of solutions

$$x = p_{2ts}, \quad y = q_{2ts} \qquad \text{(A)},$$

for the equation $x^2 - Cy^2 = 1$.

If the number of quotients in the period be odd, $= 2s - 1$ say, then $(-)^{2s-1}M_{2s-1}$ will be -1, but $(-)^{4s-2}M_{4s-2}$, $(-)^{8s-4}M_{8s-4}$, . . . will each be $+1$. Hence we shall have the system of solutions

$$x = p_{4ts-2t}, \quad y = q_{4ts-2t} \qquad \text{(B)},$$

for the equation $x^2 - Cy^2 = 1$.

Cor. 2. *The equation* $x^2 - Cy^2 = -1$ *admits of an infinite number of integral solutions provided there be an odd number of quotients in the period of* $\sqrt{C}$.

§ 16.] In dealing with the equation

$$x^2 - Cy^2 = \pm H \qquad (1)$$

we may always confine ourselves to what are called *primitive solutions*, that is, those for which x is prime to y. For, if x and y have a common factor θ, then θ^2 must be a factor in H, and we could reduce (1) to $x'^2 - Cy'^2 = \pm H/\theta^2$. In this way, we could make the complete solution of (1) depend on the primitive solutions of as many equations like $x'^2 - Cy'^2 = \pm H/\theta^2$ as H has square divisors.

We shall therefore, in all that follows, suppose that x is prime to y, from which it results that x and y are prime to H.

With this understanding, we can prove the following important theorem:—

If $H < \sqrt{C}$, *all the solutions of* (1) *are furnished by the convergents to* $\sqrt{C}$ *according to the method of* § 15.

This amounts to proving that, if $x = p$, $y = q$ be any primitive integral solution of (1), then p/q is a convergent to $\sqrt{C}$.

Brouncker and Wallis. The complete theory, of which the solution of this equation is merely a part, was given by Lagrange in a series of memoirs which form a landmark in the theory of numbers. See especially *Œuvres*, t. ii., p. 377.

Now we have, if the upper sign be taken,

$$p^2 - Cq^2 = H.$$

Hence
$$\begin{aligned} p/q - \sqrt{C} &= H/q(p + \sqrt{C}q), \\ &< \sqrt{C}/q(p + \sqrt{C}q), \\ &< 1/q^2(p/q\sqrt{C} + 1) \end{aligned} \qquad (2).$$

Now $p/q - \sqrt{C}$ is positive, therefore $p/q\sqrt{C} > 1$. Hence

$$p/q - \sqrt{C} < 1/2q^2 \qquad (3).$$

It follows, therefore, by chap. xxxii., § 9, Cor. 4, that p/q is one of the convergents to $\sqrt{C}$.

If the lower sign be taken, we have

$$q^2 - (1/C)p^2 = H/C,$$

where $H/C < \sqrt{(1/C)}$. We can therefore prove, as before, that q/p is one of the convergents to $\sqrt{(1/C)}$, from which it follows that p/q is one of the convergents to $\sqrt{C}$.

Cor. 1. *All the solutions of*

$$x^2 - Cy^2 = 1 \qquad (4)$$

are furnished by the penultimate convergents in the successive or alternate periods of $\sqrt{C}$.

Cor. 2. *If the number of quotients in the period of* $\sqrt{C}$ *be even, the equation*

$$x^2 - Cy^2 = -1 \qquad (5)$$

has no integral solution. If the number of quotients in the period be odd, all the integral solutions are furnished by the penultimate convergents in the alternate periods of $\sqrt{C}$.

§ 17.] We have seen that all the integral solutions of the equation (4) are derivable from the convergents to $\sqrt{C}$; it is easy to give a general expression for all the solutions in terms of the first one, say (p, q). If we put

$$\left.\begin{aligned} x + y\sqrt{C} &= (p + q\sqrt{C})^n \\ x - y\sqrt{C} &= (p - q\sqrt{C})^n \end{aligned}\right\} \qquad (6),$$

we have

$$x^2 - Cy^2 = (p^2 - Cq^2)^n = 1.$$

Hence (6) gives a solution of (4).

In like manner, if n be any integer, and (p, q) the first solution of (5), a more general solution is given by

$$\left.\begin{aligned} x + y\sqrt{C} &= (p + q\sqrt{C})^{2n-1} \\ x - y\sqrt{C} &= (p - q\sqrt{C})^{2n-1} \end{aligned}\right\} \qquad (7).$$

Finally, if (p, q) be the first solution of (1), we may express all the solutions derivable therefrom* by means of the general solution (6) of the equation (4). For, if (r, s) be any solution whatever of (4), we have

$$\begin{gathered} p^2 - Cq^2 = \pm H, \\ r^2 - Cs^2 = 1 ; \\ (p^2 - Cq^2)(r^2 - Cs^2) = \pm H, \\ (pr \pm Cqs)^2 - C(ps \pm qr)^2 = \pm H. \end{gathered}$$

Therefore

$$\left.\begin{aligned} x &= pr \pm Cqs \\ y &= ps \pm qr \end{aligned}\right\} \qquad (8)$$

is a solution of (1).

The formulæ (6), (7), (8) may be established by means of the relations which connect the convergents of $\sqrt{C}$ (see Exercises XXXI., 25, and Serret, *Alg. Sup.*, § 27 *et seq.*). This method of demonstration, although more tedious, is much more satisfactory, because, taken in conjunction with what we have established in § 16, it shows that (6), (7), and (8) *contain all the solutions in question.*

Example 1. Find the integral solutions of $x^2 - 13y^2 = 1$.

If we refer to chap. xxxii., § 5, we find the following table of values for $\sqrt{13}$:—

n	a_n	p_n	q_n	M_n
1	3	3	1	4
2	1	4	1	3
3	1	7	2	3
4	1	11	3	4
5	1	18	5	1
6	6	119	33	4
7	1	137	38	3
8	1	256	71	3
9	1	393	109	4
10	1	649	180	1
11	6	4287	1189	4

Hence the smallest solution of $x^2 - 13y^2 = 1$ is $x = 649$, $y = 180$. We have, in fact,

$$649^2 - 13.180^2 = 421201 - 421200 = 1.$$

* It must not be forgotten that there may be more than one solution in the first period. For every such primary solution there will be a general group like (8).

From (6) above, we see that the general solution is given by

$$x = \tfrac{1}{2}\{(649 + 180\sqrt{13})^n + (649 - 180\sqrt{13})^n\},$$
$$y = \tfrac{1}{2}\{(649 + 180\sqrt{13})^n - (649 - 180\sqrt{13})^n\}/\sqrt{13},$$

where n is any positive integer.

In particular, taking $n = 2$, we get the solution

$$x = 649^2 + 13.180^2 = 842401,$$
$$y = 2.649.180 = 233640.$$

Example 2. Find the integral solutions of $x^2 - 13y^2 = -1$.

The primary solution is given by the 5th convergent to $\sqrt{13}$, as may be seen by the table given in last example.

The general solution is, by (7),

$$x = \frac{1}{2}\{(18 + 5\sqrt{13})^{2n-1} + (18 - 5\sqrt{13})^{2n-1}\},$$

$$y = \frac{1}{2\sqrt{13}}\{(18 + 5\sqrt{13})^{2n-1} - (18 - 5\sqrt{13})^{2n-1}\},$$

when n is any positive integer.

Example 3. Find all the integral solutions of $x^2 - 13y^2 = 3$.

The primary solution is $x = 4$, $y = 1$, as may be seen from the table above.

The general solution is therefore, by (8),

$$x = 4r \pm 13s, \quad y = 4s \pm r,$$

where (r, s) is any solution whatever of $x^2 - 13y^2 = 1$.

In particular, taking $r = 649$ and $s = 180$, we get the two solutions, $x = 256$, $y = 71$, and $x = 4936$, $y = 1369$.

§ 18.] Let us next consider the equation

$$x^2 - Cy^2 = \pm H \qquad (9),$$

where C is positive and not a perfect square, and H is positive but $> \sqrt{C}$.

We propose to show that the solution of (9) can always be made to depend on the solution of an equation of the same form in which $H < \sqrt{C}$; that is, upon the case already completely solved in §§ 15-17.

Let (x, y) be any primitive solution of (9), so that x is prime to y. Then we can always determine (x_1, y_1) so that

$$xy_1 - yx_1 = \pm 1 \qquad (10).^*$$

In fact, if p/q be the penultimate convergent to x/y when converted into a simple continued fraction, we have, by § 12,

$$x_1 = tx \pm p, \quad y_1 = ty \pm q \qquad (11).$$

* There is no connection between the double signs here and in (9).

If we multiply both sides of (9) by $x_1^2 - Cy_1^2$, and rearrange the left-hand side, we get

$$(xx_1 - Cyy_1)^2 - C(xy_1 - yx_1)^2 = \pm H(x_1^2 - Cy_1^2).$$

This gives, by (10),

$$(xx_1 - Cyy_1)^2 - C = \pm H(x_1^2 - Cy_1^2) \qquad (12).$$

Now

$$xx_1 - Cyy_1 = t(x^2 - Cy^2) \pm (xp - Cyq) \qquad (13).$$

But we may put $xp - Cyq = SH \pm K_1$, where $K_1 \not> \frac{1}{2}H$. Hence

$$xx_1 - Cyy_1 = (t \pm S)H \pm (\pm K_1) \qquad (14).$$

Now t and the double sign in (13) are both at our disposal; and we may obviously so choose them that

$$xx_1 - Cyy_1 = K_1 \qquad (15),$$

where

$$K_1 \not> \tfrac{1}{2}H. \qquad (16).$$

We therefore have, from (12),

$$K_1^2 - C = \pm H(x_1^2 - Cy_1^2) \qquad (17).$$

Now, by hypothesis, $\sqrt{C} < H$, therefore $C < H^2$ and $K_1^2 \sim C < H^2$.

Since (x_1, y_1) are integers, it follows from (17) that, if (9) have an integral solution, then it must be possible to find an integer $K_1 \not> \frac{1}{2}H$ such that

$$(K_1^2 - C)/H = H_1 \qquad (18),$$

where H_1 is some integer which is less than H^2/H, that is, $< H$.

If no value of $K_1 < \frac{1}{2}H$ can be found to make $(K_1^2 - C)/H$ integral (and, be it observed, we have only a limited number of possible values to try, since $K_1 \not> \frac{1}{2}H$), then the equation (9) has no integral solution.

Let us suppose that one or more such values of K_1, say K_1, K_1', K_1'', . . ., *can* be found, and let the corresponding values of H_1 be H_1, H_1', H_1'', . . . Then it follows from our analysis that for every integral solution of (9) we must be able to find an integral solution of one of the limited group of equations

$$\left.\begin{aligned} x_1^2 - Cy_1^2 &= \pm H_1 \\ x_1^2 - Cy_1^2 &= \pm H_1' \\ x_1^2 - Cy_1^2 &= \pm H_1'' \\ &\cdots\cdots \end{aligned}\right\} \qquad (19),$$

where H_1, H_1', H_1'', . . . are all less than H.

If it also happens that in all the equations (19) the numerical value of the right-hand side is $< \sqrt{C}$, then these equations can all be completely solved, as already explained.

If (x_1, y_1) be a solution of any one of them, we see, by (10) and (15), that

$$x = (K_1 x_1 \mp C y_1)/H_1, \quad y = (K_1 y_1 \mp x_1)/H_1 \qquad (20),$$

or $$x = (K_1' x_1 \mp C y_1)/H_1', \quad y = (K_1' y_1 \mp x_1)/H_1',$$

. . . . . .

If in any of the equations (19), say, for instance, in the first, the condition $H_1 < \sqrt{C}$ is not yet fulfilled, we can repeat the above transformation, and deduce from it a new system,

$$\left.\begin{aligned} x_2^2 - C y_2^2 &= \pm H_2 \\ x_2^2 - C y_2^2 &= \pm H_2' \\ &\cdots \end{aligned}\right\} \qquad (21),$$

where H_2 and H_2' are each less than H_1; and we have

$$\left.\begin{aligned} x_1 &= (K_2 x_2 \mp C y_2)/H_2, \quad y_1 = (K_2 y_2 \mp x_2)/H_2 \\ x_1 &= (K_2' x_2 \mp C y_2)/H_2', \quad y_1 = (K_2' y_2 \mp x_2)/H_2' \\ &\cdots \end{aligned}\right\} \qquad (22).$$

Since the H's are all integers, the chain of successive operations thus indicated must finally come to an end in every branch.

Thus we see that *any integral solution of* (9) *must be deducible from the solution of one or other of a finite group of equations of the type*

$$x^2 - C y^2 = H_n^{(m)} \qquad (23),$$

where $H_n^{(m)} < \sqrt{C}$.

The practical method of solution thus suggested is as follows:—

Find all the integral values of $K_1 < \frac{1}{2}H$ for which $(K_1^2 - C)/H$ is an integer. Take any one of these, say K_1; and let H_1 be the corresponding value of $(K_1^2 - C)/H$. Then, if $H_1 < \sqrt{C}$, solve the equation $x_1^2 - C y_1^2 = \pm H_1$ generally; take the formula (20); and find which of the solutions (x_1, y_1), if any, make (x, y) integral. We thus get a group of solutions of (9). If $H_1 > \sqrt{C}$, then we find all the values of $K_2 < \frac{1}{2}H_1$ for which $(K_2^2 - C)/H_1$ is integral,

$= H_2$ say, and, if $H_2 < \sqrt{C}$, solve the equation $x_2^2 - Cy_2^2 = \pm H_2$; then pass back to x through the two transformations (20) and (22); and, finally, select the integral values of x and y thus resulting, if there be any.

By proceeding in this way until each branch and twig, as it were, of the solution is traced to its end, we shall get all the possible integral solutions of (9), or else satisfy ourselves that there are none.

The straightforward application of these principles is illustrated in the following example. Into the various devices for shortening the labour of calculation we cannot enter here.

Example. Find the integral solutions of

$$x^2 - 15y^2 = 61 \qquad (9').$$

Let $$(K_1^2 - 15)/61 = H_1 \qquad (18'),$$

where $K_1 \not> 30$.

Then $$K_1^2 = 15 + 61H_1.$$

Since $K_1^2 \not> 900$, we have merely to select the perfect squares among the numbers 15, 76, 137, 198, 259, 320, 381, 442, 503, 564, 625, 686, 747, 808, 869. The only one is 625, corresponding to which we have $K_1 = 25$ and $H_1 = 10$.

Since $H_1 > \sqrt{15}$, we must repeat the process, and put

$$(K_2^2 - 15)/10 = H_2 \qquad (18''),$$

where $K_2 \not> 5$, and therefore $K_2^2 \not> 25$.

Since $K_2^2 = 15 + 10H_2$, the only values of K_2^2 to be examined here are 5, 15, 25. Of these the last only is suitable, corresponding to which we have $K_2 = 5$, $H_2 = 1$.

We have now arrived at the equation

$$x_2^2 - 15y_2^2 = 1 \qquad (21'),$$

the first solution of which is easily seen to be (4, 1). Hence the general solution of (21') is

$$\left.\begin{aligned} x_2 &= \tfrac{1}{2}\{(4+\sqrt{15})^n + (4-\sqrt{15})^n\} \\ y_2 &= \frac{1}{2\sqrt{15}}\{(4+\sqrt{15})^n - (4-\sqrt{15})^n\} \end{aligned}\right\} \qquad (24).$$

The general solution of (9') is connected with this by the relations

$$x_1 = (5x_2 \mp 15y_2)/1, \quad y_1 = (5y_2 \mp x_2)/1 \qquad (22');$$
$$x = (25x_1 \mp 15y_1)/10, \quad y = (25y_1 \mp x_1)/10 \qquad (20').$$

Hence

$$x = 14x_2 \mp 45y_2, \qquad y = \mp 3x_2 + 14y_2 \qquad (25),$$

where x_2 and y_2 are given by (24). The question regarding the integrality of x and y does not arise in this case.

As a verification put $x_2 = 4$, $y_2 = 1$, and we get the solutions (11, 2) and (101, 26) for (9'), which are correct.

§ 19.] There remain two cases of the binomial equation $x^2 - Cy^2 = \pm H$ which are not covered by the above analysis—

$$x^2 - Cy^2 = \pm H \qquad (26),$$

where C is a perfect square, say $C = R^2$; and

$$x^2 + Cy^2 = +H \qquad (27).$$

The equation (26) may be written

$$(x - Ry)(x + Ry) = \pm H.$$

Hence we must have

$$\left.\begin{aligned} x - Ry &= u \\ x + Ry &= v \end{aligned}\right\} \qquad (28),$$

where u and v are any pair of complementary factors of $\pm H$. We have therefore simply to solve every such pair as (28), and select the integral solutions. The number of such solutions is clearly limited, and there may be none.

In the case of equation (27) also the number of solutions is obviously limited, since each of the two terms on the left is positive, and their sum cannot exceed H. The simplest method of solution is to give y all integral values $\not> \sqrt{(H/C)}$, and examine which of these, if any, render $H - Cy^2$ a perfect square.

§ 20.] In conclusion, we shall briefly indicate how the solution of the general equation of the 2nd degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad (29),$$

can be made to depend on the solution of a binomial equation.

By a slight modification of the analysis of chap. vii., § 13, the reader will easily verify that, provided a and b be not both zero, and c be not zero, (29) may be thrown into one or other of the forms

$$(Cy + F)^2 - C(ax + hy + g)^2 = -a\Delta \qquad (30);$$

or

$$(Cx + G)^2 - C(hx + by + f)^2 = -b\Delta \qquad (31),$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$, $C = h^2 - ab$, $F = gh - af$, $G = hf - bg$; say into the form (30). If, then, we put

$$\left.\begin{aligned} Cy + F &= \xi \\ ax + hy + g &= \eta \end{aligned}\right\} \qquad (32),$$

(30) reduces to

$$\xi^2 - C\eta^2 = -a\Delta \qquad (33),$$

which is a binomial form, and may be treated by the methods already explained.

If $h^2 > ab$, then C is positive, and, provided C be not a perfect square, we fall upon cases (1) or (9).

If C be a positive and a perfect square, we have case (26).

It should be noticed that, if either $a = 0$ or $b = 0$, or both $a = 0$ and $b = 0$, we get the leading peculiarity of this case, which is that the left-hand side of the equation breaks up into rational factors (see Example 2 below).

If $h^2 < ab$, then C is negative, and we have case (27).

If $h^2 = ab$, then $C = 0$, and the equation (29) may be written

$$(ax + hy)^2 + 2agx + 2afy + ac = 0 \qquad (34),$$

which can in general by an obvious transformation be made to depend upon the equation

$$\eta^2 = Q\xi \qquad (35),$$

which can easily be solved.

Example 1. Find all the positive integral solutions of

$$3x^2 - 8xy + 7y^2 - 4x + 2y = 109.$$

This equation may be written

$$(3x - 4y - 2)^2 + 5(y - 1)^2 = 336,$$

say

$$\xi^2 + 5\eta^2 = 336.$$

Here we have merely to try all values of η from 0 to 8, and find which of them makes $336 - 5\eta^2$ a perfect square. We thus find

$$\xi = \pm 16, \quad \eta = \pm 4;$$
$$\xi = \pm 4, \quad \eta = \pm 8.$$

Hence

$$3x - 4y - 2 = \pm 16, \quad y - 1 = \pm 4 \qquad (1);$$
$$3x - 4y - 2 = \pm 4, \quad y - 1 = \pm 8 \qquad (2).$$

It is at once obvious that in order to get positive values of y the upper sign must be taken in the second equation in each case. Hence $y = 5$ or $y = 9$. To get corresponding positive integral values of x, we must take the lower sign in the first of (1), and the upper sign in the first of (2). Hence the only positive integral solutions are

$$x = 2, \quad y = 5, \quad \text{and } x = 14, \quad y = 9.$$

Example 2. Find the positive integral solutions of

$$3xy+2y^2-4x-3y=12.$$

This is a case where the terms of the 2nd degree break up into two rational factors. We may put the equation into the form

$$(9x+6y-1)(3y-4)=112.$$

Since $3y-4$ is obviously less than $9x+6y-1$ when both x and y are positive, $3y-4$ must be equal to a minor factor of 112, that is, to 1, 2, 4, 7, or 8; the second and the last of these alone give integral values for y, namely, $y=2$ and $y=4$. To get the corresponding values of x, we have $9x+6y-1=56$ and $9x+6y-1=14$, that is to say, $9x=45$ and $9x=-9$. Hence the only positive integral solution is $x=5$, $y=2$.

Example 3. Find all the integral solutions of

$$9x^2-12xy+4y^2+3x+2y=12.$$

Here the terms of the 2nd degree form a complete square, and we may write the equation thus—

$$(3x-2y)^2+(3x-2y)+4y=12,$$

or

$$4(3x-2y)^2+4(3x-2y)+1+16y=49;$$

that is,

$$(6x-4y+1)^2=49-16y.$$

Hence, if

$$u=6x-4y+1 \tag{1},$$

so that u is certainly integral, we must have

$$y=(49-u^2)/16 \tag{2}.$$

Now we may put $u=16\mu\pm s$, where s is a positive integer ≥ 8.

It then appears that y will not be integral unless $(49-s^2)/16$ be integral. The only value of s for which this happens is $s=1$. Therefore

$$u=16\mu\pm 1 \tag{3}.$$

Hence, by (1), (2), and (3), we must have

$$x=2+4\mu(1-8\mu)/3, \quad y=3-2\mu-16\mu^2 \tag{4},$$

or

$$x=4\mu+(5-32\mu^2)/3, \quad y=3+2\mu-16\mu^2 \tag{5}.$$

It remains to determine μ so that x shall be integral.

Taking (4), we see that $\mu(1-8\mu)/3$ will be integral when and only when $\mu=3\nu$ or $\mu=3\nu-1$.

Using these forms for μ, we get

$$x=2+4\nu-96\nu^2, \quad y=3-6\nu-144\nu^2 \tag{6};$$

$$x=-10+68\nu-96\nu^2, \quad y=-11+90\nu-144\nu^2 \tag{7}.$$

Taking (5), we find that $(5-32\mu^2)/3$ is integral when and only when $\mu=3\nu+1$ or $\mu=3\nu-1$.

Using these forms, we get from (5)

$$x=-5-52\nu-96\nu^2, \quad y=-11-90\nu-144\nu^2 \tag{8};$$

$$x=-13+76\nu-96\nu^2, \quad y=-15+102\nu-144\nu^2 \tag{9}.$$

The formulæ (6), (7), (8), (9), wherein ν may have any integral value, positive or negative, contain all the integral solutions of the given equation.

EXERCISES XXXII.

Find all the integral and also all the positive integral solutions of the following equations :—

(1.) $5x+7y=29$. (2.) $16x-17y=27$.

(3.) $11x+7y=1103$. (4.) $1367x-1013y=16246$.

(5.) If £x, ys. be double £y, xs., find x and y.

(6.) Find the greatest integer which can be formed in nine different ways and no more, by adding together a positive integral multiple of 5 and a positive integral multiple of 7.

(7.) In how many ways can £2 : 15 : 6 be paid in half-crowns and florins?

(8.) A has 200 shilling-coins, and B 200 franc-coins. In how many ways can A pay to B a debt of 4s.?

(9.) 4 apples cost the same as 5 plums, 3 pears the same as 7 apples, 8 apricots the same as 15 pears, and 5 apples cost twopence. How can I buy the same number of each fruit so as to spend an exact number of pence and spend the least possible sum?

(10.) A woman has more than 5 dozen and less than 6 dozen of eggs in her basket. If she counts them by fours there is one over, if by fives there are four over. How many eggs has she?

(11.) A woman counted her eggs by threes and found that there were two over; and again by sixes and found there were three over. Show that she made a mistake.

(12.) Find the least number which has 3 for remainder when divided by 8, and 5 for remainder when divided by 7.

(13.) Find the least number which, when divided by 28, 19, 15 respectively, gives the remainders 15, 12, 10 respectively.

(14.) In how many ways can £2 be paid in half-crowns, shillings, and sixpences?

(15.) A bookcase which will hold 250 volumes is to be filled with 3-volumed novels, 5-volumed poems, 12-volumed histories. In how many ways can this be done? If novels cost 10s. 6d. per volume, poems 7s. 6d., and histories 5s., show that the cheapest way of doing it will cost £129, 15s.

Solve the following systems, and find the positive integral solutions :—

(16.) $x+2y+3z=120$.

(17.) $\left.\begin{aligned} x+y+z+u&=4,\\ 5y+6z+9u&=18.\end{aligned}\right\}$

(18.) $\left.\begin{aligned} 2x+5y+3z&=324,\\ 6x-4y+14z&=190.\end{aligned}\right\}$

(19.) $\left.\begin{aligned} 5x-6y+7z&=173,\\ 17x-4y+3z&=510.\end{aligned}\right\}$

(20.) $17x+19y+21z=400$.

(21.) $\left.\begin{aligned} x+y+z+u&=26,\\ 3x+2y+4z+u&=63,\\ 2x+3y+2z+4u&=74.\end{aligned}\right\}$

(22.) Show how to express the general integral solution of the system

$$\begin{aligned} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=d_1,\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=d_2,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot&\\ a_{n-1,1}x_1+a_{n-1,2}x_2+\ldots+a_{n-1,n}x_n&=d_{n-1}\end{aligned}$$

by means of determinants, when a particular solution is known.

Find the values of x which make the values of the following functions integral squares:—

(23.) $2x^2+2x$. (24.) $(x^3-x)/5$. (25.) $x+11$ and $x+20$, simultaneously.
(26.) $7x+6$ and $4x+3$, simultaneously. (27.) x^3+x+8.

Solve the following equations, giving in each case the least integral solution, and indicating how all the other integral solutions may be found:—

(28.) $x^2-44y^2=-8$. (29.) $x^2-44y^2=+5$.
(30.) $x^2-44y^2=-7$. (31.) $x^2-44y^2=+4$.
(32.) $x^2+3y^2=628$. (33.) $x^2-69y^2=-11$.
(34.) $x^2-47y^2=+1$. (35.) $x^2-47y^2=-1$.
(36.) $x^2-26y^2=-1105$. (37.) $x^2-7y^2=186$.
(38.) $x^2-(a^2+1)y^2=1$. (39.) $x^2-(a^2-1)y^2=1$.
(40.) $x^2-(a^2+a)y^2=1$. (41.) $x^2-(a^2-a)y^2=1$.
(42.) $x^2+5xy-2x+3y=853$. (43.) $xy-2x-3y=15$.
(44.) $x^2-y^2+4x-5y=27$. (45.) $3x^2+2xy+5y^2=390$.
(46.) $x^2+4xy-11y^2+2x-86y-140=0$.
(47.) $x^2-xy-72y^2+2x-440y-659=0$.
(48.) $x^2+2xy-17y^2+72y-75=0$.
(49.) $61x^2+28xy+251y^2+264x+526y+260=0$.

(50.) Show that all the primitive solutions of $Dx^2-Cy^2=\pm H$ are furnished by the convergents to $\sqrt{(C/D)}$, provided $H<\sqrt{(CD)}$. Show also how to reduce the equation $Dx^2-Cy^2=\pm H$, when $H>\sqrt{(CD)}$.

(51.) Find all the solutions of

$$4x^2-7y^2=-3,$$

and of

$$4x^2-7y^2=53.$$

(52.) If D, E, F, H be integers, and $H<\sqrt{(E^2-DF)}$ (real), show that all the solutions of

$$Dx^2-2Exy+Fy^2=\pm H$$

are furnished by the convergents to one of the roots of

$$Dz^2-2Ez+F=0.$$

(See Serret, *Alg. Sup.*, § 35.)

(53.) If $U_n=p_n-xq_n$, where x is a periodic fraction having a cycle of c quotients, and p_n and q_n have their usual meanings, then

$$U_{mc+r}=(\alpha-\beta x_{r+1})^m U_r,$$

where

$$x_{r+1}=a_{r+1}+\underset{*}{\frac{1}{a_{r+2}+}}\cdots\underset{*}{\frac{1}{a_{r+c}+}}\cdots,$$

and

$$\frac{\alpha}{\beta}=a_{r+1}+\frac{1}{a_{r+2}+}\cdots\frac{1}{a_{r+c}}.$$

In particular, if $x=\sqrt{(C/D)}$, then

$$D(p_{mc+r}-\sqrt{(CD)}q_{mc+r})=\{\alpha M_r-\beta L_r-\beta\sqrt{(CD)}\}^m(Dp_r-\sqrt{(CD)}q_r)/M_r^m.$$

Point out the bearing of this result on the solution of $Dx^2-Cy^2=\pm H$.

CHAPTER XXXIV.

General Continued Fractions.

FUNDAMENTAL FORMULÆ.

§ 1.] The theory of the general continued fraction

$$x_1 = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \qquad \text{(A)},$$

where $a_1, a_2, a_3, \ldots, b_2, b_3, \ldots$ are any quantities whatever, is inferior in importance to the theory of the simple continued fraction, and it is also much less complete. There are, however, a number of theorems regarding such fractions so closely analogous to those already established for simple continued fractions that we give them here, leaving the demonstrations, where they are like those of chap. xxxii., as exercises for the reader. There are also some analytical theories closely allied to the general theory of continued fractions which will find an appropriate place in the present chapter.

In dealing with the general continued fraction, where the numerators are not all positive units, and the denominators not necessarily positive, it must be borne in mind that the chain of operations indicated in the primary definition of the right-hand side of (A) may fail to have any definite meaning even when the number of the operations is finite. Thus in forming the third convergent of $1 + \frac{1}{1-} \frac{1}{1-} \frac{1}{1-} \cdots$ we are led to $1 + 1/(1 - 1)$; and in forming the fourth to $1 + 1/\{1 - 1/(1 - 1)\}$. It is obvious that we could not suppose the convergents of this fraction formed by the direct process of chap. xxxii., § 6 (α), (β),

(γ). It must also be remembered that no piece of reasoning that involves the use of the value of a *non-terminating* continued fraction is legitimate till we have shown that the value in question is finite and definite.

In cases where any difficulty regarding the meaning or convergency of the continued fraction taken in its primary sense arises, we regard the form on the right of (A) *merely as representing the assemblage of convergents* p_1/q_1, p_2/q_2, . . ., p_n/q_n *whose denominators are constructed by means of the recurrence-formulæ* (2) *and* (3) *below.*

That is to say, when the primary definition fails, we make the formulæ (2) and (3) the definition of the continued fraction.

In what follows we shall be most concerned with two varieties of continued fraction, namely,

$$a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \qquad \text{(B)},$$

and

$$a_1 + \frac{b_2}{a_2 -} \frac{b_3}{a_3 -} \cdots \qquad \text{(C)},$$

where $a_1, a_2, a_3, \ldots, b_2, b_3, \ldots$ are all real and positive. We shall speak of (B) and (C) as *continued fractions of the first and second class* respectively.

§ 2.] If p_1/q_1, p_2/q_2, &c. be the successive convergents to

$$x_1 = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \qquad (1),$$

then

$$p_n = a_n p_{n-1} + b_n p_{n-2} \qquad (2);$$
$$q_n = a_n q_{n-1} + b_n q_{n-2} \qquad (3),$$

with the initial conditions $p_0 = 1$, $p_1 = a_1$; $q_1 = 1$, $q_2 = a_2$.

Cor. 1. *In a continued fraction of the first class* p_n *and* q_n *are both positive; and, provided* $a_n \not< 1$, *each of them continually increases with* n.*

In a continued fraction of the second class, subject to the restriction $a_n \not< 1 + b_n$, p_n *and* q_n *are positive, and each of them continually increases with* n.*

* It does not necessarily follow that $Lp_n = \infty$ and $Lq_n = \infty$, for the successive increments here are not positive integral numbers, as in the case of simple continued fractions.

These conclusions follow very readily by induction from such formulæ as

$$p_n - p_{n-1} = (a_n - 1)p_{n-1} + b_n p_{n-2} \qquad (4).$$

Cor. 2.

$$\frac{p_n}{p_{n-1}} = a_n + \frac{b_n}{a_{n-1}+}\,\frac{b_{n-1}}{a_{n-2}+}\cdots\frac{b_2}{a_1} \qquad (5);$$

$$\frac{q_n}{q_{n-1}} = a_n + \frac{b_n}{a_{n-1}+}\,\frac{b_{n-1}}{a_{n-2}+}\cdots\frac{b_2}{a_2} \qquad (6).$$

§ 3.] From (2) and (3) we deduce

$$p_n q_{n-1} - p_{n-1}q_n = (-)^n b_2 b_3 \ldots b_n \qquad (1).$$

Cor. 1. *The convergents, as calculated by the recurrence-rule, are not necessarily at their lowest terms.*

Cor. 2.

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-)^n \frac{b_2 b_3 \ldots b_n}{q_n q_{n-1}} \qquad (2).$$

Cor. 3.

$$\frac{p_n}{q_n} = a_1 + \frac{b_2}{q_1 q_2} - \frac{b_2 b_3}{q_2 q_3} + \cdots (-)^n \frac{b_2 b_3 \ldots b_n}{q_{n-1} q_n} \qquad (3).$$

Cor. 4.

$$p_n q_{n-2} - p_{n-2} q_n = (-)^{n-1} a_n b_2 b_3 \ldots b_{n-1} \qquad (4);$$

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-)^{n-1} \frac{a_n b_2 b_3 \ldots b_{n-1}}{q_n q_{n-2}} \qquad (5).$$

Cor. 5.

$$\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right) \Big/ \left(\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}}\right) = -\frac{b_n q_{n-2}}{q_n},$$

$$= -\frac{b_n q_{n-2}}{a_n q_{n-1} + b_n q_{n-2}} \qquad (6).$$

Cor. 6. *In a continued fraction of the first class, the odd convergents form an increasing series, and the even convergents a decreasing series; and every odd convergent is less than, and every even convergent greater than following convergents.*

In a continued fraction of the second class, subject to the restriction $a_n \nless 1 + b_n$, *all the convergents are positive, and form an increasing series.*

These conclusions follow at once from (2) and (5), if we remember that, for a fraction of the second class, we have to replace $b_2, \ldots, b_n$ by $-b_2, \ldots, -b_n$.

CONTINUANTS.

§ 4.] The functions p_n, q_n of $a_1, a_2, \ldots, a_n$; $b_2, b_3, \ldots, b_n$ which constitute the numerators and denominators of the continued fraction

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots \cfrac{b_n}{a_n}}}$$

belong to a common class of rational integral functions.*

In fact, p_n is determined by the set of equations

$$p_2 = a_2 p_1 + b_2 p_0, \quad p_3 = a_3 p_2 + b_3 p_1, \quad \ldots, \quad p_n = a_n p_{n-1} + b_n p_{n-2} \tag{1},$$

together with the initial conditions $p_0 = 1$, $p_1 = a_1$; while q_n is determined by the system

$$q_3 = a_3 q_2 + b_3 q_1, \quad q_4 = a_4 q_3 + b_4 q_2, \quad \ldots, \quad q_n = a_n q_{n-1} + b_n q_{n-2} \tag{2},$$

together with the initial conditions $q_1 = 1$, $q_2 = a_2$.

It is obvious, therefore, that q_n *is the same function of* $a_2, a_3, \ldots, a_n$; $b_3, b_4, \ldots, b_n$ *as* p_n *is of* $a_1, a_2, \ldots, a_n$; $b_2, b_3, \ldots, b_n$.

We denote the function p_n by

$$p_n = \mathrm{K}\begin{pmatrix} b_2, & \ldots, & b_n \\ a_1, & \ldots, & a_n \end{pmatrix} \tag{3},$$

and speak of it as a *continuant of the nth order* whose denominators are $a_1, a_2, \ldots, a_n$, and whose numerators are $b_2, \ldots, b_n$. We have then

$$q_n = \mathrm{K}\begin{pmatrix} b_3, & \ldots, & b_n \\ a_2, & \ldots, & a_n \end{pmatrix} \tag{4}.$$

* This was first pointed out by Euler in his memoir entitled "Specimen Algorithmi Singularis," *Nov. Comm. Petrop.* (1764). Elegant demonstrations of Euler's results were given by Möbius, *Crelle's Jour.* (1830). The theory has been treated of late in connection with determinants by Sylvester and Muir.

When the numerators of the continuant are all unity, it is usual to omit them altogether, and write simply $\mathrm{K}(a_1, a_2, \ldots, a_n)$. A continuant of this kind is called a *simple continuant.*

When it is not necessary to express the numerators and denominators it is convenient to abbreviate both

$$\mathrm{K}\begin{pmatrix} b_2, & \ldots, & b_n \\ a_1, & \ldots, & a_n \end{pmatrix} \text{ and } \mathrm{K}(a_1, a_2, \ldots, a_n)$$

into $\mathrm{K}(1, n)$. In this notation we should have, if $r < s$,

$$\mathrm{K}(r, s) = \mathrm{K}\begin{pmatrix} b_{r+1}, & \ldots, & b_s \\ a_r, & \ldots, & a_s \end{pmatrix} \qquad (5);$$

$$\mathrm{K}(s, r) = \mathrm{K}\begin{pmatrix} b_s, & \ldots, & b_{r+1} \\ a_s, & \ldots, & a_r \end{pmatrix} \qquad (6).$$

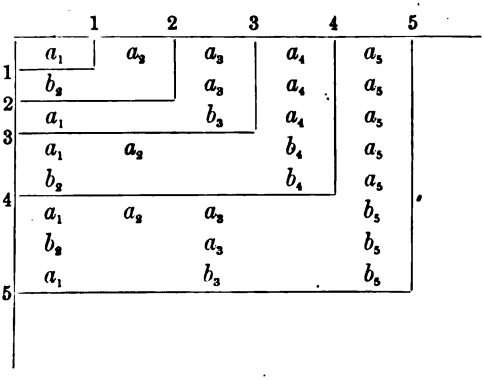
In particular, $\mathrm{K}(r, r)$ means simply a_r, so that $p_1 = \mathrm{K}(1, 1) = a_1$. To make the notation complete, we shall denote p_0 and q_1 by $\mathrm{K}(\ )$, which therefore stands for unity; and, in general, when the statement of any rule requires us to form a continuant for which the system of numerators and denominators under consideration furnishes no constituents, we shall denote that continuant by $\mathrm{K}(\ )$ and understand its value to be unity. It will be found that this convention introduces great simplicity into the enunciation of theorems regarding continuants.

§ 5.] *A continuant of the nth order is an integral function of the nth degree of its constituents.*

This follows at once from the definition of the function, for we have, by § 4 (1),

$$\left.\begin{aligned} \mathrm{K}(l, n) &= a_n \mathrm{K}(l, n-1) + b_n \mathrm{K}(l, n-2), \\ \mathrm{K}(l, n-1) &= a_{n-1} \mathrm{K}(l, n-2) + b_{n-1} \mathrm{K}(l, n-3), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{K}(l, l+1) &= a_{l+1} \mathrm{K}(l, l) + b_{l+1} \mathrm{K}(\ ), \\ \mathrm{K}(l, l) &= a_l, \quad \mathrm{K}(\ ) = 1. \end{aligned}\right\} \qquad (7).$$

The following rule of Hindenburg's gives a convenient process for writing down the terms of a series of continuants, say $\mathrm{K}(1, 1)$, $\mathrm{K}(1, 2)$, $\mathrm{K}(1, 3)$, . . . :—



1st. Write down a_1, and enclose it in the rectangle 1, 1. The term in 1, 1 is K(1, 1).

2nd. Write a_2 to the right of all the rows in 1, 1 ; and write b_2 underneath. Enclose all the rows thus constructed in the rectangle 2, 2. Then the rows in 2, 2 give the products in K(1, 2), namely, $a_1a_2 + b_2$.

3rd. Write a_3 at the ends of all the rows of 2, 2; repeat under 2, 2 all the rows in 1, 1, and write b_3 at the end of each of them. Enclose all the rows thus constructed in 3, 3. Then the rows in 3, 3 give the products in K(1, 3), namely, $a_1a_2a_3 + b_2a_3 + a_1b_3$.

The law for continuing the process will now be obvious. The scheme is, in fact, merely a graphic representation of the continual application of the recurrence-formula

$$K(1, n) = a_nK(1, n-1) + b_nK(1, n-2) \qquad (8).$$

By considering Hindenburg's scheme we are led to the following rule of Euler's* for writing down all the terms of a continuant of the nth order.

Write down $a_1a_2a_3 \ldots a_{n-1}a_n$. This is the first term. To get the rest, omit from this product in every possible way one or more pairs of consecutive a's, always replacing the second a of the pair by a b of the same order.

* Euler (*l.c.*) gave the rule for the simple continuant merely. Cayley (*Phil. Mag.*, 1853) gave the more general form.

For example, to get the terms of K(1, 4). The first is $a_1a_2a_3a_4$. By omitting from this, first a_1a_2, then a_2a_3, then a_3a_4, and replacing by b_2, b_3, b_4 respectively, we get three more terms, $b_2a_3a_4$, $a_1b_3a_4$, $a_1a_2b_4$. Then, omitting two pairs, we get b_2b_4. We thus get all the terms of K(1, 4).

It is easy to verify this rule up to K(1, 5); and a glance at the recurrence-formula (8) shows that, if it holds for any two consecutive orders of continuants, it will hold for all orders.

From Euler's rule we deduce at once the following :—

Cor. 1. *The value of a continuant is not altered by reversing the order of its constituents, that is to say,*

$$\mathrm{K}\begin{pmatrix} b_2, & \ldots, & b_n \\ a_1, & \ldots, & a_n \end{pmatrix} = \mathrm{K}\begin{pmatrix} b_n, & \ldots, & b_2 \\ a_n, & \ldots, & a_1 \end{pmatrix} \qquad (9).$$

We could obviously form the continuant K(1, n) by starting with $a_na_{n-1}\ldots a_2a_1$ instead of $a_1a_2\ldots a_{n-1}a_n$, and replacing each consecutive pair of a's in every possible way by a b of the same order as the first a of the pair. In this way we should get precisely the same terms as before. Hence the theorem. We may express it in the form

$$\mathrm{K}(l, m) = \mathrm{K}(m, l) \qquad (10).$$

Cor. 2. *We have the following recurrence-formula* :—

$$\mathrm{K}(l, m) = a_l\mathrm{K}(l+1, m) + b_{l+1}\mathrm{K}(l+2, m) \qquad (11).$$

For, by Cor. 1,

$$\begin{aligned}\mathrm{K}(l, m) &= \mathrm{K}(m, l),\\ &= a_l\mathrm{K}(m, l+1) + b_{l+1}\mathrm{K}(m, l+2), \text{ by (7)},\\ &= a_l\mathrm{K}(l+1, m) + b_{l+1}\mathrm{K}(l+2, m), \text{ by Cor. 1.}\end{aligned}$$

§ 6.] The theorems (1) and (4) of § 3 may be written in continuant notation as follows :—

$$\mathrm{K}(1, n)\mathrm{K}(2, n-1) - \mathrm{K}(1, n-1)\mathrm{K}(2, n) = (-)^n b_2b_3\ldots b_n\mathrm{K}(\,)\mathrm{K}(\,) \qquad (12),$$

$$\mathrm{K}(1, n)\mathrm{K}(2, n-2) - \mathrm{K}(1, n-2)\mathrm{K}(2, n) = (-)^{n-3} b_2b_3\ldots b_{n-1}\mathrm{K}(\,)\mathrm{K}(n, n) \qquad (13).$$

These are particular cases of the following general theorem, originally due to Euler* :—

* Euler stated it, however, only for simple continuants. It has been stated in the above general form and proved by Stern, Muir, and others.

$$K(1, n)K(l, m) - K(1, m)K(l, n)$$
$$= (-)^{m-l+1} b_l b_{l+1} \dots b_{m+1} K(1, l-2)K(m+2, n) \quad (14),$$

where $1 < l < m < n$.

This theorem is easily remembered by means of the following elegant memoria technica, given by its discoverer :—

$$\overset{\cdot\quad\cdot}{1, 2, \dots, l-2, l-1, |l, \dots, m, |m+1, m+2, \dots, n}.$$

Draw two vertical lines enclosing the indices belonging to $K(l, m)$; then two horizontal lines as above ; and put dots over the indices immediately outside the two vertical lines. The indices for the first continuant on the left of (14) are the whole row ; those of the second are inside the vertical lines ; those of the third and fourth under the upper and over the lower horizontal lines ; those of the two continuants on the right outside the two vertical lines, the dotted indices being omitted. The b's are the b's of $K(l, m)$ with one more at the end ; and the index of the minus sign is the number of constituents in $K(l, m)$.

The proof of the theorem is very simple. We can show, by means of the recurrence-formulæ (7) and (11), that, if the formula hold for $l, m+2$, and for $l, m+1$, or for $l-2, m$, and for $l-1, m$, it will hold for l, m. Now (12) asserts the truth of the theorem for $l = 2, m = n-1$; and it is easy to deduce from (12), by means of (7) and (11), that the theorem holds for $l = 3, m = n-1$, and also for $l = 2, m = n-2$. The general case is therefore established by a double mathematical induction based on the particular case (12).

The theorem (14) might be made the basis of the whole theory of continued fractions ; and it leads at once to a variety of important particular results, some of which have already been given in the two preceding chapters. Among these we shall merely mention the following regarding what may be called *reciprocal* simple continuants :—

$$K(a_1, a_2, \dots, a_i, a_i, \dots, a_2, a_1)$$
$$= K(a_1, a_2, \dots, a_i)^2 + K(a_1, a_2, \dots, a_{i-1})^2 \quad (A);$$

$$K(a_1, a_2, \dots, a_{i-1}, a_i, a_{i-1}, \dots, a_2, a_1)$$
$$= K(a_1, a_2, \dots, a_{i-1})\{K(a_1, a_2, \dots, a_i) + K(a_1, a_2, \dots, a_{i-2})\} \quad (B).$$

Example. Show that every prime p of the form $4\lambda+1$ can be exhibited as the sum of two integral squares.*

Let, $\mu_1, \mu_2, \ldots, \mu_n$ be all the integers prime to p and $< \frac{1}{2}p$; and let simple continued fractions be formed for $p/\mu_1, p/\mu_2, \ldots, p/\mu_n$, each terminating so that the last partial quotient >1. Then each of these continued fractions has for its last convergent the value $K(a_1, a_2, \ldots, a_n)/K(a_2, a_3, \ldots, a_n)$, where the two continuants are of course prime to each other, and $a_1>1$, $a_n>1$.

From this it appears that there are as many ways, and no more, of representing p by a simple continuant (whose constituents are positive integers the first and the last of which are each greater than unity) as there are integers prime to p and $< \frac{1}{2}p$.

Now, since $K(a_1, a_2, \ldots, a_n)=K(a_n, \ldots, a_2, a_1)$, and $a_n>1$, it is obvious that $K(a_n, \ldots, a_2, a_1)$ must arise from one of the other fractions p/μ. Hence, given any fraction p/μ, it is possible to find another also belonging to the series which shall have the same partial quotients in the reverse order.

Let p be a prime of the form $4\lambda+1$, then the greatest integer in $\frac{1}{2}p$ is 2λ, which is even. Since, therefore, the number of continuants which are equal to p must be even, and since $K(p)$ is one of them, there must, among the remaining odd number, be one at least which gives rise to no *new* fraction when we reverse its constituents, that is to say, which is reciprocal. Now the reciprocal continuant in question cannot be of the form $K(a_1, a_2, \ldots, a_{i-1}, a_i, a_{i-1}, \ldots, a_2, a_1)$, for it follows from (B) that such a continuant cannot represent a prime, unless $i=1$, or else $i=2$, and $a_1=1$, all of which are obviously excluded.

We must therefore have an equation of the form

$$p=K(a_1, a_2, \ldots, a_i, a_i, \ldots, a_2, a_1),$$
$$K(a_1, a_2, \ldots, a_i)^2+K(a_1, a_2, \ldots, a_{i-1})^2,$$

by (A), which proves the theorem in question.

As an example, take $13=3\times4+1$.

We have $\frac{13}{1}=13$; $\frac{13}{2}=6+\frac{1}{2}$; $\frac{13}{3}=4+\frac{1}{3}$; $\frac{13}{4}=3+\frac{1}{4}$; $\frac{13}{5}=2+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}$; $\frac{13}{6}=2+\frac{1}{6}$. So that $13=K(13)=K(6, 2)=K(4, 3)=K(3, 4)=K(2, 1, 1, 2)=K(2, 6)$; and, in particular, $13=K(2, 1, 1, 2)=K(2, 1)^2+K(2)^2=3^2+2^2$.

§ 7.] By considering the system of equations (1) of § 4, it is easy to see that, if we multiply a_r, b_r, b_{r+1} by c_r, the result is the same as if we multiplied the continuant $K(1, n)$ $(n>r)$ by c_r. Hence we have

$$K\begin{pmatrix} c_2b_2, & c_2c_3b_3, & c_3c_4b_4, & \ldots, & c_{n-1}c_nb_n \\ a_1, & c_2a_2, & c_3a_3, & c_4a_4, & \ldots, & c_na_n \end{pmatrix}$$
$$= c_2c_3\ldots c_nK\begin{pmatrix} b_2, & \ldots, & b_n \\ a_1, & a_2, & \ldots, & a_n \end{pmatrix} \quad (15).$$

* The following elegant proof of this well-known theorem of Fermat's was given by the late Professor Henry Smith of Oxford (*Crelle's Jour.*, 1855).

We may so determine $c_2, c_3, \ldots, c_n$ that all the numerators of the continuant become equal. In fact, if we put

$$c_2 b_2 = \lambda, \quad c_2 c_3 b_3 = \lambda, \quad \ldots, \quad c_{n-1} c_n b_n = \lambda,$$

we get

$$c_2 = \lambda/b_2, \quad c_3 = b_2/b_3, \quad c_4 = \lambda b_3/b_4 b_2, \quad c_5 = b_4 b_2/b_5 b_3,$$
$$c_6 = \lambda b_5 b_3/b_6 b_4 b_2, \ \ldots \ .$$

Hence

$$\mathrm{K}\begin{pmatrix} b_2, & \ldots, & b_n \\ a_1, & \ldots, & a_n \end{pmatrix}$$
$$= (1/\lambda)^\rho b_n b_{n-2} b_{n-4} \ldots \times \mathrm{K}\begin{pmatrix} \lambda, & \lambda, & \lambda, & \ldots \\ a_1, & \lambda a_2/b_2, & a_3 b_2/b_3, & \lambda a_4 b_3/b_4 b_2, \ \ldots \end{pmatrix} \tag{16},$$

where ρ is the number of even integers (excluding 0) which do not exceed n.

Cor. *Every continuant can be reduced to a simple continuant, or to a continuant each of whose numerators is* -1.

Thus, if we put $\lambda = +1$ and $\lambda = -1$, we have

$$\mathrm{K}\begin{pmatrix} b_2, & \ldots, & b_n \\ a_1, & \ldots, & a_n \end{pmatrix}$$
$$= b_n b_{n-2} \ldots \times \mathrm{K}(a_1, \ a_2/b_2, \ a_3 b_2/b_3, \ a_4 b_2/b_4 b_2, \ \ldots,$$
$$a_n b_{n-1} b_{n-3} \ldots / b_n b_{n-2} \ldots) \tag{17},$$
$$= (-)^\rho b_n b_{n-2} \ldots \times \mathrm{K}\begin{pmatrix} -1, & -1, & -1, & \ldots, \\ a_1, & -a_2/b_2, & a_3 b_2/b_3, & -a_4 b_2/b_4 b_2, \ \ldots, \\ & & & (-)^{n-1} a_n b_{n-1} b_{n-3} \ldots / b_n b_{n-2} \ldots \end{pmatrix} \tag{18}.$$

§ 8.] *The connection between a continuant and a continued fraction follows readily from* (11). For we have, provided $\mathrm{K}(2, n)$, $\mathrm{K}(3, n)$, $\mathrm{K}(4, n)$, ... are all different from zero,

$$\frac{\mathrm{K}(1, n)}{\mathrm{K}(2, n)} = a_1 + \frac{b_2}{\mathrm{K}(2, n)/\mathrm{K}(3, n)},$$
$$\frac{\mathrm{K}(2, n)}{\mathrm{K}(3, n)} = a_2 + \frac{b_3}{\mathrm{K}(3, n)/\mathrm{K}(4, n)}.$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Hence

$$\frac{\mathrm{K}(1, n)}{\mathrm{K}(2, n)} = a_1 + \frac{b_2}{a_2 +} \ \frac{b_3}{a_3 +} \ \ldots \ \frac{b_r}{\mathrm{K}(r, n)/\mathrm{K}(r+1, n)} \tag{19}.$$

If in this last equation we put $r = n$, and remember that here $K(n+1, n) = K(\ \ ) = 1$, we get

$$\frac{K(1, n)}{K(2, n)} = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{a_n} \qquad (20),$$

a result which was obvious from the considerations of § 4.

§ 9.] When the continuant equation

$$K(1, n) = a_n K(1, n-1) + b_n K(1, n-2),$$

or

$$p_n = a_n p_{n-1} + b_n p_{n-2},$$

which may be regarded as a finite difference equation of the second order, can be solved, we can at once derive from (20) an expression for

$$a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{a_n}.$$

When a_n and b_n are constants, the problem is simply that of finding the general term of a recurring series, already solved in chap. xxxi., § 7.

Example. To find an expression for the nth convergent to

$$F = 1 + \frac{1}{1+} \frac{1}{1+} \cdots \frac{1}{1+} \cdots .$$

Here we have to solve the equation $p_n = p_{n-1} + p_{n-2}$, with the initial conditions $p_0 = 1$, $p_1 = 1$. The result is

$$K(1, n) = p_n = \{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\}/2^{n+1}\sqrt{5}.$$

Hence

$$\frac{p_n}{q_n} = \frac{K(1, n)}{K(2, n)} = \frac{\{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\}/2^{n+1}\sqrt{5}}{\{(1+\sqrt{5})^n - (1-\sqrt{5})^n\}/2^n\sqrt{5}},$$

$$= \tfrac{1}{2} \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n}.$$

From the expression for $K(1, n)$ (all the terms in which reduce in this case to $+1$) we see incidentally that the number of different terms in a continuant of the nth order is

$$\frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}} = \frac{1}{2^n}\{{}_{n+1}C_1 + 5\,{}_{n+1}C_3 + 5^2\,{}_{n+1}C_5 + \ldots\}.$$

§ 10.] When two continued fractions F and F′ are so related that every convergent of F is equal to the convergent of F′ of the same order, they are said to be *equivalent*.*

* We may also have an (m, n)-equivalence, that is, $p_{rm}/q_{rm} = p_{rn}'/q_{rn}'$. See Exercises XXXIII., 2, 17, &c.

It follows at once from §§ 7 and 8 (and is, indeed, otherwise obvious, provided the continued fraction has a definite meaning according to its primary definition) that we may multiply a_r, b_r, and b_{r+1} by any quantity m ($\neq 0$) without disturbing the equivalence of the fraction. Hence we may reduce every continued fraction to an equivalent one which has all its numerators equal to $+1$ or to -1. Thus we have

$$a_1 + \frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\cdots\frac{b_n}{a_n+}\cdots$$
$$= a_1 + \frac{1}{a_2/b_2+}\,\frac{1}{a_3b_2/b_3+}\,\frac{1}{a_4b_3/b_4b_2+}\cdots\frac{1}{a_nb_{n-1}b_{n-3}\ldots/b_nb_{n-2}\ldots+}\cdots \quad (21).$$

§ 11.] If we treat the equations (1) as a linear system to determine K(1, 1), K(1, 2), . . ., K(1, n), and use the determinant notation, we get

$$\mathrm{K}(1,n) = \begin{vmatrix} a_1 & b_2 & 0 & 0 & 0\ldots & 0 & 0 & 0 \\ -1 & a_2 & b_3 & 0 & 0\ldots & 0 & 0 & 0 \\ 0 & -1 & a_3 & b_4 & 0\ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & a_4 & b_5\ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0\ldots & -1 & a_{n-1} & b_n \\ 0 & 0 & 0 & 0 & 0\ldots & 0 & -1 & a_n \end{vmatrix},$$

which gives an expression for a continuant as a determinant. The theory of continuants has been considered from this point of view by Sylvester and Muir;* and many of the theorems regarding them can thus be proved in a very simple and natural manner.

Exercises XXXIII.

(1.) Assuming that both the fractions

$$x = \frac{a}{a+}\,\frac{b}{b+}\,\frac{c}{c+}\cdots, \qquad y = \frac{a}{b+}\,\frac{b}{c+}\,\frac{c}{d+}\cdots$$

are convergent, show that

$$x(a+1+y) = a+y.$$

* See Muir's *Theory of Determinants*, chap. iii.

(2.) If p/q and p'/q' be the ultimate and penultimate convergents to $a+\frac{1}{b+}\ \ldots\ \frac{1}{k}$, show that

$$a+\frac{1}{b+}\ \ldots\ \frac{1}{\overset{*}{k}+}\ \ldots \text{ to } n \text{ periods} = \frac{1}{q}\left[p \mp \frac{1}{q'+p\mp}\ \frac{1}{q'+p\mp}\ \ldots\ \frac{1}{q'}\right],$$

where the quotient $q'+p$ is repeated $n-1$ times, and the upper or the lower sign is to be taken according as p/q is an even or an odd convergent.

(3.) Evaluate $a+\frac{1}{a+}\ \frac{1}{a+}\ \ldots$ to n quotients, a being any real quantity positive or negative. Show from your result that the continued fraction in question always converges to the numerically greatest root of $x^2-ax-1=0$.*

(4.) Deduce from the results of (2) and (3) that a recurring continued fraction whose numerators and denominators are real quantities in general converges to a finite limit; and indicate the nature of the exceptional cases.

(5.) Evaluate $2-\frac{1}{2-}\ \frac{1}{2-}\ \frac{1}{2-}\ \ldots$ to n terms.

(6.) Show that the nth convergent to $\frac{1}{3-}\ \frac{4}{3-}\ \frac{2}{3-}\ \frac{2}{3-}\ \frac{2}{3-}\ \ldots$, every subsequent component being $\frac{2}{3}$, is $(2^n-1)/(2^n+1)$.

(7.) Show that $\frac{x}{x+1-}\ \frac{x}{x+1-}\ \ldots$ to n terms $=\frac{x^{n+1}-x}{x^{n+1}-1}$.

(8.) $$\frac{1}{1-}\ \frac{a}{a+1-}\ \frac{a+1}{a+2-}\ \ldots \text{ } (\overline{n+1} \text{ components})$$
$$=1+a+a(a+1)+\ \ldots\ +a(a+1)\ldots(a+n-1).$$

(9.) If $\phi(n)=\frac{1}{a+}\ \frac{1}{a+}\ \ldots$ n quotients, then

$$\phi(m+n)=\{\phi(m)+\phi(n)-a\phi(m)\phi(n)\}/\{1+\phi(m)\phi(n)\}.$$

(Clausen.)

(10.) Show that

$$\mathrm{K}(0,\ a_2,\ a_3,\ \ldots,\ a_n)=\mathrm{K}(a_2,\ \ldots,\ a_n);$$
$$\mathrm{K}(\ldots\ a,\ b,\ c,\ 0,\ e,\ f,\ g,\ \ldots)=\mathrm{K}(\ldots\ a,\ b,\ c+e,\ f,\ g,\ \ldots);$$
$$\mathrm{K}(\ldots\ a,\ b,\ c,\ 0,\ 0,\ e,\ f,\ g,\ \ldots)=\mathrm{K}(\ldots\ a,\ b,\ c+e,\ f,\ g,\ \ldots);$$
$$\mathrm{K}(\ldots\ a,\ b,\ c,\ 0,\ 0,\ e,\ f,\ \ldots)=\mathrm{K}(\ldots\ a,\ b,\ c,\ e,\ f,\ \ldots).$$

(Muir, *Determinants*, p. 159.)

(11.) Show that the number of terms in a continuant of the nth order is

$$1+(n-1)+\frac{(n-2)(n-3)}{2!}+\frac{(n-3)(n-4)(n-5)}{3!}+\ \ldots\ .$$

(Sylvester.)

(12.) If $p_n=\mathrm{K}\begin{pmatrix} b_2,\ b_3,\ \ldots,\ b_n \\ a_1,\ a_2,\ a_3,\ \ldots,\ a_n \end{pmatrix}$, show that there exists a relation of the form

$$\mathrm{A}p_n^2+\mathrm{B}p_{n-1}^2+\mathrm{C}p_{n-2}^2+\mathrm{D}p_{n-3}^2=0,$$

where A, B, C, D are integral functions of a_n, b_n, a_{n-1}, b_{n-1}.

---

* This is a particular case of the theorem (due to Euler?) that the numerically greatest root of $x^2-px+q=0$ is $p-\frac{q}{p-}\ \frac{q}{p-}\ \ldots$.

(13.) Show that

$$\mathrm{K}\begin{pmatrix} b_1, & (b_1+a_1)b_2, & (b_2+a_2)b_3, & \dots \\ 1, & a_1, & a_2, & a_3, \dots \end{pmatrix} = (b_1+a_1)(b_2+a_2)(b_3+a_3)\dots;$$

and deduce the theorem of § 19. (Muir, *l.c.*)

Taking $(a, b, c, \dots, k)$ to denote the continued fraction $\frac{1}{a-}\frac{1}{b-}\frac{1}{c-}\dots\frac{1}{k}$, and $[a, b, c, \dots, k]$, or, when no confusion is likely, $[a, k]$, to denote $\mathrm{K}\begin{pmatrix} -1, & -1, & \dots, & -1 \\ a, & b, & \dots, & k \end{pmatrix}$, prove the following theorems *:—

(14.) If $x=(a, b, c, \dots, e, y)$, then $y=(e, \dots, c, b, a, x)$;

$$xy-(e, \dots, a)x-(a, \dots, e)y+(c, \dots, a)(a, \dots, d)=0;$$

$$(a, \dots, c)(c, \dots, b)=(e, \dots, a)(a, \dots, d);$$

$$\{x-(a, \dots, e)\}\{y-(e, \dots, a)\}$$
$$=(e, \dots, a)^2(d, \dots, a)^2(c, \dots, a)^2\dots(a)^2$$

(15.) $(a, \dots, e)-(a, \dots, d)=(e, \dots, a)(d, \dots, a)^2(c, \dots, a)^2\dots(a)^2$.

(16.) $[a, b, c, d, e]=1/(a, b, c, d, e)(b, c, d, e)(c, d, e)(d, e)(e)$.

(17.) Prove the following equivalence theorem:—

$$(a, \dots, e, f,\ a', \dots, e', f',\ a'', \dots, e'', f'',\ a''', \dots, e''', f''')$$
$$=\frac{1}{[a, e]}\left\{[b, e]+\frac{[a', e']}{[a, e']-}\frac{[a, e][a'', e'']}{[a', e'']-}\frac{[a', e'][a''', e''']}{[a'', e''']-}\frac{[a'', e']}{[a''', e''']f'''-[a''', d''']}\right\}.$$

(18.) $(a, f,\ a', f',\ a'', f'',\ a''', f''', \dots)$

$$=\frac{1}{a}\left\{1+\frac{a'}{afa'-a-a'-}\frac{aa''}{a'f'a''-a'-a''-}\frac{a'a'''}{a''f''a'''-a''-a'''-}\dots\right\}.$$

(19.) $a+\frac{1}{m+}\frac{1}{b+}\frac{1}{m+}\frac{1}{c+}\frac{1}{m+}\dots$

$$=\frac{1}{m}\left\{am+1-\frac{1}{2+bm-}\frac{1}{2+cm-}\dots\right\}.$$

(20.) $\sqrt{2}=1+\frac{1}{2+}\frac{1}{2+}\dots=\frac{1}{5}\left\{7+\frac{1}{14+}\frac{1}{14+}\frac{1}{14+}\dots\right\}.$

(21.) $(a, \dots, e, f,\ a, \dots, e, f',\ a, \dots, e, f'', \dots \text{ad } \infty)$
$-(e, \dots, a, f,\ e, \dots, a, f',\ e, \dots, a, f'', \dots \text{ad } \infty)$
$=(a, \dots, e)-(e, \dots, a).$

(22.) Show that the successive constituents $\alpha, \beta, \gamma, \dots, \lambda, \mu, \nu$ may be omitted from the continued fraction $(\dots a, b, a, \beta, \gamma, \dots, \lambda, \mu, \nu, c, d, \dots)$ without altering its value, provided $[\beta, \dots, \mu]=\pm 1$, $\alpha=\pm[\gamma, \dots, \mu]$, and $\nu=\pm[\beta, \dots, \lambda]$; and construct examples.

(23.) If $x=(\overset{*}{a}, \dots, e, \overset{*}{f}, \dots)$, the other root of the quadratic equation to which this leads is $x=(\overset{*}{f}, c, \dots, \overset{*}{a}, \dots)$.

(24.) If $b+\frac{1}{b_1+}\dots\frac{1}{b_m+}\frac{1}{\overset{*}{a}+}\dots\frac{1}{\overset{*}{a_m}+}\dots$ be one root of a quadratic

* The notation and the order of ideas used in (14) to (23), as well as some of the special results, are due to Möbius (*Crelle's Jour.*, 1830).

equation, the other is

$$b+\frac{1}{b_1+}\cdots\frac{1}{b_m-a_m-}\;\frac{1}{a_{m-1}+}\;\frac{1}{a_{m-2}+}\cdots\frac{1}{a+}\;\frac{1}{a_m+}\cdots\frac{1}{a+}\cdots.$$

(Stern, *Crelle's Jour.*, 1827.) *

(25.) If $q>p$, show that

$$1=\frac{q-p}{q-}\;\frac{pq(q-p)}{q^2-p^2-}\;\frac{pq(q-p)^2}{q^2-p^2-}\cdots;$$

$$(q-p)q=q^2-p^2-\frac{pq(q-p)^2}{q^2-p^2-}\cdots.$$

CONVERGENCE OF INFINITE CONTINUED FRACTIONS.

§ 12.] By the value or limit of an infinite continued fraction is meant the limit, if any such exist, towards which the convergent p_n/q_n approaches when n is made infinitely great. It may happen that this limit is finite and definite; the fraction is then said to be *convergent*. It may happen that $\mathrm{L}_{n=\infty}\, p_n/q_n$ fluctuates between a certain number of finite values according to the integral character of n; the fraction is then said to *oscillate*. Finally, it may happen that $\mathrm{L}_{n=\infty}\, p_n/q_n$ tends constantly towards $\pm\infty$; in this case the fraction is said to be divergent.

We have already seen that all simple continued fractions are convergent. The fraction $1-\frac{1}{1-}\;\frac{1}{1-}\;\frac{1}{1-}\cdots$ is an obvious example of oscillation, its value being 1, 0, or $-\infty$ according as $n=3m+1$, $3m+2$, or $3m+3$.

The fraction $1-\frac{1}{-\frac{1}{2}+\frac{1}{2}\sqrt{5}-}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\cdots$ diverges to $-\infty$, for $\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\cdots$ converges to $-\frac{1}{2}+\frac{1}{2}\sqrt{5}$, as may be easily seen from the expression for its nth convergent given in § 9.

The last example brings into view a fact which it is important to notice, namely, that the divergence of an infinite continued fraction is something quite different from the divergence of an infinite series. The divergence of the fraction is, in fact, an accidental phenomenon, and will in general disappear if we modify the fraction by omitting a constituent. It is therefore

* (23) and (24) are generalisations of an older theorem of Galois'. See *Gergonne Ann. d. Math.*, t. xix.

not safe in general to argue that a continued fraction does not diverge because the continued fraction formed by taking all its constituents after a certain order converges.

With the exception of simple continued fractions and recurring continued fractions (whether simple or not), the only cases where rules of any generality have been found for testing convergency are continued fractions of the "first" and "second class." To these we shall confine ourselves in what follows.*

§ 13.] *A continued fraction of the first class cannot be divergent; and it will be convergent or oscillating if any one of the residual fractions* $x_2, x_3, \ldots, x_n, \ldots$ *converge or oscillate.*

The latter part of this proposition is at once obvious from the equation

$$x_1 = a_1 + \frac{b_2}{a_2 +} \frac{b_3}{a_3 +} \cdots \frac{b_n}{x_n}.$$

Again, since (§ 3, Cor. 6) the odd convergents continually increase and the even convergents continually decrease, while any even convergent is greater than any following odd convergent, it follows that $\mathrm{L}p_{2n}/q_{2n} = \mathrm{A}$ and $\mathrm{L}p_{2n-1}/q_{2n-1} = \mathrm{B}$, where A and B are two finite quantities, and $\mathrm{A} \leqslant \mathrm{B}$. If $\mathrm{A} = \mathrm{B}$, the fraction is convergent; if $\mathrm{A} > \mathrm{B}$, it oscillates; and no other case can arise.

§ 14.] *A continued fraction of the first class is convergent if the series* $\Sigma a_{n-1}a_n/b_n$ *be divergent.*

We have, since all the quantities involved are positive,

$$\begin{aligned}
q_n &= a_n q_{n-1} + b_n q_{n-2}; \\
q_{n-1} &= a_{n-1} q_{n-2} + b_{n-1} q_{n-3}, \quad q_{n-1} > a_{n-1} q_{n-2}; \\
q_{n-2} &= a_{n-2} q_{n-3} + b_{n-2} q_{n-4}, \quad q_{n-2} > a_{n-2} q_{n-3}; \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
q_4 &= a_4 q_3 + b_4 q_2, \quad q_4 > a_4 q_3; \\
q_3 &= a_3 q_2 + b_3 q_1, \quad q_3 > a_3 q_2; \\
q_2 &= a_2 q_1.
\end{aligned}$$

* Our knowledge of the convergence of continued fractions is chiefly due to Schlömilch, *Handb. d. Algebraische Analysis* (1845); Arndt, *Disquisitiones Nonnullæ de Fractionibus Continuis*, Sundiæ (1845); Seidel, *Untersuchungen über die Convergenz und Divergenz der Kettenbrüche* (Habilitationsschrift München, 1846); also *Abhandlungen d. Math. Classe d. K. Bayerischen Akad. d. Wiss.*, Bd. vii. (1855); and Stern, *Crelle's Jour.*, xxxvii. (1848).

Hence

$$q_n > (a_n a_{n-1} + b_n) q_{n-2},$$
$$q_{n-1} > (a_{n-1} a_{n-2} + b_{n-1}) q_{n-3},$$
$$q_{n-2} > (a_{n-2} a_{n-3} + b_{n-2}) q_{n-4},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$q_4 > (a_4 a_3 + b_4) q_2,$$
$$q_3 = (a_3 a_2 + b_3) q_1.$$

Therefore

$$q_n q_{n-1} > q_1 q_2 (b_2 + a_2 a_3)(b_4 + a_3 a_4) \ldots (b_n + a_{n-1} a_n),$$

and, since $q_1 = 1$, $q_2 = a_2$,

$$\frac{q_n q_{n-1}}{b_2 b_3 \ldots b_n} > \frac{a_2}{b_2}\left(1 + \frac{a_2 a_3}{b_3}\right)\left(1 + \frac{a_3 a_4}{b_4}\right) \ldots \left(1 + \frac{a_{n-1} a_n}{b_n}\right) \quad (1).$$

Now, since $\Sigma a_{n-1} a_n / b_n$ is divergent, $\Pi(1 + a_{n-1} a_n / b_n)$ diverges to $+\infty$ (chap. xxvi., § 23), therefore $\mathrm{L} q_n q_{n-1} / b_2 b_3 \ldots b_n = +\infty$. Hence

$$\mathrm{L}\left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right) = \mathrm{L}\frac{b_2 b_3 \ldots b_{2n}}{q_{2n} q_{2n-1}} = 0,$$

that is, the continued fraction is convergent.

Cor. 1. *The fraction in question is convergent if* $\mathrm{L} a_{n-1} a_n / b_n > 0$.

Cor. 2. *Also if* $\mathrm{L} a_n / b_n > 0$, *and* Σa_n *be divergent.*

Cor. 3. *Also if* $\mathrm{L} a_{n+1} b_n / a_{n-1} b_{n+1} > 1$.

The above criterion is simple in practice; but it is not complete, inasmuch as it is not proved that oscillation follows if $\Sigma a_{n-1} a_n / b_n$ be convergent. The theorem of next paragraph supplies this defect.

§ 15.] *If a continued fraction of the first class be reduced to the form*

$$d_1 + \frac{1}{d_2 +} \frac{1}{d_3 +} \frac{1}{d_4 +} \cdots \frac{1}{d_n +} \cdots \quad (4),$$

so that

$$d_1 = a_1, \quad d_2 = \frac{a_2}{b_2}, \quad d_3 = \frac{a_3 b_2}{b_3}, \quad d_4 = \frac{a_4 b_3}{b_4 b_2}, \quad \ldots,$$

$$d_n = \frac{a_n b_{n-1} b_{n-3} \cdots}{b_n b_{n-2} \cdots} \quad (5),$$

then it is convergent if at least one of the series

$$d_3 + d_5 + d_7 + \ldots \quad (6)$$
$$d_2 + d_4 + d_6 + \ldots \quad (7)$$

be divergent, oscillating if both these series be convergent.

This proposition depends on the following inequalities between the q's and d's of the fraction (4):—

$$0 < q_n < (1 + d_2)(1 + d_3) \ldots (1 + d_n) \qquad (8);$$

$$q_{2n} > d_2 + d_4 + \ldots + d_{2n} \qquad (9);$$

$$q_{2n-1} > 1 \qquad (10).$$

These follow at once from Euler's law for the formation of the terms in q_n, which, in the present case, runs as follows:— Write down $d_2 d_3 \ldots d_n$ and all the terms that can be formed therefrom by omitting any number of pairs of consecutive d's. We thus see that q_n contains fewer terms than the product $(1 + d_2)(1 + d_3) \ldots (1 + d_n)$; and, since the terms are all positive, (8) follows. Again, in forming the terms of the 1st degree in q_{2n}, we can only have letters that stand in odd places in the succession $d_2 d_3 d_4 \ldots d_{2n}$; hence (9); and (10) is obvious from a similar consideration.

To apply this to our present purpose, we observe that, since the numerators are all equal to 1, we have

$$\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}} = \frac{1}{q_{2n} q_{2n-1}} \qquad (11).$$

If we suppose $d_2 \neq 0$, neither q_{2n} nor q_{2n-1} can vanish. Hence, if both $\mathrm{L}q_{2n}$ and $\mathrm{L}q_{2n-1}$ be finite, the fraction will oscillate, and if one of them be infinite it will converge.

Now, if both the series (6) and (7) converge, the series $d_2 + d_3 + d_4 + \ldots + d_n$ will converge; and the product on the right of (8) will be finite when $n = \infty$. In this case, therefore, both q_{2n} and q_{2n-1} will be finite; and the fraction (4) will oscillate.

If the series $d_2 + d_4 + d_6 + \ldots$ diverge, then by (9) $\mathrm{L}q_{2n} = \infty$, and the fraction (4) will converge.

By the same reasoning, if the series $d_3 + d_5 + d_7 + \ldots$ diverge, then the fraction

$$d_2 + \frac{1}{d_3 +} \frac{1}{d_4 +} \cdots \frac{1}{d_n +} \cdots$$

will converge; and consequently the fraction (4) will converge.

Remark.—We might deduce the criterion of last paragraph from the above. For we have

$$d_1 d_2 = a_1 a_2/b_2, \quad d_2 d_3 = a_2 a_3/b_3, \quad \ldots, \quad d_{n-1} d_n = a_{n-1} a_n/b_n.$$

Now, if the series Σd_n converge, the series formed by adding together the products of every possible pair of its terms must, by chap. xxx., § 2, converge: *a fortiori*, the series $\Sigma d_{n-1} d_n$, that is, $\Sigma a_{n-1} a_n/b_n$, must converge. Hence, if this last series diverge, Σd_n cannot converge. Σd_n must therefore diverge, since it cannot oscillate, all its terms being positive. Therefore either (6) or (7) must diverge, that is to say, the fraction (4) must converge.

Example 1. Consider the fraction

$$1 + \frac{1^2}{2+}\frac{2^2}{2+}\frac{3^2}{2+} \cdots .$$

Here
$$d_{2n+1} = \frac{2(2n-1)^2(2n-3)^2 \ldots 3^2.1^2}{(2n)^2(2n-2)^2 \ldots 4^2.2^2}.$$

It may be shown, by the third criterion of chap. xxvi., § 6, Cor. 5, that the series Σd_{2n+1} is divergent. Or we may use Stirling's Theorem. Thus, when n is very great, we have very nearly

$$\begin{aligned} d_{2n+1} &= 2(2n\,!)^2/2^{4n}(n\,!)^4, \\ &= 2[\{\sqrt{(2\pi 2n)}\,(2n/e)^{2n}\}/\{2^{2n}(2\pi n)\,(n/e)^{2n}\}]^2, \\ &= 2/\pi n. \end{aligned}$$

The convergence of Σd_{2n+1} is therefore comparable with that of $\Sigma 1/n$, which is divergent.

Hence the continued fraction in question converges.

Example 2.

$$a + \frac{x}{a+}\frac{x^2}{a+}\frac{x^3}{a+} \cdots$$

oscillates or converges according as $x > 1$ or $\not> 1$.

Example 3.

$$\frac{1}{2+}\frac{2}{3+}\frac{3}{4+} \cdots .$$

Here
$$\mathrm{L} a_{n-1} a_n/b_n = \mathrm{L}(n-1)n/(n+1) = \infty,$$

therefore the fraction is convergent.

§ 16.] There is no comprehensive criterion for the convergence of fractions of the second class; but the following theorem embraces a large number of important cases:—

If an infinite continued fraction of the second class of the form

$$\mathrm{F} = \frac{b_2}{a_2 -}\frac{b_3}{a_3 -} \cdots \frac{b_n}{a_n -} \cdots \qquad (1)$$

be such that

$$a_n \geqq b_n + 1 \qquad (2)$$

for all values of n, it converges to a finite limit F *not greater than unity.*

If the sign > *occur at least among the conditions* (2), *then* $F < 1$.

If the sign = *alone occur, then* $F = 1 - 1/S$, *where*

$$S = 1 + b_2 + b_2b_3 + b_2b_3b_4 + \ldots + b_2b_3 \ldots b_n + \ldots \text{ ad } \infty \quad (A),$$

so that F = *or* < 1 *according as the series in* (A) *is divergent or convergent.*

These results follow from the following characteristic properties of the restricted fraction (1):—

$$p_n - p_{n-1} \geqq b_2b_3 \ldots b_n \qquad (3);$$

$$p_n \geqq b_2 + b_2b_3 + b_2b_3b_4 + \ldots + b_2b_3 \ldots b_n \qquad (4);$$

$$q_n - q_{n-1} \geqq b_2b_3 \ldots b_n \qquad (5);$$

$$q_n \geqq 1 + b_2 + b_2b_3 + \ldots + b_2b_3 \ldots b_n \qquad (6);$$

$$q_n - p_n \geqq q_{n-1} - p_{n-1} \geqq \ldots \geqq q_2 - p_2 \geqq 1 \qquad (7).$$

To prove (3) we observe that

$$p_n - p_{n-1} = (a_n - 1)p_{n-1} - b_np_{n-2}.$$

Hence, since p_n, q_n are positive and increase with n (§ 2, Cor. 1),

$p_n - p_{n-1} \geqq b_n(p_{n-1} - p_{n-2})$,	acc. as $a_n \geqq b_n + 1$;
$p_{n-1} - p_{n-2} \geqq b_{n-1}(p_{n-2} - p_{n-3})$,	acc. as $a_{n-1} \geqq b_{n-1} + 1$;
. . . . .	. . . . .
$p_3 - p_2 \geqq b_3b_2$.	acc. as $a_3 \geqq b_3 + 1$.

Therefore $p_n - p_{n-1} \geqq b_2b_3 \ldots b_n$, where the upper sign must be taken if it occur anywhere among the conditions to the right of the vertical line.

To prove (4), we have merely to put in (3) $n-1$, $n-2$, . . ., 3 in place of n, adjoin the equation $p_2 = b_2$, and add all the resulting equations.

(5) and (6) are established in precisely the same way.

It follows, of course, that p_n and q_n both remain finite or both become infinite when $n = \infty$, according as the series in (6) is convergent or divergent.

To prove (7), we have

$$q_n - p_n = a_n(q_{n-1} - p_{n-1}) - b_n(q_{n-2} - p_{n-2}),$$
$$\geqq (q_{n-1} - p_{n-1}) + b_n\{(q_{n-1} - p_{n-1}) - (q_{n-2} - p_{n-2})\},$$

according as $a_n \geqq b_n + 1$, provided $q_{n-1} - p_{n-1}$ is positive.

This shows that, if any one of the relations in (7) hold, the next in order follows. Now $q_2 - p_2 = a_2 - b_2 \geqq 1$, according as $a_2 \geqq b_2 + 1$; and $q_3 - p_3 = a_2a_3 - b_3 - b_2a_3 \geqq (a_3 - b_3)(b_2 + 1) - b_3 \geqq (a_2 - b_2) + b_3(a_2 - b_2 - 1)$, according as $a_3 \geqq b_3 + 1$; hence the theorem. It is important to observe that the first $>$ that occurs among the relations $a_2 \geqq b_2 + 1$, $a_3 \geqq b_3 + 1$, . . . determines the first $>$ that occurs among the relations (7): all the signs to the right of this one will be $=$, all those to the left $>$.

The convergency theorems for the restricted fraction of the second class follow at once. In the first place, as we have already seen in § 3, the convergents to (1) form an increasing series of positive quantities, so that there can be no oscillation. Also, since $q_n - p_n \geqq 1$, it follows that

$$p_n/q_n \leqq 1 - 1/q_n \qquad (8).$$

Therefore, since $q_n > 1$, it follows that F converges to a finite limit $\not> 1$.

If the sign $>$ occur at least once among the relations (2), the sign $<$ must be taken in (8); that is, $F < 1$.

If the sign $=$ occur throughout, we have

$$\mathrm{L}p_n/q_n = 1 - \mathrm{L}1/q_n = 1 - 1/\mathrm{S},$$

where S is the sum to infinity of the series (6). Hence, if (6) converge, $F < 1$; if it diverge, $F = 1$.

If we dismiss from our minds the question of convergency, and therefore remove the restriction that $b_2, b_3, \ldots, b_n$ be positive, but still put $a_n = b_n + 1$, $a_{n-1} = b_{n-1} + 1$, . . ., $a_3 = b_3 + 1$, $a_2 = b_2 + 1$, we get by the above reasoning

$$p_n/q_n = 1 - 1/q_n \qquad (8');$$
$$q_n = 1 + b_2 + b_2b_3 + \ldots + b_2b_3 \ldots b_n \qquad (6').$$

Now (8′) gives us $q_n = 1/(1 - p_n/q_n)$. Hence the following remarkable transformation theorem :—

Cor. 1. *If* $b_2, \ldots, b_n$ *be any quantities whatsoever, then*

$$1 + b_2 + b_2b_3 + \ldots + b_2b_3 \ldots b_n = \frac{1}{1 - \dfrac{b_2}{b_2 + 1 - \dfrac{b_3}{b_3 + 1 - \cdots \dfrac{b_n}{b_n + 1}}}} \quad (9),$$

from which, putting $u_1 = b_2$, $u_2 = b_2b_3$, $\ldots$, $u_n = b_2b_3 \ldots b_{n+1}$, we readily derive

$$1 + u_1 + u_2 + \ldots + u_n = \frac{1}{1 - \dfrac{u_1}{1 + u_1 - \dfrac{u_2}{u_1 + u_2 - \dfrac{u_1u_3}{u_2 + u_3 - \dfrac{u_2u_4}{u_3 + u_4 - \cdots \dfrac{u_{n-3}u_{n-1}}{u_{n-2} + u_{n-1} - \dfrac{u_{n-2}u_n}{u_{n-1} + u_n}}}}}} \quad (10),$$

an important theorem of Euler's to which we shall return presently.

INCOMMENSURABILITY OF CERTAIN CONTINUED FRACTIONS.

§ 17.] *If* $a_2, a_3, \ldots, a_n, b_2, b_3, \ldots, b_n$ *be all positive integers, then*

I. *The infinite continued fraction*

$$\frac{b_2}{a_2 + \dfrac{b_3}{a_3 + \cdots \dfrac{b_n}{a_n + \cdots}}} \quad (1)$$

converges to an incommensurable limit provided that after some finite value of n *the condition* $a_n \lessdot b_n$ *be always satisfied.*

II. *The infinite continued fraction*

$$\frac{b_2}{a_2 - \dfrac{b_3}{a_3 - \cdots \dfrac{b_n}{a_n - \cdots}}} \quad (2)$$

converges to an incommensurable limit provided that after some finite value of n *the condition* $a_n \geqq b_n + 1$ *be always satisfied, where the sign* $>$ *need not always occur but must occur infinitely often.**

To prove II., let us first suppose that the condition $a_n \geqq b_n + 1$ holds from the first. Then (2) converges, by § 16,

* These theorems are due to Legendre, *Éléments de Géométrie*, note iv.

to a positive value < 1. Let us assume that it converges to a commensurable limit, say λ_2/λ_1, where λ_1, λ_2 are positive integers, and $\lambda_1 > \lambda_2$.

Let now

$$\rho_3 = \frac{b_3}{a_3 -}\,\frac{b_4}{a_4 -}\ \cdots\ .$$

Since the sign $>$ must occur among the conditions $a_3 \geqq b_3 + 1$, $a_4 \geqq b_4 + 1$, . . ., ρ_3 must be a positive quantity < 1. Now, by our hypothesis,

$$\lambda_2/\lambda_1 = b_2/(a_2 - \rho_3),$$

therefore

$$\rho_3 = (a_2\lambda_2 - b_2\lambda_1)/\lambda_2,$$
$$= \lambda_3/\lambda_2, \text{ say,}$$

where $\lambda_3 = a_2\lambda_2 - b_2\lambda_1$ is an integer, which must be positive and $< \lambda_2$, since ρ_3 is positive and < 1.

Next, put

$$\rho_4 = \frac{b_4}{a_4 -}\,\frac{b_5}{a_5 -}\ \cdots\ .$$

Then, exactly as before, we can show that $\rho_4 = \lambda_4/\lambda_3$, where λ_4 is a positive integer $< \lambda_3$.

Since the sign $>$ occurs infinitely often among the conditions $a_n \geqq b_n + 1$, this process can be repeated as often as we please. The hypothesis that the fraction (2) is commensurable therefore requires the existence of an infinite number of positive integers λ_1, λ_2, λ_3, λ_4, . . . such that $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \ldots$; but this is impossible, since λ_1 is finite. Hence (2) is incommensurable.

Next suppose the condition $a_n \geqq b_n + 1$ to hold after $n = m$. Then, by what has been shown,

$$y = \frac{b_{m+1}}{a_{m+1} -}\,\frac{b_{m+2}}{a_{m+2} -}\ \cdots$$

is incommensurable.

Now we have

$$\mathrm{F} = \frac{b_2}{a_2 -}\,\frac{b_3}{a_3 -}\ \cdots\ \frac{b_m}{a_m - y},$$

consequently

$$\mathrm{F} = \frac{(a_m - y)p_{m-1} - b_m p_{m-2}}{(a_m - y)q_{m-1} - b_m q_{m-2}},$$

$$= \frac{p_m - y p_{m-1}}{q_m - y q_{m-1}} \qquad (3),$$

where p_m/q_m, p_{m-1}/q_{m-1} are the ultimate and penultimate convergents of

$$\frac{b_2}{a_2-}\ \frac{b_3}{a_3-}\ \cdots\ \frac{b_m}{a_m}.$$

It results from (3) that

$$y(\mathrm{F}q_{m-1}-p_{m-1})=\mathrm{F}q_m-p_m \qquad (4).$$

Now $\mathrm{F}q_{m-1}-p_{m-1}$ and $\mathrm{F}q_m-p_m$ cannot both be zero, for that would involve the equality $p_m/q_m=p_{m-1}/q_{m-1}$, which is inconsistent with the equation (2) of § 3. Hence, if F were commensurable, (4) would give a commensurable value for the incommensurable y. F must therefore be incommensurable.

The proof of I. is exactly similar, for the condition $a_n \not< b_n$ secures that each of the residual fractions of (1) shall be positive and less than unity.

These two theorems do not by any means include all cases of incommensurability in convergent infinite continued fractions. Brouncker's fraction, for example, $1+\dfrac{1^2}{2+}\ \dfrac{3^2}{2+}\ \dfrac{5^2}{2+}\ \cdots$, converges to the incommensurable value $4/\pi$, and yet violates the condition of Proposition I.

CONVERSION OF SERIES AND CONTINUED PRODUCTS INTO CONTINUED FRACTIONS.

§ 18.] *To convert the series*

$$u_1+u_2+\ \cdots\ +u_n+\ \cdots$$

into an "equivalent" continued fraction of the form

$$\frac{b_1}{a_1-}\ \frac{b_2}{a_2-}\ \cdots\ \frac{b_n}{a_n-}\ \cdots \qquad (1).$$

A continued fraction is said to be "equivalent" to a series when the nth convergent of the former is equal to the sum of n terms of the latter for all values of n.

Since the convergents merely are given, we may leave the denominators $q_1, q_2, \ldots, q_n$ arbitrary (we take $q_0=1$, as usual).

For the fraction (1) we have

$$p_n/q_n - p_{n-1}/q_{n-1} = b_1 b_2 \ldots b_n/q_{n-1}q_n \qquad (2);$$

$$q_1 = a_1, \quad q_2 = a_2 q_1 - b_2, \quad \ldots, \quad q_n = a_n q_{n-1} - b_n q_{n-2} \qquad (3);$$

$$p_1/q_1 = b_1/q_1 \qquad (4).$$

Since

$$p_n/q_n = u_1 + u_2 + \ldots + u_n \qquad (5),$$

we get from (2) and (5)

$$\left.\begin{aligned} u_n &= b_1 b_2 \ldots b_n/q_{n-1}q_n, \\ u_{n-1} &= b_1 b_2 \ldots b_{n-1}/q_{n-2}q_{n-1}, \\ &\cdots\cdots\cdots \\ u_2 &= b_1 b_2/q_1 q_2, \\ u_1 &= b_1/q_1. \end{aligned}\right\} \qquad (6).$$

From (6), by using successive pairs of the equations, we get

$$b_1 = q_1 u_1, \quad b_2 = q_2 u_2/u_1, \quad b_3 = q_3 u_3/q_1 u_2, \quad \ldots, \quad b_n = q_n u_n/q_{n-2}u_{n-1} \qquad (7).$$

Combining (3) with (7), we also find

$$a_1 = q_1, \quad a_2 = q_2(u_1 + u_2)/q_1 u_1, \quad a_3 = q_3(u_2 + u_3)/q_2 u_2, \quad \ldots,$$
$$a_n = q_n(u_{n-1} + u_n)/q_{n-1}u_{n-1} \qquad (8).$$

Hence

$$S_n = u_1 + u_2 + \ldots + u_n,$$
$$= \cfrac{q_1 u_1}{q_1 - \cfrac{q_2 u_2/u_1}{q_2(u_1+u_2)/q_1 u_1 - \cfrac{q_3 u_3/q_1 u_2}{q_3(u_2+u_3)/q_2 u_2 - \ldots \cfrac{q_n u_n/q_{n-2}u_{n-1}}{q_n(u_{n-1}+u_n)/q_{n-1}u_{n-1}}}}} \qquad (9).$$

It will be observed that the q's may be cleared out of the fraction. Thus, for example, we get rid of q_1 by multiplying the first and second numerators and the first denominator by $1/q_1$, and the second and third numerators and the second denominator by q_1; and so on. We thus get for S_n the equivalent fraction

$$S_n = \cfrac{u_1}{1 - \cfrac{u_2/u_1}{(u_1+u_2)/u_1 - \cfrac{u_3/u_2}{(u_2+u_3)/u_2 - \ldots \cfrac{u_n/u_{n-1}}{(u_{n-1}+u_n)/u_{n-1}}}}} \qquad (10),$$

which may be thrown into the form

$$S_n = \cfrac{u_1}{1 - \cfrac{u_2}{u_1 + u_2 - \cfrac{u_1 u_3}{u_2 + u_3 - \ldots \cfrac{u_{n-2}u_n}{u_{n-1}+u_n}}}} \qquad (11).$$

This formula is practically the same as the one obtained incidentally in § 16; it was first given, along with many applications, by Euler in his memoir, "De Transformatione Serierum in Fractiones Continuas," *Opuscula Analytica*, t. ii. (1785).

It is important to remark that, since the continued fraction (10) or (11) is *equivalent* to the series, it must converge if the series converges, and that to the same limit.

By giving to $u_1, u_2, \ldots, u_n$ various values, and modifying the fraction by introducing multipliers as above, we can deduce a variety of results, among which the following are specially useful:—

$$v_1x + v_2x^2 + \ldots + v_nx^n = \frac{v_1x}{1 -} \frac{v_2x}{v_1 + v_2x -} \frac{v_1v_3x}{v_2 + v_3x -} \cdots \frac{v_{n-2}v_nx}{v_{n-1} + v_nx} \quad (12);$$

$$\frac{x}{v_1} + \frac{x^2}{v_2} + \ldots + \frac{x^n}{v_n} = \frac{x}{v_1 -} \frac{v_1^2x}{v_1x + v_2 -} \frac{v_2^2x}{v_2x + v_3 -} \cdots \frac{v_{n-1}^2x}{v_{n-1}x + v_n} \quad (13);$$

$$\frac{a_1}{b_1}x + \frac{a_1a_2}{b_1b_2}x^2 + \ldots + \frac{a_1a_2\ldots a_n}{b_1b_2\ldots b_n}x^n = \frac{a_1x}{b_1 -} \frac{b_1a_2x}{b_2 + a_2x -} \frac{b_2a_3x}{b_3 + a_3x -} \cdots \frac{b_{n-1}a_nx}{b_n + a_nx} \quad (14).$$

Example 1. If $-\frac{1}{4}\pi < x < \frac{1}{4}\pi$, then

$$\tan^{-1}x = x - x^3/3 + x^5/5 - x^7/7 + \ldots, = \frac{x}{1 +} \frac{1^2x^2}{3 - x^2 +} \frac{3^2x^2}{5 - 3x^2 +} \frac{5^2x^2}{7 - 5x^2 +} \cdots;$$

and, in particular,

$$\frac{\pi}{4} = \frac{1}{1 +} \frac{1^2}{2 +} \frac{3^2}{2 +} \frac{5^2}{2 +} \cdots,$$

which is Brouncker's formula for the quadrature of the circle.

Example 2. If $x < 1$,

$$(1+x)^m = 1 + \frac{mx}{1 -} \frac{1(m-1)x}{2 + (m-1)x -} \frac{2(m-2)x}{3 + (m-2)x -} \frac{3(m-3)x}{4 + (m-3)x -} \cdots$$

Also, if $m > -1$,

$$2^m = 1 + \frac{m}{1-} \frac{1(m-1)}{m+1-} \frac{2(m-2)}{m+1-} \frac{3(m-3)}{m+1-} \cdots;$$

and, if $m > 0$,

$$0 = 1 - \frac{m}{1+} \frac{1(m-1)}{3-m+} \frac{2(m-2)}{5-m+} \frac{3(m-3)}{7-m+} \cdots.$$

§ 19.] *The analysis of last paragraph enables us to construct a continued fraction, say of the form* (1), *whose first n convergents shall be any given quantities* $f_1, f_2, \ldots, f_n$ *respectively.*

All we have to do is to replace $u_1, u_2, \ldots, u_n$ in (10) or (11) by $f_1, f_2 - f_1, \ldots, f_n - f_{n-1}$ respectively.

The required fraction is, therefore,

$$\frac{f_1}{1-} \frac{f_2 - f_1}{f_2 -} \frac{f_1(f_3 - f_2)}{f_3 - f_1 -} \frac{(f_2 - f_1)(f_4 - f_3)}{f_4 - f_2 -} \cdots \frac{(f_{n-2} - f_{n-3})(f_n - f_{n-1})}{f_n - f_{n-2}} \quad (15).$$

Cor. *Hence we can express any continued product, say*

$$P_n = \frac{d_1 d_2 \ldots d_n}{e_1 e_2 \ldots e_n},$$

as a continued fraction.

We have merely to put $f_1 = d_1/e_1$, $f_2 = d_1 d_2/e_1 e_2$, . . ., effect some obvious reductions, and we find

$$P_n = \frac{d_1}{e_1 -} \frac{e_1(d_2 - e_2)}{d_2 -} \frac{d_2 e_2 (d_3 - e_3)}{d_2 d_3 - e_2 e_3 -} \frac{d_3 e_3 (d_2 - e_2)(d_4 - e_4)}{d_3 d_4 - e_3 e_4 -} \frac{d_4 e_4 (d_3 - e_3)(d_5 - e_5)}{d_4 d_5 - e_4 e_5 -} \cdots \frac{d_{n-1} e_{n-1} (d_{n-2} - e_{n-2})(d_n - e_n)}{d_{n-1} d_n - e_{n-1} e_n} \quad (16).*$$

§ 20.] Instead of requiring that the continued fraction be equivalent to the series, or to the function $f(n, x)$, which it is to represent, we may require that the sum to infinity of the series (or $f(\infty, x)$) be reduced to *a fraction of a given form*, say

$$\frac{\beta_0}{1-} \frac{\beta_1 x}{1-} \frac{\beta_2 x}{1-} \cdots \frac{\beta_n x}{1-} \cdots \quad (1),$$

where $\beta_0, \beta_1, \ldots, \beta_n$ are all independent of x.

There is a process, originally given in Lambert's *Beyträge*

* A similar formula, given by Stern, *Crelle's Jour.*, x., p. 267 (1833), may be obtained by a slight modification of the above process.

(th. ii., p. 75), *for reducing to the form* (1) *the quotient of two convergent series, say* $F(1, x)/F(0, x)$.

We suppose that the absolute terms of $F(1, x)$ and $F(0, x)$ do not vanish, and, for simplicity, we take each of these terms to be 1. Then we can establish an equation of the form

$$F(1, x) - F(0, x) = \beta_1 x F(2, x) \qquad (2_1),$$

where $F(2, x)$ is a convergent series whose absolute term we suppose again not to vanish, and β_1 is the coefficient of x in $F(1, x) - F(0, x)$, which also is supposed not to vanish.*

In like manner we establish the series of equations

$$F(2, x) - F(1, x) = \beta_2 x F(3, x) \qquad (2_2),$$
$$F(3, x) - F(2, x) = \beta_3 x F(4, x) \qquad (2_3),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$F(n+1, x) - F(n, x) = \beta_{n+1} x F(n+2, x) \qquad (2_{n+1}).$$

Let us, in the meantime, suppose that none of the functions F becomes 0 for the value of x in question. We may then put

$$G(n, x) = F(n+1, x)/F(n, x) \qquad (3),$$

where $G(n, x)$ is a definite function of n and x which becomes neither 0 nor ∞ for the value of x in question.

The equation (2_{n+1}) may now be written

$$G(n, x) - 1 = \beta_{n+1} x G(n+1, x) G(n, x),$$

that is,
$$G(n, x) = 1/\{1 - \beta_{n+1} x G(n+1, x)\} \qquad (4).$$

If in (4) we put successively $n = 0$, $n = 1$, . . ., we derive the following:—

$$G(0, x) = \frac{1}{1 -} \; \frac{\beta_1 x}{1 -} \; \cdots \; \frac{\beta_n x}{1 - (1 - 1/G(n, x))} \qquad (5);$$

$$1 - \frac{1}{G(n, x)} = \frac{\beta_{n+1} x}{1 -} \; \cdots \; \frac{\beta_{n+m} x}{1 - (1 - 1/G(n+m, x))} \qquad (6).$$

* The vanishing of one or more of these coefficients would lead to a more general form than (1), namely,

$$\frac{\beta_0 x^{\alpha_0}}{1 -} \; \frac{\beta_1 x^{\alpha_1}}{1 -} \; \cdots \; .$$

General expressions have been found for β_0, β_1, . . . by Heilermann, *Crelle's Jour.* (1846), and by Muir, *Proc.* L.M.S. (1876).

In order that we may be able to assert the equality

$$G(0, x) = \frac{1}{1-}\ \frac{\beta_1 x}{1-}\ \cdots\ \frac{\beta_n x}{1-}\ \cdots \text{ ad } \infty \qquad (7),$$

it is necessary, and it is sufficient, that it be possible by making m sufficiently great to cause $1 - 1/G(n, x)$ to differ from the mth convergent of the residual fraction

$$\frac{\beta_{n+1} x}{1-}\ \frac{\beta_{n+2} x}{1-}\ \cdots\ \frac{\beta_{n+m} x}{1-}\ \cdots \qquad (8)$$

by as little as we please.

Let us denote the convergents of (8) by p_1/q_1, p_2/q_2, . . ., p_m/q_m. Then, from (6), we see that

$$\{1 - 1/G(n, x)\} - p_m/q_m$$

$$= \frac{p_m - p_{m-1}\{1 - 1/G(n+m, x)\}}{q_m - q_{m-1}\{1 - 1/G(n+m, x)\}} - \frac{p_m}{q_m},$$

$$= \frac{\{1 - 1/G(n+m, x)\}(p_m/q_m - p_{m-1}/q_{m-1})}{q_m q_{m-1} - \{1 - 1/G(n+m, x)\}} \qquad (9),$$

$$= \frac{\{1 - 1/G(n+m, x)\}\beta_{n+1}\beta_{n+2} \cdots \beta_{n+m} x^m}{q_m[q_m - q_{m-1}\{1 - 1/G(n+m, x)\}]} \qquad (10).$$

The necessary and sufficient condition for the subsistence of (7) *is, therefore, that the right-hand side of* (9), *or of* (10), *shall vanish when* $m = \infty$.

Concerning these conditions it should be remarked that while either of them secures the convergence of the infinite continued fraction in (7), the convergence of the fraction is not necessarily by itself a sufficient condition for the subsistence of the equation (7).

In what precedes we have supposed that none of the functions $F(n, x)$ vanish. This restriction may be partly removed. It is obvious that no two consecutive F's can vanish, for then (by the equations (2)) all the preceding F's would vanish, and $G(0, x)$ would not be determinate. Suppose, however, that $F(r+1, x') = 0$, so that $G(r, x') = 0$; then (5) furnishes for $G(0, x')$ the closed continued fraction

$$G(0, x') = \frac{1}{1-}\ \frac{\beta_1 x'}{1-}\ \cdots\ \frac{\beta_{r-1} x'}{1}.$$

In order that this may be identical with the value given by (7), it is necessary and sufficient that $G(r+1, x')$, as given by (6), should become ∞, that is, it is necessary and sufficient that the residual fraction

$$\frac{\beta_{r+2}x'}{1-}\ \frac{\beta_{r+3}x'}{1-}\ \ldots \text{ ad } \infty$$

should converge to 1; but this condition will in general be satisfied if the relation (4) subsist for all values of n, and the condition (9) be also satisfied when $n \not< r+2$.

§ 21.] As an example of the process of last paragraph, let

$$F(n, x) = 1 + \frac{x}{1!(\gamma+n)} + \frac{x^2}{2!(\gamma+n)(\gamma+n+1)} + \ldots \quad (11).$$

Then

$$F(n+1, x) - F(n, x) = -\frac{x}{(\gamma+n)(\gamma+n+1)}F(n+2, x) \quad (2');$$

and

$$G(n, x) = 1/\left\{1 + \frac{x}{(\gamma+n)(\gamma+n+1)}G(n+1, x)\right\} \quad (4'),$$

where $G(n, x) = F(n+1, x)/F(n, x)$.

Hence

$$G(0, x) = \frac{1}{1+}\ \frac{x/\gamma(\gamma+1)}{1+}\ \frac{x/(\gamma+1)(\gamma+2)}{1+}\ \ldots\ \frac{x/(\gamma+n-1)(\gamma+n)}{1-\{1-1/G(n, x)\}} \quad (5');$$

and

$$1 - \frac{1}{G(n, x)} = -\frac{x/(\gamma+n)(\gamma+n+1)}{1+}\ \ldots\ \frac{x/(\gamma+n+m-1)(\gamma+n+m)}{1-\{1-1/G(n+m, x)\}} \quad (6').$$

The series (11) will be convergent for all finite values of x, and for all positive integral values of n, including 0, provided γ be not 0 or a negative integer. Hence we have obviously, for all finite values of x, $LG(n+m, x) = 1$ when $m = \infty$.

Let us suppose that x is positive. Then the residual continued fraction

$$\frac{x/(\gamma+n)(\gamma+n+1)}{1+}\ \frac{x/(\gamma+n+1)(\gamma+n+2)}{1+}\ \cdots$$
$$\frac{x/(\gamma+n+m-1)(\gamma+n+m)}{1+}\ \cdots \quad (8')$$

is (by the criterion of § 14) evidently convergent. Hence the factor $p_m/q_m - p_{m-1}/q_{m-1}$ in the expression (9) vanishes when $m = \infty$.

Also, since the q's continually increase, $\mathrm{L}q_m/q_{m-1} \not< 1$.

Therefore we may continue the fraction to infinity when x is positive.

Next suppose x negative, $= -y$ say; we then have

$$\mathrm{G}(0,-y) = \frac{1}{1-}\ \frac{y/\gamma(\gamma+1)}{1-}\ \frac{y/(\gamma+1)(\gamma+2)}{1-}\ \cdots$$
$$\frac{y/(\gamma+n-1)(\gamma+n)}{1-\{1-1/\mathrm{G}(n,-y)\}} \quad (5'');$$

and

$$1 - \frac{1}{\mathrm{G}(n,-y)} = \frac{y/(\gamma+n)(\gamma+n+1)}{1-}\ \cdots$$
$$\frac{y/(\gamma+n+m-1)(\gamma+n+m)}{1-\{1-1/\mathrm{G}(n+m,-y)\}} \quad (6'').$$

The fraction (8) in this case is "equivalent" to

$$\frac{1}{\gamma+n}\left\{\frac{y}{\gamma+n+1-}\ \frac{y}{\gamma+n+2-}\ \cdots\ \frac{y}{\gamma+n+m-}\ \cdots\right\} \quad (8''),$$

which is obviously convergent (by § 16), if y have any finite value whatever. Hence the factor $p_m/q_m - p_{m-1}/q_{m-1}$ belonging to the equivalent fraction (8) must vanish.

Again, by § 2 (6),

$$\frac{q_m}{q_{m-1}} = 1 - \frac{y/(\gamma+n+m-1)(\gamma+n+m)}{1-}\ \frac{y/(\gamma+n+m-2)(\gamma+n+m-1)}{1-}$$
$$\cdots\ \frac{y/(\gamma+n)(\gamma+n+1)}{1}$$
$$= 1 - \frac{1}{\gamma+n+m}\left\{\frac{y}{\gamma+n+m-1-}\ \frac{y}{\gamma+n+m-2-}\ \cdots\ \frac{y}{\gamma+n}\right\} \quad (12).$$

If only n be taken large enough, the fraction inside the brackets satisfies the condition of § 16 throughout: its value is therefore < 1, however great m may be; and it follows from (12) that $Lq_m/q_{m-1} = 1$ when $m = \infty$.

Since $LG(n+m, -y) = 1$ when $m = \infty$, it follows that all the requisite conditions are fulfilled in the present case also.

We have thus shown that

$$\frac{F(1,x)}{F(0,x)} = \frac{1}{1+}\ \frac{x/\gamma(\gamma+1)}{1+}\ \frac{x/(\gamma+1)(\gamma+2)}{1+}\ \cdots \frac{x/(\gamma+n-1)(\gamma+n)}{1+}\ \cdots \text{ ad } \infty \quad (13),$$

whence, by an obvious reduction,

$$\frac{F(1,x)}{F(0,x)} = \frac{\gamma}{\gamma+}\ \frac{x}{\gamma+1+}\ \frac{x}{\gamma+2+}\ \cdots \frac{x}{\gamma+n+}\ \cdots \quad (14),$$

a result which holds for all finite real values of x, except such as render $F(0, x)$ zero,* and for all values of γ, except zero and negative integers.

If we put $\pm x^2/4$ in place of x in the functions $F(0, x)$ and $F(1, x)$, and at the same time put $\gamma = \frac{1}{2}$, we get

$$F(0, -x^2/4) = \cos x, \quad F(1, -x^2/4) = \sin x/x;$$
$$F(0, x^2/4) = \cosh x, \quad F(1, x^2/4) = \sinh x/x.$$

Cor. 1. *Hence, from* (14), *we get at once*

$$\tan x = \frac{x}{1-}\ \frac{x^2}{3-}\ \frac{x^2}{5-}\ \cdots \frac{x^2}{2n+1-}\ \cdots \quad (15);$$

$$\tanh x = \frac{x}{1+}\ \frac{x^2}{3+}\ \frac{x^2}{5+}\ \cdots \frac{x^2}{2n+1+}\ \cdots \quad (16).$$

Cor. 2. *The numerical constants π and π^2 are incommensurable.*

For, if π were commensurable, $\pi/4$ would be commensurable, say $= \lambda/\mu$. Hence we should have, by (15),

* In a sense it will hold even then, for the fraction

$$\frac{1}{\gamma}\left\{\gamma + \frac{x}{\gamma+1+}\ \frac{x}{\gamma+2+}\ \cdots\right\}$$

which represents $F(0, x)/F(1, x)$ will converge to 0. Of course, two consecutive functions $F(n, x)$, $F(n+1, x)$ cannot vanish for the same value of x; otherwise we should have $F(\infty, x) = 0$, which is impossible, since $F(\infty, x) = 1$.

$$1 = \frac{\lambda/\mu}{1-}\,\frac{\lambda^2/\mu^2}{3-}\,\frac{\lambda^2/\mu^2}{5-}\cdots\frac{\lambda^2/\mu^2}{2n+1-}\cdots,$$

$$= \frac{\lambda}{\mu-}\,\frac{\lambda^2}{3\mu-}\,\frac{\lambda^2}{5\mu-}\cdots\frac{\lambda^2}{(2n+1)\mu-}\cdots \qquad (17).$$

Now, since λ and μ are fixed finite integers, if we take n large enough we shall have $(2n+1)\mu > \lambda^2 + 1$. Hence, by § 17, the fraction in (17) converges to an incommensurable limit, which is impossible since 1 is commensurable.

That π^2 is also incommensurable follows in like manner very readily from (15).

By using (16) in a similar way we can easily show that

Cor. 3. *Any commensurable power of e is incommensurable.**

§ 22.] The development of last paragraph is in reality a particular case of the following general theorem regarding the hypergeometric series, given by Gauss in his classical memoir on that subject (1812)† :—

If

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \cdots,$$

and

$$G(\alpha, \beta, \gamma, x) = F(\alpha, \beta+1, \gamma+1, x)/F(\alpha, \beta, \gamma, x),$$

then

$$G(\alpha, \beta, \gamma, x) = \frac{1}{1-}\,\frac{\beta_1 x}{1-}\,\frac{\beta_2 x}{1-}\cdots\frac{\beta_{2n}x}{1/G(\alpha+n, \beta+n, \gamma+2n)} \qquad (18),$$

where

$$\beta_1 = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}, \qquad \beta_2 = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)},$$

$$\beta_3 = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)}, \qquad \beta_4 = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)},$$

$$\beta_{2n-1} = \frac{(\alpha+n-1)(\gamma+n-1-\beta)}{(\gamma+2n-2)(\gamma+2n-1)}, \qquad \beta_{2n} = \frac{(\beta+n)(\gamma+n-\alpha)}{(\gamma+2n-1)(\gamma+2n)}.$$

* The results of this paragraph were first given by Lambert in a memoir which is very important in the history of continued fractions (*Hist. d. l. Ac. Roy. d. Berlin*, 1761). The arrangement of the analysis is taken from Legendre (*l.c.*), the general idea of the discussion of the convergence of the fraction from Schlömilch.

† *Werke*, Bd. iii., p. 134.

After what has been done, the proof of this theorem should present no difficulty.

The discussion of the question of convergence is also comparatively simple when x is positive; but presents some difficulty in the case where x is negative. In fact, we are not aware that any *complete* elementary discussion of this latter point has been given.

Cor. *If in* (18) *we put* $\beta = 0$, *and write* $\gamma - 1$ *in place of* γ, *we get the transformation*

$$1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \ldots$$

$$= \frac{1}{1-}\ \frac{\beta_1 x}{1-}\ \frac{\beta_2 x}{1-}\ \ldots \quad (19),$$

where

$$\beta_1 = \frac{\alpha}{\gamma}, \qquad \beta_2 = \frac{\gamma - \alpha}{\gamma(\gamma+1)},$$

$$\beta_3 = \frac{(\alpha+1)\gamma}{(\gamma+1)(\gamma+2)}, \qquad \beta_4 = \frac{2(\gamma+1-\alpha)}{(\gamma+2)(\gamma+3)},$$

. . . . . .

$$\beta_{2n-1} = \frac{(\alpha+n-1)(\gamma+n-2)}{(\gamma+2n-3)(\gamma+2n-2)}, \quad \beta_{2n} = \frac{n(\gamma+n-1-\alpha)}{(\gamma+2n-2)(\gamma+2n-1)}.$$

Gauss's Theorem is a very general one; for the hypergeometric series includes nearly all the ordinary elementary series.

Thus, for example, we have, as the reader may easily verify,

$$(1+x)^m = \mathrm{F}(-m, \beta, \beta, -x);$$

$$\log(1+x) = x\mathrm{F}(1, 1, 2, -x);$$

$$\sinh x = x \operatorname*{L}_{k=\infty} \operatorname*{L}_{k'=\infty} \mathrm{F}(k, k', \tfrac{3}{2}, x^2/4kk');$$

$$\sin x = x \operatorname*{L}_{k=\infty} \operatorname*{L}_{k'=\infty} \mathrm{F}(k, k', \tfrac{3}{2}, -x^2/4kk');$$

$$\sin^{-1}x = x\mathrm{F}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x^2);$$

$$= x\sqrt{(1-x^2)}\mathrm{F}(1, 1, \tfrac{3}{2}, x^2);$$

$$\tan^{-1}x = x\mathrm{F}(\tfrac{1}{2}, 1, \tfrac{3}{2}, -x^2).$$

EXERCISES XXXIV.

Examine the convergence of the following:—

(1.) $1+\frac{1}{1^2+}\frac{1}{2^2+}\frac{1}{3^2+}\cdots$ (2.) $1+\frac{1^2}{3+}\frac{2^2}{5+}\frac{3^2}{7+}\cdots$

(3.) $1+\frac{1^2}{1+}\frac{1^2.2^2}{1+}\frac{2^2.3^2}{1+}\cdots$ (4.) $1+\frac{1}{1+}\frac{1.2}{1+}\frac{2.3}{1+}\cdots$

(5.) $1+\frac{1\cdot}{1+}\frac{2}{1+}\frac{3}{1+}\cdots$ (6.) $\frac{m^2}{n+}\frac{(m+n)^2}{n+}\frac{(m+2n)^2}{n+}\cdots$

(7.) $x+\frac{1^\alpha}{x+}\frac{2^\alpha}{x+}\frac{3^\alpha}{x+}\cdots$ (8.) $1+\frac{1.3}{1+}\frac{3.5}{1+}\frac{5.7}{1+}\cdots$

(9.) $\frac{2}{1+}\frac{1^3.3}{1+}\frac{2^3.4}{1+}\frac{3^3.5}{1+}\cdots$ (10.) $\frac{2}{1+}\frac{2^2}{1+}\frac{2^3}{1+}\frac{2^4}{1+}\cdots$

(11.) Show that the fraction of the second class, $a_1-\frac{b_2}{a_2-}\frac{b_3}{a_3-}\cdots$, converges to a positive limit if, for all values of n,

$$a_2/b_1\,b_2+a_3/b_2\,b_3+\ldots+a_{n+1}/b_n\,b_{n+1}\not>1.$$

(Stern, *Gött. Nach.*, 1845.)

(12.) Show that $\frac{a_1}{a_1-}\frac{a_2}{a_2-}\frac{a_3}{a_3-}\cdots$, where $a_n>0$, converges if $a_{n+1}\not>a_n+1$.

(13.) Show that the series of fractions $(p_n-p_{n-1})/(q_n-q_{n-1})$ forms a *descending* series of convergents to the infinite continued fraction of the second class, provided $a_n\geqq b_n+1$, and the sign $>$ occurs at least once among these conditions.

(14.) Show that

$$\frac{x}{x+1-}\frac{x}{x+1-}\frac{x}{x+1-}\cdots,$$

where $x>0$, is equal to x or 1 according as $x<$ or $\not<1$.

(15.) Evaluate $\frac{1}{2-}\frac{2}{3-}\frac{3}{4-}\cdots;$

and $$\frac{m}{m+1-}\frac{m+1}{m+2-}\frac{m+2}{m+3-}\cdots,$$

where m is any integer.

Show that

(16.) $1+\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\ldots=1+\frac{a}{b-}\frac{(a+1)b}{a+b+2-}\frac{(a+2)(b+1)}{a+b+4-}\cdots$

(17.) $\sin x=\frac{x}{1+}\frac{x^2}{2.3-x^2+}\frac{2.3x^2}{4.5-x^2+}\frac{4.5x^2}{6.7-x^2+}\cdots$

(18.) $\log(1+x)=\frac{x}{1+}\frac{1^2x}{2-x+}\frac{2^2x}{3-2x+}\frac{3^2x}{4-3x+}\cdots$

* Exercises (5) to (10) are taken from Stern's memoir, *Crelle's Jour.*, xxxvii.

(19.) $1=\dfrac{1^2}{3-}\dfrac{2^2}{5-}\dfrac{3^2}{7-}\cdots$.

(20.) $\log\dfrac{1+x}{x}=\dfrac{x}{1+}\dfrac{1^2x}{2x-1+}\dfrac{2^2x}{3x-2+}\dfrac{3^2x}{4x-3+}\cdots$.

(21.) $\log y=\dfrac{y^2-1}{1-}\dfrac{1^2(y^2-1)^2}{4(y^2+y+1)-}\dfrac{3^2(y^2-1)^2}{4(2y^2+y+2)-}\cdots$
$\dfrac{(2n-1)^2(y^2-1)^2}{4(ny^2+y+n)-}\cdots$.

(22.) $e^x=\dfrac{1}{1-}\dfrac{x}{1+x-}\dfrac{x}{2+x-}\dfrac{2x}{3+x-}\dfrac{3x}{4+x-}\cdots$.

Evaluate the following:—

(23.) $1+\dfrac{1}{1-}\dfrac{1}{3-}\dfrac{2}{4-}\dfrac{3}{5-}\dfrac{4}{6-}\cdots$.

(24.) $\dfrac{1}{1+}\dfrac{2}{2+}\dfrac{3}{3+}\dfrac{4}{4+}\cdots$.

(25.) $\dfrac{1}{1+}\dfrac{1^2}{1+}\dfrac{2^2}{1+}\dfrac{3^2}{1+}\cdots$.

(26.) $\dfrac{1}{2+}\dfrac{2}{3+}\dfrac{3}{4+}\dfrac{4}{5+}\cdots$.

(27.) Show that $\tan x$ and $\tanh x$ are incommensurable if x be commensurable.

Establish the following transformations:—

(28.) $e^x=\dfrac{1}{1-}\dfrac{x}{1+}\dfrac{x}{2-}\dfrac{x}{3+}\dfrac{x}{2-}\dfrac{x}{5+}\dfrac{x}{2-}\dfrac{x}{7+}\cdots$.

(29.) $\log(1+x)+\dfrac{x}{1+}\dfrac{1^2x}{2+}\dfrac{1^2x}{3+}\dfrac{2^2x}{4+}\dfrac{2^2x}{5+}\dfrac{3^2x}{6+}\dfrac{3^2x}{7+}\cdots$.

(30.) $\tan^{-1}x=\dfrac{x}{1+}\dfrac{1^2x^2}{3+}\dfrac{2^2x^2}{5+}\dfrac{3^2x^2}{7+}\cdots$.

$\tanh^{-1}x=\dfrac{x}{1-}\dfrac{1^2x^2}{3-}\dfrac{2^2x^2}{5-}\dfrac{3^2x^2}{7-}\cdots$.

(31.) $\tan nx=\dfrac{n\tan x}{1-}\dfrac{(n^2-1^2)\tan^2x}{3-}\dfrac{(n^2-2^2)\tan^2x}{5-}\cdots$.

(Euler, *Mem. Acad. Pet.*, 1813.)

(32.) $\dfrac{\sin(n+1)x}{\sin nx}=2\cos x\,\dfrac{1}{2\cos x-}\dfrac{1}{2\cos x-}\cdots$,

where there are n partial quotients.

(33.) If

$$\phi(\alpha,\beta,\gamma,x)=1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}x^2+\cdots$$

then

$$\frac{\phi(\alpha,\beta+1,\gamma+1,x)}{\phi(\alpha,\beta,\gamma,x)}=\frac{1}{1-}\frac{\beta_1x}{1-}\frac{\beta_2x}{1-}\cdots,$$

where

$$\beta_{2r}=\frac{(q^{\beta+r}-1)(q^{\gamma+r-\alpha}-1)}{(q^{\gamma+2r-1}-1)(q^{\gamma+2r}-1)}q^{\alpha+r-1},$$

$$\beta_{2r+1}=\frac{(q^{\alpha+r}-1)(q^{\gamma+r-\beta}-1)}{(q^{\gamma+2r}-1)(q^{\gamma+2r-1}-1)}q^{\beta+r}.$$

(Heine, *Crelle's Jour.*, xxxii.)

(34.) Show that

$$a^2=\left\{a-1+\frac{1}{2(a-1)+}\,\frac{3^2}{2(a-1)+}\,\frac{5^2}{2(a-1)+}\cdots\right\}$$
$$\times\left\{a+1+\frac{1}{2(a+1)+}\,\frac{3^2}{2(a+1)+}\,\frac{5^2}{2(a+1)+}\cdots\right\}.$$

Wallis (see Muir, *Phil. Mag.*, 1877).

CHAPTER XXXV.

General Properties of Integral Numbers.

NUMBERS WHICH ARE CONGRUENT WITH RESPECT TO A GIVEN MODULUS.

§ 1.] *If m be any positive integer whatever, which we call the modulus, two integers,* M *and* N, *which leave the same remainder when divided by m are said to be congruent with respect to the modulus m.**

In other words, if $M = pm + r$, and $N = qm + r$, M and N are said to be congruent with respect to the modulus m. Gauss, who made the notion of congruence the fundamental idea in his famous *Disquisitiones Arithmeticæ*, uses for this relation between M and N the symbolism

$$M \equiv N \pmod{m};$$

or simply

$$M \equiv N,$$

if there is no doubt about the modulus, and no danger of confusion with the use of $\equiv$ to denote algebraical identity.

Cor. 1. *If two numbers* M *and* N *be congruent with respect to modulus m, then they differ by a multiple of m; so that we have, say,* $M = N + pm$.

Cor. 2. *If either* M *or* N *have any factor in common with m, then the other must also have that factor; and if either be prime to m, the other must be prime to m also.*

In the present chapter we shall use only the most elementary consequences of the theory of congruent numbers.

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* To save repetition, let it be understood, when nothing else is indicated, that throughout this chapter every letter stands for a positive or negative integer.

Our object here is simply to give the reader a conspectus of the more elementary methods of demonstration which are employed in establishing properties of integral numbers; and to illustrate these methods by proving some of the elementary theorems which he is likely to meet with in an ordinary course of mathematical study. Further developments must be sought for in special treatises on the theory of numbers.

§ 2.] If we select any "modulus" m, then it follows, from chap. iii., § 11, that *all integral numbers can be arranged into successive groups of m, such that each of the integers in one of these groups is congruent with one and with one only of the set*

$$0,\ 1,\ 2,\ \ldots,\ (m-2),\ (m-1) \qquad \text{(A)},$$

or, if we choose, of the set

$$0,\ 1,\ 2,\ \ldots,\ -2,\ -1. \qquad \text{(B)},$$

where there are m integers.

Another way of expressing the above is to say that, *if we take any m consecutive integers whatever, and divide them by m, their remainders taken in order will be a cyclical permutation of the integers* (A).

Example. If we take $m=5$, the set (A) is 0, 1, 2, 3, 4. Now if we take the 5 consecutive integers 63, 64, 65, 66, 67 and divide them by 5, the remainders are 3, 4, 0, 1, 2, which is a cyclical permutation of 0, 1, 2, 3, 4.

§ 3.] A large number of curious properties of integral numbers can be directly deduced from the simple principle of classification just explained.

Example 1. Every integer which is a perfect cube is of the form $7p$, or $7p\pm1$. Bearing in mind that every integer N has one or other of the forms

$$7m,\quad 7m\pm1,\quad 7m\pm2,\quad 7m\pm3,$$

also that

$$\begin{aligned}(7m\pm r)^3 &= (7m)^3 \pm 3(7m)^2 r + 3(7m)r^2 \pm r^3,\\ &= (7^2m^3 \pm 21m^2r + 3mr^2)7 \pm r^3,\\ &= \mathrm{M}7 \pm r^3,\end{aligned}$$

we see that in the four possible cases we have

$$\mathrm{N}^3 = (7m)^3 = (7^2m^3)7\ ;$$

$$\mathrm{N}^3 = (7m\pm1)^3 = \mathrm{M}7 \pm 1\ ;$$

$$\begin{aligned}\mathrm{N}^3 &= (7m\pm2)^3,\\ &= \mathrm{M}7 \pm 8 = (\mathrm{M}\pm1)7 \pm 1\ ;\end{aligned}$$

$$\mathrm{N}^3 = (7m\pm3)^3 = (\mathrm{M}\pm4)7 \mp 1.$$

In every case, therefore, the cube has one or other of the forms $7p$ or $7p \pm 1$.

Example 2. Prove that $3^{2n+1} + 2^{n+2}$ is divisible by 7 (Wolstenholme).

We have $3^{2n+1} + 2^{n+2} = (7-4)^{2n+1} + 2^{n+2}$.

Now (see above, Example 1, or below, § 4)

$$(7-4)^{2n+1} = \mathrm{M}7 - 4^{2n+1}.$$

Hence
$$3^{2n+1} + 2^{n+2} = \mathrm{M}7 - 4^{2n+1} + 2^{n+2},$$
$$= \mathrm{M}7 - 2^{n+2}(2^{3n} - 1).$$

But $2^{3n} - 1$ is divisible by $2^3 - 1$ (see chap. v., § 17), that is, by 7. Hence

$$2^{n+2}(2^{3n} - 1) = \mathrm{N}7.$$

Finally, therefore, $3^{2n+1} + 2^{n+2} = (\mathrm{M} - \mathrm{N})7$,

which proves the theorem.

Example 3. The product of 3 successive integers is always divisible by 1.2.3.

Let the product in question be $m(m+1)(m+2)$. Then, since m must have one or other of the three forms, $3m$, $3m+1$, $3m-1$, we have the following cases to consider :—

$$3m(3m+1)(3m+2) \quad (1);$$
$$(3m+1)(3m+2)(3m+3) \quad (2);$$
$$(3m-1)3m(3m+1) \quad (3).$$

In (1) the proposition is at once evident; for $3m$ is divisible by 3, and $(3m+1)(3m+2)$ by 2. The same is true in (2).

In case (3) we have to show that $(3m-1)m(3m+1)$ is divisible by 2. Now this must be so; because, if m is even, m is divisible by 2; and if m be odd, both $3m-1$ and $3m+1$ are even; that is, both $3m-1$ and $3m+1$ are divisible by 2.

In all cases, therefore, the theorem holds.

Example 4. To show that the product of p successive integers is always divisible by $1.2.3 \ldots p$.

Let us suppose that it has been shown, 1st, that the product of any $p-1$ successive integers whatever is divisible by $1.2.3 \ldots \overline{p-1}$; 2nd, that the product of p successive integers beginning with any integer up to x is divisible by $1.2.3 \ldots \overline{p-1}.p$.

Consider the product of p successive integers beginning with $x+1$. We have

$$(x+1)(x+2)\ldots(x+p-1)(x+p)$$
$$= p(x+1)(x+2)\ldots(x+p-1) + x(x+1)(x+2)\ldots(x+p-1) \ldots \quad (1).$$

Now, by our first supposition, $(x+1)(x+2)\ldots(x+p-1)$ is divisible by $1.2 \ldots \overline{p-1}$; and, by our second, $x(x+1)(x+2)\ldots(x+p-1)$ is divisible by $1.2.3 \ldots p$.

Hence each member on the right of (1) is divisible by $1.2.3 \ldots p$.

It follows, therefore, that, if our two suppositions be right, then the product of p successive integers beginning with $x+1$ is divisible by $1.2.3 \ldots p$.

But we have shown in Example 3 that the product of 3 consecutive integers is always divisible by 1.2.3; and it is self-evident that the product of 4 con-

secutive integers beginning with 1 is divisible by 1.2.3.4. It follows, therefore, that the product of 4 consecutive integers beginning with 2 is divisible by 1.2.3.4. Using Example 3 again, and the result just established, we prove that 4 consecutive integers beginning with 3 is divisible by 1.2.3.4; and thus we finally establish that the product of any 4 consecutive integers whatever is divisible by 1.2.3.4.

Proceeding in exactly the same way, we next show that our theorem holds when $p=5$; and so on. Hence it holds generally.

This demonstration is a good example of "mathematical induction."

Example 5. If a, b, c be three integers such that $a^2+b^2=c^2$, then they are represented in the most general way possible by the forms

$$a=\lambda(m^2-n^2), \quad b=2\lambda mn, \quad c=\lambda(m^2+n^2).$$

First of all, it is obvious, on account of the relation $a^2+b^2=c^2$, that, if any two of the numbers have a common factor λ, then that factor must occur in the other also; so that we may write $a=\lambda a'$, $b=\lambda b'$, $c=\lambda c'$, where a', b', c' are prime to each other, and we have

$$a'^2+b'^2=c'^2 \tag{1}.$$

No two of the three, a', b', c', therefore, can be even; also both a' and b' cannot be odd, for then $a'^2+b'^2$ would be of the form $4n+2$, which is an impossible form for the number c'^2.

It appears, then, that one of the two, a', b', say b' $(=2\beta)$, must be even, and that a' and c' must be odd. Hence $(c'+a')/2$ and $(c'-a')/2$ must be integers; and these integers must be prime to each other; for, if they had a common factor, it must divide their sum which is c' and their difference which is a'; but c' and a' have by hypothesis no common factor.

Now we have from (1)

$$c'^2-a'^2=b'^2=4\beta^2,$$

whence

$$\left(\frac{c'+a'}{2}\right)\times\left(\frac{c'-a'}{2}\right)=\beta^2 \tag{2}.$$

Therefore, since $(c'+a')/2$ is prime to $(c'-a')/2$, each of these must be a perfect square; so that we must have

$$\frac{c'+a'}{2}=m^2 \tag{3},$$

$$\frac{c'-a'}{2}=n^2 \tag{4},$$

$$\beta=mn \tag{5},$$

where m is prime to n.

From (3) and (4), we have, by subtraction and addition,

$$a'=m^2-n^2, \quad c'=m^2+n^2;$$

and, from (5), $$b'=2\beta=2mn.$$

Returning, therefore, to our original case, we must have generally

$$a=\lambda(m^2-n^2), \quad b=2\lambda mn, \quad c=\lambda(m^2+n^2).$$

This is the complete analytical solution of the famous Pythagorean problem—to find a right-angled triangle whose sides shall be commensurable.

§ 4.] The following theorem may be deduced very readily from the principles of § 2. Let $f(x)$ stand for $p_0 + p_1x + p_2x^2 + \ldots + p_nx^n$, where $p_0, p_1, \ldots, p_n$ are positive or negative integers, and x any positive integer; then, *if x be congruent with r with respect to the modulus m, $f(x)$ will be congruent with $f(r)$ with respect to modulus m.*

By the binomial expansion, we have

$$\begin{aligned}(qm + r)^n &= (qm)^n + {}_nC_1(qm)^{n-1}r + \ldots + {}_nC_{n-1}(qm)r^{n-1} + r^n,\\ &= (q^nm^{n-1} + {}_nC_1q^{n-1}m^{n-2}r + \ldots + {}_nC_{n-1}qr^{n-1})m + r^n,\\ &= M_nm + r^n;\end{aligned}$$

where M_n is some integer, since all the numbers ${}_nC_1, {}_nC_2, \ldots, {}_nC_{n-1}$ are, by § 3, Example 4, or by their law of formation (see chap. iv., § 14) necessarily integers.

Similarly

$$(qm + r)^{n-1} = M_{n-1}m + r^{n-1},$$

. . . . .

Hence, if $x = qm + r$,

$$\begin{aligned}f(x) &= p_0 + p_1r + p_2r^2 + \ldots + p_nr^n + (p_1M_1 + p_2M_2 + \ldots + p_nM_n)m,\\ &= f(r) + Mm.\end{aligned}$$

Hence $f(x)$ is congruent with $f(r)$ with respect to modulus m.

Cor. 1. Since all integers are congruent (with respect to modulus m) with one or other of the series

$$0, 1, 2, \ldots, \overline{m-1},$$

it follows that to test the divisibility of $f(x)$ by m for all integral values of x, we need only test the divisibility by m of $f(0), f(1), f(2), \ldots, f(m-1)$.

Example 1. Let $f(x) = x(x+1)(2x+1)$; and let it be required to find when $f(x)$ is divisible by 6. We have $f(0) = 0$, $f(1) = 6$, $f(2) = 30$, $f(3) = 84$, $f(4) = 180$, $f(5) = 330$. Each of these is divisible by 6; and every integer is congruent (mod 6) with one of the six numbers 0, 1, 2, 3, 4, 5; hence $x(x+1)(2x+1)$ is *always* divisible by 6.

Cor. 2. *$f\{qf(r) + r\}$ is always divisible by $f(r)$;* for $f\{qf(r) + r\} = Mf(r) + f(r) = (M + 1)f(r)$.

Hence *an infinite number of values of x can always be found which will make $f(x)$ a composite number.*

This result is sometimes stated by saying that *no integral function of x can furnish prime numbers only.*

Example 2. Show that x^4-1 is divisible by 5 if x be prime to 5, but not otherwise.

With modulus 5 all integral values of x are congruent with 0, ±1, ±2. If $f(x)=x^4-1$, $f(0)=-1$, $f(\pm1)=0$, $f(\pm2)=15$. Now 0 and 15 are each divisible by 5; but -1 is not divisible by 5. Hence x^4-1 is divisible by 5 when x is prime to 5, but not otherwise.

Example 3. To show that x^2+x+17 is not divisible by any number less than 17, and that it is divisible by 17 when and only when x is of the form $17m$ or $17m-1$.

Here

$f(0)=17$, $f(+1)=19$, $f(+2)=23$, $f(+3)=29$, $f(+4)=37$, $f(+5)=47$, $f(+6)=59$, $f(+7)=73$, $f(+8)=89$, $f(-1)=17$, $f(-2)=19$, $f(-3)=23$, $f(-4)=29$, $f(-5)=37$, $f(-6)=47$, $f(-7)=59$, $f(-8)=73$.

These numbers are all primes, hence no number less than 17 will divide x^2+x+17, whatever the value of x may be; and 17 will do so only when $x=m17$ or $x=m17-1$.

§ 5.] *Method of Differences.*—There is another method for testing the divisibility of integral functions, which may be given here, although it belongs, strictly speaking, to an order of ideas somewhat different from that which we are now following.

Let $f_n(x)$ denote an integral function of the nth degree.

$$f_n(x+1)-f_n(x)=p_0+p_1(x+1)+\ldots+p_{n-1}(x+1)^{n-1}+p_n(x+1)^n - p_0-p_1x-\ldots-p_{n-1}x^{n-1}-p_nx^n \quad (1).$$

Now on the right-hand side the highest power of x, namely x^n, disappears; and the whole becomes an integral function of the $n-1$th degree, $f_{n-1}(x)$, say. Thus, if m be the divisor, we have

$$\frac{f_n(x+1)-f_n(x)}{m}=\frac{f_{n-1}(x)}{m} \quad (2).$$

It may happen that the question of divisibility can be at once settled for the simpler function $f_{n-1}(x)$. Suppose, for example, that it turns out that $f_{n-1}(x)$ is always divisible by m, whatever x may be; then $f_n(x+1)-f_n(x)$ is always divisible by m, whatever x may be. Suppose, further, that $f_n(0)$ is divisible by m; then, since $f_n(1)-f_n(0)$, as we have just seen, is divisible by m, it follows that $f_n(1)$ is divisible by m. Similarly, it may be shown that $f_n(2)$ is divisible by m; and so on.

If the divisibility or non-divisibility of $f_{n-1}(x)$ be not at once evident, we may proceed with $f_{n-1}(x)$ as we did before with $f_n(x)$, and make the question depend on a function of still lower degree; and so on.

Example 1. $f_5(x)=x^5-x$ is always divisible by 5.

$$f_5(x+1)-f_5(x)=(x+1)^5-(x+1)-x^5+x,$$
$$=5x^4+10x^3+10x^2+5x,$$
$$=\mathrm{M}5.$$

Now $f_5(1)=0,$

therefore $f_5(2)-f_5(1)=\mathrm{M}_0 5,$

and $f_5(2)=\mathrm{M}_0 5.$

Similarly, $f_5(3)-f_5(2)=\mathrm{M}_1 5,$

therefore $f_5(3)=(\mathrm{M}_0+\mathrm{M}_1)5;$

and so on.

Thus we prove that $f_5(1)$, $f_5(2)$, $f_5(3)$, &c., are all divisible by 5; in other words, that x^5-x is always divisible by 5.

EXERCISES XXXV.

(1.) The sum of two odd squares cannot be a square.

(2.) Every prime greater than 3 is of the form $6n\pm1$.

(3.) Every prime, except 2, has one or other of the forms $4n\pm1$.

(4.) Every integer of the form $4n-1$ which is not prime has an odd number of factors of the form $4n-1$.

(5.) Every prime greater than 5 has the form $30m+n$, where $n=1$, 7, 11, 13, 17, 19, 23, or 29.

(6.) The square of every prime greater than 3 is of the form $24m+1$; and the square of every integer which is not divisible by 2 or 3 is of the same form.

(7.) If two odd primes differ by a power of 2, their sum is a multiple of 3.

(8.) The difference of the squares of any two odd primes is divisible by 24.

(9.) None of the forms $(3m+2)n^2+3$, $4mn-m-1$, $4mn-m-n$ can represent a square integer. (Goldbach and Euler.)

(10.) The nth power of an odd number greater than unity can be presented as the difference of two square numbers in n different ways.

(11.) If N differ from the two successive squares between which it lies by x and y respectively, prove that $\mathrm{N}-xy$ is a square.

(12.) The cube of every rational number is the difference of the squares of two rational numbers.

(13.) Any uneven cube, n^3, is the sum of n consecutive uneven numbers, of which n^2 is the middle one.

(14.) There can always be found n consecutive integers, each of which is not a prime, however great n may be.

(15.) In the scale of 7 every square integer must have 0, 1, 2, or 4 for its unit digit.

(16.) The scale in which 34 denotes a square integer has a radix of the form $n(3n+4)$ or $(n+2)(3n+2)$.

(17.) There cannot in any scale be found three different digits such that the three integers formed by placing each digit differently in each integer shall be in Arithmetical Progression, unless the radix of the scale be of the form $3p+1$. If this condition be satisfied, there are $2(p-1)$ such sets of digits; and the common difference of the A.P. is the same in all cases.

(18.) If $x>2$, $x^4-4x^3+5x^2-2x$ is divisible by 12.

(19.) $x^5/5+x^4/2+x^3/3-x/30$, and $x^6/6+x^5/2+5x^4/12-x^2/12$ are both integral for all integral values of x.

(20.) If x, y, z be three consecutive integers, $(\Sigma x)^3-3\Sigma x^3$ is divisible by 108.

(21.) x^3-x is divisible by 6.

(22.) Find the form of x in order that x^4+1 may be divisible by 17.

(23.) Examine how far the forms x^2+x+41, $2x^2+29$ represent prime numbers.

(24.) Find the least value of x for which 2^x-1 is divisible by 47.

(25.) Find the least value of x for which 2^x-1 is divisible by 23.

(26.) Find the values of x and y for which 7^x-y is divisible by 22.

(27.) Show that the remainder of $2^{2^{x+p}}+1$ with respect to $2^{2^x}+1$ is 2.

(28.) $3^{2x}\sim 2^{2y}$ is divisible by 5, if $x\sim y=2$.

(29.) Show that $2^{2n+1}+1$ is always divisible by 3.

(30.) $4^{3x\pm1}+2^{3x\pm1}+1$ is divisible by 7.

(31.) $x^{4m}+x^{2m}+1$ never represents a prime unless $x=0$ or $x=1$.

(32.) If P be prime and $=a^2+b^2$, show that P^n can be resolved into the sum of two squares in $\frac{1}{2}n$ ways or $\frac{1}{2}(n+1)$ ways, according as n is even or odd, and give one of these resolutions.

(33.) If $x^2+y^2=z^2$, x, y, z being integers, then $xyz\equiv 0 \pmod{60}$; and if x be prime and >3, $y\equiv 0 \pmod{12}$. Show also that one of the three numbers $\equiv 0 \pmod 5$.

(34.) The solution in integers of $x^2+y^2=2z^2$ can be deduced from that of $x^2+y^2=z^2$. Hence, or otherwise, find the two lowest solutions in integers of the first of these equations.

(35.) If the equation $x^3+y^3=z^3$ had an integral solution, show that one of the three x, y, z must be of the form $7m$, and one of the form $3m$.

(36.) The area of a right-angled triangle with commensurable sides cannot be a square number.

(37.) The sum of two integral fourth powers cannot be an integral square.

(38.) Show that $(3+\sqrt{5})^x+(3-\sqrt{5})^x$ is divisible by 2^x.

(39.) If x be any odd integer, not divisible by 3, prove that the integral part of $4^x-(2+\sqrt{2})^x$ is a multiple of 112.

(40.) If n be odd, show that $1+{}_nC_4+{}_nC_8+{}_nC_{16}+\ldots$ is divisible by $2^{(n-3)/2}$.

ON THE DIVISORS OF A GIVEN INTEGER.

§ 6.] We have already seen (chap. iii., § 7) that every composite integer N can be represented in the form $a^{\alpha}b^{\beta}c^{\gamma}\ldots$, where a, b, c, . . . are primes. If N be a perfect square, all the indices must be even, and we have $N = a^{2\alpha'}b^{2\beta'}c^{2\gamma'}\ldots$; so that $\sqrt{N} = a^{\alpha'}b^{\beta'}c^{\gamma'}\ldots$.

In this case N is divisible by $\sqrt{N}$.

If N be not a perfect square, then one at least of the indices must be odd; and we have, say,

$$N = a^{2\alpha'+1}b^{2\beta'}c^{2\gamma'}\ldots = a^{\alpha'}b^{\beta'}c^{\gamma'}\ldots a^{\alpha'+1}b^{\beta'}c^{\gamma'}\ldots,$$

so that N is divisible by $a^{\alpha'}b^{\beta'}c^{\gamma'}\ldots$, which is obviously less than $\sqrt{N}$. Hence

Every composite number has a factor which is not greater than its square root.

This proposition is useful as a guide in finding the least factors of large numbers. This has been done, once for all, in a systematic, but more or less tentative, manner, and the results published for the first nine million integers in the Factor Tables of Burckhard, Dase, and the British Association.*

§ 7.] The divisors of any given number $N = a^{\alpha}b^{\beta}c^{\gamma}\ldots$ are all of the form $a^{\alpha'}b^{\beta'}c^{\gamma'}\ldots$, where α', β', γ', . . . may have any values from 0 up to α, from 0 up to β, from 0 up to γ, . . . respectively. Hence, if we include 1 and N itself among the divisors, *the divisors of* $N = a^{\alpha}b^{\beta}c^{\gamma}\ldots$ *are the various terms obtained by distributing the product*

$$\begin{aligned}&(1 + a + a^2 + \ldots + a^{\alpha})\\ &\times (1 + b + b^2 + \ldots + b^{\beta})\\ &\times (1 + c + c^2 + \ldots + c^{\gamma})\\ &\times \quad . \quad . \quad . \quad . \quad .\end{aligned} \qquad (1).$$

* For an interesting account of the construction and use of these tables, see J. W. L. Glaisher's Report, *Rep. Brit. Assoc.* (1877).

Cor 1.
Since

$$1 + a + a^2 + \ldots + a^\alpha = \frac{a^{\alpha+1} - 1}{a - 1},$$

$$1 + b + b^2 + \ldots + b^\beta = \frac{b^{\beta+1} - 1}{b - 1},$$

and so on,

It follows that the sum of the divisors of $N = a^\alpha b^\beta c^\gamma \ldots$ *is*

$$\frac{(a^{\alpha+1} - 1)(b^{\beta+1} - 1)\ldots}{(a - 1)(b - 1)\ldots}.$$

If in (1) we put $a = 1$, $b = 1$, $c = 1$, . . ., each divisor, that is, each term of the distributed product, becomes unity; and the sum of the whole is simply the number of the different divisors. Hence, since there are $\alpha + 1$ terms in the first bracket, $\beta + 1$ in the second, and so on, it follows that

Cor. 2. *The number of the divisors of* $N = a^\alpha b^\beta c^\gamma \ldots$ *is* $(\alpha + 1)(\beta + 1)(\gamma + 1)\ldots$.

Cor. 3. *The number of ways in which* * $N = a^\alpha b^\beta c^\gamma \ldots$ *can be resolved into two factors is* $\frac{1}{2}\{1 + (\alpha + 1)(\beta + 1)(\gamma + 1)\ldots\}$, *or* $\frac{1}{2}(\alpha + 1)(\beta + 1)(\gamma + 1)\ldots$, *according as* N *is or is not a square number.*

For every factor has a complementary factor, that is to say, every factorisation corresponds to *two* divisors; unless N be a square number, and then one factor, namely $\sqrt{N}$, has itself for complementary factor, and therefore the factorisation $N = \sqrt{N} \times \sqrt{N}$ corresponds to only one divisor.

Cor. 4. *The number of ways in which* $N = a^\alpha b^\beta c^\gamma \ldots$ *can be resolved into two factors that are prime to each other is* 2^{n-1}, *n being the number of prime factors* a^α, b^β, c^γ, . . .

For, in this kind of resolution, no single prime factor, a^α for example, can be split between the two factors. The number of different divisors is therefore the same as if α, β, γ, . . .

* This result is given by Wallis in his *Discourse of Combinations, Alternations, and Aliquot Parts* (1685), chap. iii., § 12. In the same work are given most of the results of §§ 6 and 7 above.

were each equal to unity. Hence the number of ways is $\frac{1}{2}(1+1)(1+1)(1+1)\ldots(n \text{ factors}) = \frac{1}{2}.2^n = 2^{n-1}$.

Example 1. Find the different divisors of 360, their sum, and their number.

We have $360 = 2^3 3^2 5$.

The divisors are therefore the terms in the distributed product $(1+2+2^2+2^3)(1+3+3^2)(1+5)$; that is to say,

1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72, 5, 10, 20, 40, 15, 30, 60, 120, 45, 90, 180, 360.

Their sum is $(2^4-1)(3^3-1)(5^2-1)/(2-1)(3-1)(5-1) = 1170$.

Their number is $(1+3)(1+2)(1+1) = 24$.

Example 2. Find the least number which has 30 divisors. Let the number be $N = a^\alpha b^\beta c^\gamma$. There cannot be more than three prime factors; for $30 = 2 \times 3 \times 5$, which has at most three factors, must $= (\alpha+1)(\beta+1)(\gamma+1)$. There might of course be only two, and then we must have $30 = (\alpha+1)(\beta+1)$; or there might be only one, and then $30 = \alpha+1$.

In the first case $\alpha=1$, $\beta=2$, $\gamma=4$. Taking the three least primes, 2, 3, 5, and putting the larger indices to the smaller primes, we have $N = 2^4.3^2.5 = 720$.

In the second case we should get $2^{14}.3$, $2^5.3^4$, or $2^9.3^2$.

In the last case, 2^{29}.

It will be found that the least of all these is $2^4.3^2.5$; so that 720 is the required number.

Example 3. Show that, if $2^n - 1$ be a prime number, then $2^{n-1}(2^n-1)$ is equal to the sum of its divisors (itself excluded).*

Since $2^n - 1$ is supposed to be prime, the prime factors of the given number are 2^{n-1} and $2^n - 1$. Hence the sum of its divisors, excluding itself, is, by Cor. 1 above,

$$\frac{(2^n-1)\{(2^n-1)^2-1\}}{(2-1)\{(2^n-1)-1\}} - 2^{n-1}(2^n-1) = (2^n-1)\{(2^n-1)+1\} - 2^{n-1}(2^n-1),$$
$$= (2^n-1)\{2^n - 2^{n-1}\},$$
$$= 2^{n-1}(2^n-1)\{2-1\},$$
$$= 2^{n-1}(2^n-1);$$

as was to be shown.

ON THE NUMBER OF INTEGERS LESS THAN A GIVEN INTEGER AND PRIME TO IT.

§ 8.] If we consider all the integers less than a given one, N, a certain number of these have factors in common with N, and the rest have none. The number of the latter is usually denoted

* In the language of the ancients such a number was called a Perfect Number. 6, 28, 496, 8128 are perfect numbers.

by $\phi(N)$. Thus $\phi(N)$ *is taken to denote the number of integers* (*including* 1) *which are less than* N *and prime to* N.

We have the following important theorem, first given by Euler:—

If $N = a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \ldots a_n^{\alpha_n}$, *then*

$$\phi(N) = N\left(1-\frac{1}{a_1}\right)\left(1-\frac{1}{a_2}\right)\left(1-\frac{1}{a_3}\right)\ldots\left(1-\frac{1}{a_n}\right) \quad (1).$$

The proof of this theorem which we shall give is that which follows most naturally from the principles of § 7.

Proof.—Let us find the number of all the integers, not greater than N, which have some factor in common with N. That factor must be a product of powers of one or more of the primes $a_1, a_2, a_3, \ldots, a_n$.

Now all the multiples of a_1 which do not exceed N are

$$1a_1, \quad 2a_1, \quad 3a_1, \quad \ldots, \quad (N/a_1)a_1, \quad N/a_1 \text{ in number} \quad (3);$$

all the multiples of a_2 which do not exceed N are

$$1a_2, \quad 2a_2, \quad 3a_2, \quad \ldots, \quad (N/a_2)a_2, \quad N/a_2 \text{ in number} \quad (4);$$

and so on.

All the multiples of a_1a_2 which do not exceed N are

$$1a_1a_2, \quad 2a_1a_2, \quad 3a_1a_2, \quad \ldots, \quad (N/a_1a_2)a_1a_2, \quad N/a_1a_2 \text{ in number} \quad (5);$$

and so on.

Similarly, for $a_1a_2a_3$ we have

$$1a_1a_2a_3, \quad 2a_1a_2a_3, \quad 3a_1a_2a_3, \quad \ldots, \quad (N/a_1a_2a_3)a_1a_2a_3, \quad N/a_1a_2a_3 \text{ in number} \quad (6).$$

Let us now consider the number

$$\begin{aligned}
&\frac{N}{a_1} + \frac{N}{a_2} + \frac{N}{a_3} + \ldots \\
&-\frac{N}{a_1a_2} - \frac{N}{a_1a_3} - \frac{N}{a_1a_4} - \ldots \\
&+\frac{N}{a_1a_2a_3} + \frac{N}{a_1a_2a_4} + \frac{N}{a_2a_3a_4} + \ldots \\
&-\frac{N}{a_1a_2a_3a_4} - \ldots \\
&\quad\ldots\ldots\ldots\ldots\ldots\ldots \qquad (7).
\end{aligned}$$

The number of terms in the first line is ${}_nC_1$. The number in the second line is ${}_nC_2$, since every possible group of 2 out of the n letters $a_1 a_2 \ldots a_n$ occurs among the denominators. The number in the third line is ${}_nC_3$ for a similar reason. And so on.

Now consider every multiple of the r letters $a_1 a_2 a_3 \ldots a_r$ which does not exceed N; in other words, every number, not exceeding N, that has in common with it a factor of the form $a_1^{a'_1} a_2^{a'_2} \ldots a_r^{a'_r}$. This multiple will be enumerated in the first line, once as a multiple of a_1, once as a multiple of a_2, and so on; that is, once for every letter in it, that is, ${}_rC_1$ times. In the second line the same multiple will be enumerated once as a multiple of $a_1 a_2$, once as a multiple of $a_1 a_3$, and so on; that is, once for every group of two that can be formed out of the r letters $a_1 a_2 \ldots a_r$, that is, ${}_rC_2$ times. And so on. Hence, paying attention to the signs, the multiple in question will in the whole expression (7) be enumerated

$$ {}_rC_1 - {}_rC_2 + {}_rC_3 - \ldots \pm {}_rC_{r-1} \mp {}_rC_r = 1 - (1-1)^r $$

times; that is, just once. This proof holds, of course, whatever the r letters in the group may be, and whether there be 1, 2, 3, or any number up to n in the group.

It follows, therefore, that (7) enumerates, without repetition or omission, every integer which has a factor in common with N. But, from formula (1), chap. iv., § 10, we see that (7) is simply

$$ N - N\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \ldots \left(1 - \frac{1}{a_n}\right) \qquad (8). $$

To obtain the number of integers less than N *which are prime to* N, we have merely to subtract (8) from N. We thus obtain

$$ \phi(N) = N\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right) \ldots \left(1 - \frac{1}{a_n}\right), $$

which establishes Euler's formula.

Example. $N = 100 = 2^2.5^2$; $\phi(100) = 2^2.5^2(1-\frac{1}{2})(1-\frac{1}{5}) = 40$.

§ 9.] *If* $M = PQ$, *where* P *and* Q *are prime to each other, then*

$$ \phi(M) = \phi(P)\phi(Q) \qquad (1). $$

For, since P and Q are prime to each other, we must have

$$P = a_1^{\alpha_1} a_2^{\alpha_2} \ldots,$$
$$Q = b_1^{\beta_1} b_2^{\beta_2} \ldots,$$

where none of the prime factors are common; and therefore

$$M = a_1^{\alpha_1} a_2^{\alpha_2} \ldots b_1^{\beta_1} b_2^{\beta_2} \ldots,$$

where $a_1, a_2, \ldots, b_1, b_2, \ldots$ are all primes.

But, by § 8, we then have

$$\phi(M)$$
$$= M\left(1-\frac{1}{a_1}\right)\left(1-\frac{1}{a_2}\right)\ldots\left(1-\frac{1}{b_1}\right)\left(1-\frac{1}{b_2}\right)\ldots,$$
$$= a_1^{\alpha_1} a_2^{\alpha_2}\ldots\left(1-\frac{1}{a_1}\right)\left(1-\frac{1}{a_2}\right)\ldots b_1^{\beta_1} b_2^{\beta_2}\ldots\left(1-\frac{1}{b_1}\right)\left(1-\frac{1}{b_2}\right)\ldots,$$
$$= \phi(P)\,\phi(Q).$$

Cor. *If* PQRS . . . *be prime to each other, then*

$$\phi(PQRS\ldots) = \phi(P)\,\phi(Q)\,\phi(R)\,\phi(S)\ldots \qquad (2).$$

For, since P is prime to Q, R, S, . . ., it follows that P is prime to the product QRS . . . Hence, by the above proposition,

$$\phi(PQRS\ldots) = \phi(P)\,\phi(QRS\ldots).$$

Repeating the same reasoning, we have

$$\phi(QRS\ldots) = \phi(Q)\,\phi(RS\ldots);$$

and so on.

Hence, finally,

$$\phi(PQRS\ldots) = \phi(P)\,\phi(Q)\,\phi(R)\,\phi(S)\ldots .$$

Remark.—There is no difficulty in establishing the theorem $\phi(PQ) = \phi(P)\phi(Q)$ *a priori*. This may be done, for example, by means of § 13 below (see Gross' *Algebra*, § 230). The theorem of § 8 above can then be deduced from $\phi(PQR\ldots) = \phi(P)\phi(Q)\phi(R)\ldots$ The course followed above, though not so neat, is, we think, more instructive for the learner.

Example.
$$56 = 7 \times 8,$$
$$\phi(56) = 24,$$
$$\phi(7) = 6,$$
$$\phi(8) = 4;$$
$$\phi(56) = \phi(7) \times \phi(8).$$

§ 10.] *If* $d_1, d_2, d_3, \ldots$, *&c., denote all the divisors of the integer* N, *then* *

$$\phi(d_1) + \phi(d_2) + \phi(d_3) + \ldots = N \ldots \quad (1).$$

(Gauss, *Disq. Arith.*, § 39.)

For the divisors, $d_1, d_2, d_3, \ldots$, are the terms in the distribution of the product

$$(1 + a_1 + a_1^2 + \ldots + a_1^{\alpha_1})(1 + a_2 + a_2^2 + \ldots + a_2^{\alpha_2}) \ldots .$$

If we take any one of these terms, say $d_r = a_1^{\alpha_1'} a_2^{\alpha_2'} \ldots$, then, by § 9, Cor.,

$$\phi(d_r) = \phi(a_1^{\alpha_1'} a_2^{\alpha_2'} \ldots),$$
$$= \phi(a_1^{\alpha_1'})\phi(a_2^{\alpha_2'}) \ldots ;$$

since $a_1, a_2, \ldots$ are primes.

It follows that the left-hand side of (1) is the same as

$$\begin{aligned} &\{1 + \phi(a_1) + \phi(a_1^2) + \ldots + \phi(a_1^{\alpha_1})\} \\ &\times \{1 + \phi(a_2) + \phi(a_2^2) + \ldots + \phi(a_2^{\alpha_2})\} \\ &\quad \ldots \ldots \ldots \ldots \ldots \ldots \end{aligned} \quad (2).$$

But $$\phi(a_1^r) = a_1^r\left(1 - \frac{1}{a_1}\right) = a_1^r - a_1^{r-1}.$$

Hence

$$\begin{aligned} 1 + \phi(a_1) + \phi(a_1^2) + \ldots + \phi(a_1^{\alpha_1}) &= 1 + \overline{a_1 - 1} + \overline{a_1^2 - a_1} + \ldots + \overline{a_1^{\alpha_1} - a_1^{\alpha_1 - 1}}, \\ &= a_1^{\alpha_1} ; \end{aligned}$$

and so on.

It appears, therefore, that (2) is equal to $a_1^{\alpha_1} a_2^{\alpha_2} \ldots$, that is, equal to N.

Example. $N = 315 = 3^2.5.7.$

The divisors are 1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315, and we have

$$\phi(1) + \phi(3) + \phi(5) + \ldots + \phi(315)$$
$$= 1 + 2 + 4 + 6 + 6 + 8 + 12 + 24 + 24 + 36 + 48 + 144 = 315.$$

* Here and in what follows 1 is included among the divisors, and, for convenience, $\phi(1)$ is taken to stand for 1. Strictly speaking, $\phi(1)$ has no meaning at all.

PROPERTIES OF $m!$

§ 11.] The following theorem enables us to prove some important properties of $m!$:—

The highest power of the prime p which divides $m!$ exactly is

$$\mathrm{I}\left(\frac{m}{p}\right)+\mathrm{I}\left(\frac{m}{p^2}\right)+\mathrm{I}\left(\frac{m}{p^3}\right)+\ldots,$$

where $\mathrm{I}\left(\frac{m}{p}\right)$, $\mathrm{I}\left(\frac{m}{p^2}\right)$, . . . *denote the integral parts of* m/p, m/p^2, . . .; *and the series is continued until the greatest power of* p *is reached which does not exceed* m.

To prove this, we remark that the numbers in the series

$$1, 2, \ldots, m$$

which are divisible by p are evidently

$$1p, 2p, 3p, \ldots, kp,$$

where kp is the greatest multiple of $p \not> m$. In other words, $k = \mathrm{I}(m/p)$. Hence $\mathrm{I}(m/p)$ is the number of the factors in $m!$ which are divisible by p.

If to this we add the number of those that are divisible by p^2, namely $\mathrm{I}(m/p^2)$, and again the number of those that are divisible by p^3, namely $\mathrm{I}(m/p^3)$, and so on, the sum will be the power in which p occurs in $m!$

Hence, since p is a prime, the highest power of p that will divide $m!$ exactly is

$$\mathrm{I}\left(\frac{m}{p}\right)+\mathrm{I}\left(\frac{m}{p^2}\right)+\mathrm{I}\left(\frac{m}{p^3}\right)+\ldots.$$

It is convenient for practical purposes to remark that

$$\mathrm{I}\left(\frac{m}{p^r}\right)=\mathrm{I}\left\{\mathrm{I}\left(\frac{m}{p^{r-1}}\right)\Big/p\right\}.$$

For, if

$$m/p^{r-1} = i + k/p^{r-1} \quad (k < p^{r-1}) \qquad (1),$$

then

$$m/p^r = i/p + k/p^r \qquad (2),$$

$$= j + l/p + k/p^r \quad (l < p) \qquad (3).$$

Now

$$\begin{aligned} l/p + k/p^r &> (p-1)/p + (p^{r-1}-1)/p^r, \\ &> (p^{r+1}-p)/p^{r+1}, \\ &< 1. \end{aligned}$$

Hence, by (3),

$$j = \mathrm{I}\left(\frac{m}{p^r}\right).$$

But, since $i/p = j + l/p$,

$$j = \mathrm{I}\left(\frac{i}{p}\right) = \mathrm{I}\left\{\mathrm{I}\left(\frac{m}{p^{r-1}}\right)\Big/p\right\}, \qquad \text{by (2).}$$

We may therefore proceed as follows :—*Divide m by p; take the integral quotient and divide again by p; and so on; until the integral quotient becomes zero; then add all the integral quotients, and the result is the highest power of p which will divide m! exactly.*

Example 1. To find the highest power of 7 which divides 1000! exactly.

In dividing successively by 7 the integral quotients are 142, 20, 2; the sum of these is 164. Hence 7^{164} is the power of 7 required.

Example 2. To decompose 25! into its prime factors.

Write down all the primes less than 25; write under each the successive quotients; and then add. We thus obtain

2	3	5	7	11	13	17	19	23
12	8	5	3	2	1	1	1	1
6	2	1						
3								
1								
22	10	6	3	2	1	1	1	1

Hence $25! = 2^{22}.3^{10}.5^6.7^3.11^2.13.17.19.23$.

Example 3. Express $39!/25!$ in its simplest form as a product of prime factors.

Result, $2^{13}.3^8.5^2.7^2.11.13^2.17.19.29.31.37$.

Example 4. Find the highest power of 5 that will divide $27.28.29 \ldots 100$ exactly.

Result, 5^{18}.

Example 5. If m be expressed in the scale of p, in the form

$$m = p_0 + p_1 p + p_2 p^2 + \ldots + p_n p^n,$$

the highest power of p that will divide $m!$ exactly is the

$$\frac{m - p_0 - p_1 - p_2 - \ldots - p_n}{p-1}\text{th}.$$

Example 6. If $m = 2^\alpha + 2^\beta + 2^\gamma + \ldots$ (k terms), where $\alpha < \beta < \gamma < \ldots$, the greatest power of 2 that will divide $m!$ is the $(m-k)$th.

§ 12.] *If* $f+g+h+\ldots \geqslant m$, *then* $m!/f!g!h!\ldots$ *is an integer.**

To prove this, it will be sufficient to show that, if any prime factor, p say, appear in $f!g!h!\ldots$, it will appear in at least as high a power in $m!$ In other words (§ 11), we have to show that

$$\begin{aligned} \mathrm{I}\left(\frac{m}{p}\right)+\mathrm{I}\left(\frac{m}{p^2}\right)+\ldots \leqslant & \mathrm{I}\left(\frac{f}{p}\right)+\mathrm{I}\left(\frac{f}{p^2}\right)+\ldots \\ & +\mathrm{I}\left(\frac{g}{p}\right)+\mathrm{I}\left(\frac{g}{p^2}\right)+\ldots \\ & +\mathrm{I}\left(\frac{h}{p}\right)+\mathrm{I}\left(\frac{h}{p^2}\right)+\ldots \\ & +\quad .\quad .\quad .\quad . \end{aligned} \qquad (1).$$

Now, if d be any integer whatever, we have

$$\begin{aligned} f/d &= f'+f''/d \quad (f'' \geqslant d-1), \\ g/d &= g'+g''/d \quad (g'' \geqslant d-1), \\ h/d &= h'+h''/d \quad (h'' \geqslant d-1), \\ &\quad .\quad .\quad .\quad .\quad . ; \end{aligned}$$

and we obtain by addition

$$\frac{f+g+h+\ldots}{d} = f'+g'+h'+\ldots+\frac{f''+g''+h''+\ldots}{d}.$$

Hence, if $f''+g''+h''+\ldots < d$,

$$\begin{aligned} \mathrm{I}\left(\frac{f+g+h+\ldots}{d}\right) &= f'+g'+h'+\ldots, \\ &= \mathrm{I}\left(\frac{f}{d}\right)+\mathrm{I}\left(\frac{g}{d}\right)+\mathrm{I}\left(\frac{h}{d}\right)+\ldots . \end{aligned}$$

If, on the other hand, $f''+g''+h''+\ldots > d$,† then

$$\begin{aligned} \mathrm{I}\left(\frac{f+g+h+\ldots}{d}\right) &> f'+g'+h'+\ldots, \\ &> \mathrm{I}\left(\frac{f}{d}\right)+\mathrm{I}\left(\frac{g}{d}\right)+\mathrm{I}\left(\frac{h}{d}\right)+\ldots \end{aligned} \qquad (2).$$

* This theorem might, of course, be inferred from the fact that $m!/f!g!h!\ldots$ represents the number of permutations of m things f of which are alike, g alike, h alike, &c.

† If n be the number of the letters f, g, h, . . ., the utmost value of $f''+g''+h''+\ldots$ is $n(d-1)$. Hence the utmost difference between the two sides of (2) is $\mathrm{I}\{n(d-1)/d\}$.

It appears, therefore, that, even if $m = f + g + h + \ldots$,

$$\mathrm{I}\left(\frac{m}{d}\right) \leqslant \mathrm{I}\left(\frac{f}{d}\right) + \mathrm{I}\left(\frac{g}{d}\right) + \ldots \qquad (3).$$

A fortiori is this so if $m > f + g + h + \ldots$.

If now we give d the successive values $p, p^2, \ldots$, and combine by addition the inequalities thus obtained from (3), the truth of (1) is at once established.

Cor. 1. *If* $f + g + h + \ldots \geqslant m$, *and none of the numbers* $f, g, h, \ldots$ *is equal to* m, *the integer* $m!/f!\,g!\,h!\ldots$ *is divisible by* m *if* m *be a prime.*

Cor. 2. *The product of* r *successive integers is exactly divisible by* $r!$.

The proofs of these, so far as they require proof, we leave to the reader. Cor. 2 has already been established by a totally different kind of reasoning in § 3, Example 6.

Exercises XXXVI.

(1.) What is the least multiplier that will convert 945 into a complete square?

(2.) Find the number of the divisors of 2160, and their sum.

(3.) Find the integral solutions of

$$xy = 100x + 10y + 1 \qquad (\alpha);$$
$$xy = 12^n \qquad (\beta);$$
$$y^3 = 108x \qquad (\gamma).$$

(4.) No number of the form $x^4 + 4$ except 5 is prime.

(5.) No number of the form $2^{4x+2} + 1$ except 5 is prime.

(6.) To find a number of the form $2^n.3.\alpha$ (α being prime) which shall be equal to half the sum of its divisors (itself excluded).

(7.) To find a number N of the form $2^n abc\ldots$ (a, b, c being unequal primes) such that N is one-third the sum of its divisors.

(8.) Show how to obtain two "amicable" numbers of the forms $2^n pq$, $2^n r$, where p, q, r are primes. (Two numbers are amicable when each is the sum of the divisors of the other, the number itself not being reckoned as a divisor.)

(9.) To find a cube the sum of whose divisors shall be a square.

(One of Fermat's challenges to Wallis and the English mathematicians. *Var. Op. Math.*, pp. 188, 190.)

(10.) If N be any integer, n the number of its divisors, and P the product of them all, then $\mathrm{N}^n = \mathrm{P}^2$.

(11.) The sum and the sum of the squares of all the numbers less than N and prime to it are $\frac{1}{2}N(a-1)(b-1)(c-1)\ldots$ and $\frac{1}{3}N^3(1-1/a)(1-1/b)\ldots+\frac{1}{3}N(1-a)(1-b)\ldots$ respectively. (Wolstenholme.)

(12.) If $p, q, r, \ldots$ be prime to each other, and $d(N)$ denote the sum of the divisors of N, show that

$$d(pqr\ldots)=d(p)d(q)d(r)\ldots .$$

(13.) If $N=abc$, where a, b, c are prime to each other, then the product of all the numbers less than N and prime to N is

$$(abc-1)!\,\Pi\{(a-1)!/(bc-1)!\,a^{(b-1)(c-1)}\}.$$

(Gonv. and Caius Coll., 1882.)

(14.) The number of integers less than $(r^2+1)^n$ which are divisible by r but not by r^2 is $(r-1)\{(r^2+1)^n-1\}/r^2$.

(15.) Prove that

$$\phi(1)\frac{x}{1+x^2}-\phi(3)\frac{x^3}{1+x^6}+\phi(5)\frac{x^5}{1+x^{10}}-\ldots \text{ ad } \infty=\frac{x(1-x^2)}{(1+x^2)^2}.$$

(16.) In a given set of N consecutive integers beginning with A, find the number of terms not divisible by any one of a given set of relatively prime integers. (Cayley.)

(17.) If $m-1$ be prime to $n+1$, show that ${}_mC_n$ is divisible by $n+1$.

(18.) $(a+1)(a+2)\ldots 2a\times b(b+1)\ldots 2b/(a+b)!$ is an integer.

(19.) The product of any r consecutive terms of the series $x-1, x^2-1, x^3-1, \ldots$ is exactly divisible by the product of the first r terms.

(20.) If p be prime, the highest power of p which divides $n!$ is the greatest integer in $\{n-S(n)\}/(p-1)^m$, where $S(n)$ is the sum of the digits of n when expressed in the scale of p.

If $S(m)$ have the above meaning, prove that $S(m-n)+S(m)-S(n)$ for any radix. Hence show that $(n+1)(n+2)\ldots(n+m)$ is divisible by $m!$.

(*Camb. Math. Jour.* (1839), vol. i., p. 226.)

(21.) If $f(n)$ denote the sum of the uneven, and $F(n)$ the sum of the even, divisors of n, and 1, 3, 6, 10, . . . be the "triangular numbers," then

$$f(n)+f(n-1)+f(n-3)+f(n-6)+\ldots = F(n)+F(n-1)+F(n-3)+F(n-6)+\ldots,$$

it being understood that $f(n-n)=0$, $F(n-n)=n$.

ON THE RESIDUES OF A SERIES OF INTEGERS IN ARITHMETICAL PROGRESSION.

§ 13.] *The least positive remainders of the series of numbers*

$$k,\quad k+a,\quad k+2a,\quad \ldots,\quad k+(m-1)a$$

with respect to m, where m is prime to a, are a permutation of the numbers of the series

$$0,\ 1,\ 2,\ \ldots,\ (m-1).$$

All the remainders must be different; for, if any two different numbers of the series had the same remainders, then we should have

$$k + ra = \mu m + \rho, \text{ and } k + sa = \mu' m + \rho,$$

whence

$$(r - s)a = (\mu - \mu')m, \text{ and } (r - s)a/m = \mu - \mu'.$$

Now this is impossible, since a is prime to m, and r and s are each $< m$, and therefore $r - s < m$. Hence, since there are only m possible remainders, namely, 0, 1, 2, . . ., $(m - 1)$, the proposition follows.

Cor. 1. *If the remainders of k and a with respect to m be k' and a', the remainders will occur as follows:—*

$$k', \quad k' + a', \quad k' + 2a', \quad \ldots, \quad k' + ra',$$

until we reach a number that equals or surpasses m; this we must diminish by m, and then proceed to add a' at each step as before.

Thus, if $k = 11$, $a = 25$, $m = 7$, the series is

$$11, 36, 61, 86, 111, 136, 161.$$

We have $k' = 4$ and $a' = 4$, hence the remainders are

$$4, \ 4 + 4 - 7 = 1, \ 5, \ 5 + 4 - 7 = 2, \text{ &c.};$$

in fact,

$$4, 1, 5, 2, 6, 3, 0.$$

Cor. 2. *If the progression of numbers be continued beyond m terms, the remainders will repeat in the same order as before; and in this periodic series the number of remainders intervening between two that differ by unity is always the same.*

Cor. 3. *There are as many terms in the series*

$$k, \quad k + a, \quad k + 2a, \quad \ldots, \quad k + (m - 1)a$$

which are prime to m, as there are in the series

$$0, 1, 2, \ldots (m - 1).$$

That is, the number of terms in the series in question which are prime to m is $\phi(m)$. See § 8.

This follows from the fact that two numbers which are congruent with respect to m are either both prime or both non-prime to m.

Cor. 4. *If out of the series of numbers*

$$0, 1, 2, \ldots, (m - 1)$$

we select those that are less than m and prime to it, say

$$r_1, r_2, \ldots, r_n$$

(the number n being $\phi(m)$), *then the numbers*

$$k + r_1 a, \quad k + r_2 a, \quad \ldots, \quad k + r_n a,$$

where $k = 0$ or a multiple of m, and a prime to m as before, are all prime to m; and their remainders with respect to m are a permutation of

$$r_1, r_2, \ldots, r_n.$$

For, as we have seen already, all the n remainders are unlike, and every remainder must be prime to m; for, if we had $k + r_t a = \mu m + \rho$, where ρ is not prime to m, then $r_t a = \mu m + \rho - k$ would have a factor in common with m, which is impossible, since r_t and a are both prime to m.

Hence the remainders must be the numbers $r_1, r_2, \ldots, r_n$ in some order or other.

§ 14.] *If m be not prime to a, but have with it the G.C.M. g, so that $a = ga'$, $m = gm'$, the remainders of the series*

$$k, \quad k + a, \quad k + 2a, \quad \ldots, \quad k + (m-1)a$$

with respect to m will recur in a shorter cycle of m'.

Consider any two terms of the series out of the first m', say $k + ra$, $k + sa$. These two must have different remainders, otherwise $(r - s)a$ would be exactly divisible by m; that is, $(r - s)ga'/gm'$ would be an integer; that is, $(r - s)a'/m'$ would be an integer; which is impossible, since a' is prime to m' and $r - s < m'$.

Again, consider any term beyond the m'th, say the $(m' + r)$th, then, since

$$\begin{aligned} \{k + (m' + r)a\} - \{k + ra\} &= m'a, \\ &= gm'a', \\ &= ma', \end{aligned}$$

it follows that the $(m' + r)$th term has the same remainder with respect to m as the rth.

In other words, the first m' remainders are all different, and after that they recur periodically, the increment being ga'', where a'' is the remainder of a' with respect to m', subject to diminution by m as in last article.

Example. If $k = 11$, $a = 25$, $m = 15$, we have the series

11, 36, 61, 86, 111, 136, 161, 186, 211, 236, 261, . . . ;

and here $g=5$; $a'=5$; $m'=3$; $a''=2$; $k'=11$; $ga''=10$. Hence the remainders are

$$11, 6, 1, 11, 6, 1, 11, 6, 1, 11, 6, \ldots .$$

Cor. *If the G.C.M., g, of a and m divide k exactly, and, in particular, if $k=0$, the remainders of the series*

$$k, \quad k+a, \quad k+2a, \quad \ldots$$

are the numbers

$$0g, 1g, 2g, 3g, \ldots, (m'-1)g$$

continually repeated in a certain order.

For, in this case, since $k=g\kappa$, we have $(k+ra)/m=(\kappa+ra')/m'$, hence the remainders are those of the series

$$\kappa, \quad \kappa+a', \quad \kappa+2a', \quad \ldots$$

with respect to m' which is prime to a', each multiplied by g. Hence the result follows by § 13.

Example. Let $k=10$, $a=25$, $m=15$; then the series of numbers is

$$10, 35, 60, 85, 110, 135, 160, 185, \ldots .$$

We have $g=5$; $a'=5$; $m'=3$; $\kappa=2$; and the remainders are

$$10, 5, 0, 10, 5, 0, 10, 5, \ldots;$$

that is to say,

$$2\times 5, \quad 1\times 5, \quad 0\times 5, \quad \ldots .$$

§ 15.] From § 13 we can at once deduce FERMAT'S THEOREM,* which is one of the corner-stones of the theory of numbers.

If m be a prime number, and a be prime to m, $a^{m-1}-1$ is divisible by m.

If a be prime to m, then we have

$$\begin{aligned} 1a &= \mu_1 m + \rho_1, \\ 2a &= \mu_2 m + \rho_2, \\ &\cdots \\ (m-1)a &= \mu_{m-1} m + \rho_{m-1}, \end{aligned}$$

where the numbers $\rho_1, \rho_2, \ldots, \rho_{m-1}$ are the numbers $1, 2, \ldots, (m-1)$ written in a certain order.

* Great historical interest attaches to this theorem. It was, with several other striking results in the theory of numbers, published without demonstration among Fermat's notes to an edition of Bachet de Meziriac's *Diophantus* (1670). For many years no demonstration was found. Finally, Euler (*Comment. Acad. Petrop.*, viii., 1741, and *Comment. Nov. Acad. Petrop.*, vii., 1761) gave two proofs. Another, due to Lagrange (*Nouv. Mém. de l'Ac. de Berlin*, 1771), is reproduced in § 18. The proof given above is akin to Euler's second and to that given by Gauss, *Disq. Arith.*, § 49.

Hence

$$1 \cdot 2 \ldots (m-1)\, a^{m-1} = (\mu_1 m + \rho_1)(\mu_2 m + \rho_2) \ldots (\mu_{m-1} m + \rho_{m-1}),$$
$$= \mathrm{M}m + \rho_1 \rho_2 \ldots \rho_{m-1},$$
$$= \mathrm{M}m + 1 \cdot 2 \ldots (m-1).$$

We therefore have

$$1 \cdot 2 \ldots (m-1)(a^{m-1} - 1) = \mathrm{M}m.$$

Now, m being a prime number, all the factors of $1 \cdot 2 \ldots (m-1)$ are prime to it. Hence m must divide $a^{m-1} - 1$.

It is very easy, by the method of differences, explained in § 5, to establish the following theorem :—

*If m be a prime, $a^m - a$ is exactly divisible by m.**

Since $a^m - a = a(a^{m-1} - 1)$, if a be prime to m, this is simply Fermat's Theorem in another form.

§ 16.] By using Cor. 4 of § 13 we arrive at the following generalisation of Fermat's Theorem, due to Euler :—

If m be any integer, and a be prime to m, then $a^{\phi(m)} - 1$ is exactly divisible by m.

Here $\phi(m)$ denotes, as usual, the number of integers which are less than m and prime to it.

For, if $r_1, r_2, \ldots, r_n$ be the integers less than m and prime to it, we have, by the corollary in question,

$$r_1 a = \mu_1 m + \rho_1,$$
$$r_2 a = \mu_2 m + \rho_2,$$
$$\ldots\ldots\ldots$$
$$r_n a = \mu_n m + \rho_n,$$

where the numbers $\rho_1, \rho_2, \ldots, \rho_n$ are simply $r_1, r_2, \ldots, r_n$ written in a certain order.

We have therefore, just as in last paragraph,

$$r_1 r_2 \ldots r_n (a^n - 1) = \mathrm{M}m,$$

whence, since $r_1, r_2, \ldots, r_n$ are all prime to m, it follows that $a^n - 1$, that is, $a^{\phi(m)} - 1$, is divisible by m.

§ 17.] The famous theorem of Wilson can also be established by means of the principles of § 13.

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* For another proof of this theorem see § 18 below.

Any two integers whose product has the remainder + 1 with respect to a given modulus m may be called, after Euler, *Allied Numbers*.

Consider all the integers,

$$1, 2, 3, \ldots, (m-1),$$

less than any prime number m (the number of them is of course even). We shall prove that, if we except the first and last, they can be exhaustively arranged in allied pairs.

For, take any one of them, say r, then, since r is prime to m, the remainders of

$$r.1, \quad r.2, \quad \ldots, \quad r(m-1)$$

are the numbers

$$1, 2, \ldots, (m-1)$$

in some order. Hence, some one of the series, say rr', must have the remainder 1; then r' will be allies.

The same number r cannot have two different allies, since all the remainders are different.

Nor can the two, r and r', be equal, unless $r = 1$ or $= m-1$; for, if we have

$$r^2 = \mu m + 1,$$

then $r^2 - 1 = \mu m$; that is, $(r+1)(r-1)$ must be divisible by m. But, since m is prime, this involves that either $r+1$ or $r-1$ be divisible by m, and, since r cannot be greater than m, this involves in the one case that $r = m-1$, in the other that $r = 1$.

Excluding, then, 1 and $m-1$, we can arrange the series

$$2, 3, \ldots, (m-2)$$

in allied pairs. Now every product of two allies is of the form $\mu m + 1$; hence the product $2.3 \ldots (m-2)$ is of the form $(\mu_1 m + 1)(\mu_2 m + 1) \ldots$, which reduces to the form $\mathrm{M}m + 1$.

Hence

$$2.3 \ldots (m-2) = \mathrm{M}m + 1;$$

and, multiplying by $m-1$, we get

$$1.2.3 \ldots (m-2)(m-1) = \mathrm{M}m(m-1) + m - 1.$$

Whence

$$1.2.3 \ldots (m-1) + 1 = \mathrm{N}m.$$

That is, *if m be a prime, $(m-1)!+1$ is divisible by m*, which is WILSON'S THEOREM.*

It should be observed that, *if m be not a prime, $(m-1)!+1$ is not divisible by m.*

For, if m be not a prime, its factors occur among the numbers 2, 3, . . ., $(m-1)$, each of which divides $(m-1)!$, and, therefore, none of which divide $(m-1)!+1$.

§ 18.] The following THEOREM OF LAGRANGE embraces both Fermat's Theorem and Wilson's Theorem as particular cases:—

If $(x+1)(x+2)\ldots(x+p-1)$

$$= x^{p-1} + A_1 x^{p-2} + \ldots + A_{p-2}x + A_{p-1},$$

and p be prime, then $A_1, A_2, \ldots, A_{p-2}$ are all divisible by p.

We have

$$(x+p)\{x^{p-1} + A_1 x^{p-2} + \ldots + A_{p-2}x + A_{p-1}\}$$
$$= (x+1)\{(x+1)^{p-1} + A_1(x+1)^{p-2} + \ldots + A_{p-2}(x+1) + A_{p-1}\}.$$

Hence

$$px^{p-1} + pA_1 x^{p-2} + pA_2 x^{p-3} + \ldots + pA_{p-2}x + pA_{p-1}$$
$$= \{(x+1)^p - x^p\} + A_1\{(x+1)^{p-1} - x^{p-1}\} + A_2\{(x+1)^{p-2} - x^{p-2}\} + \ldots$$

Therefore

$$pA_1 = {}_pC_2 + {}_{p-1}C_1 A_1,$$
$$pA_2 = {}_pC_3 + {}_{p-1}C_2 A_1 + {}_{p-2}C_1 A_2,$$
$$pA_3 = {}_pC_4 + {}_{p-1}C_3 A_1 + {}_{p-2}C_2 A_2 + {}_{p-3}C_1 A_3.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, since ${}_{p-1}C_1$, ${}_{p-2}C_1$, ${}_{p-3}C_1$, . . . are not divisible by p if p be prime, we get, by successive steps, the proof that A_1, A_2, A_3, . . . are all divisible by p.

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* This theorem was first published by Waring in his *Meditationes Algebraicæ* (1770). He there attributes it to Sir John Wilson, but gives no proof. The first demonstration was given by Lagrange (*Nouv. Mém. de l'Ac. de Berlin*, 1771); this is reproduced in § 18. A second proof was given by Euler in his *Opuscula Analytica* (1783), vol. i., p. 329, depending on the theory of the residues of powers.

The proof above is that given by Gauss (*Disq. Arith.*, §§ 77, 78), who generalises the theorem as follows:—"The product of all the numbers less than m and prime to it is congruent with -1, if $m = p^\mu$ or $= 2p^\mu$, where p is any prime but 2, or, again, if $m = 4$; but is congruent with $+1$ in every other case." This extension depends on the theory of quadratic residues.

Cor. 1. Put $x = 1$, and we get

$$2 \cdot 3 \ldots p = 1 + (A_1 + A_2 + \ldots + A_{p-2}) + A_{p-1}.$$

Therefore $A_{p-1} + 1$, *that is,* $(p-1)! + 1$ *is divisible by* p.

Cor. 2. Multiplying by x and transposing, we get

$$x^p - x = x(x+1)\ldots(x+p-1) - (1 + A_{p-1})x - (A_1 x^{p-1} + A_2 x^{p-2} + \ldots + A_{p-2}x^2).$$

But $x(x+1)\ldots(x+p-1)$, being the product of p consecutive integers, must be divisible by p. Also, if p be prime, $1 + A_{p-1}$ is divisible by p.

Therefore, $x^p - x$ *is divisible by* p *if* p *be prime.* From which Fermat's Theorem follows at once if x be prime to p.

EXERCISES XXXVII.

(1.) $x^{13} - x$ is divisible by 2730.

(2.) If x be prime greater than 13, $x^{12} - 1$ is divisible by 21840.

(3.) If the nth power of every number end with the same digit as the number itself, then $n = 4p + 1$.

Give a rule for determining by inspection the cube root of every perfect cube less than a million.

(4.) If the radix, r, of the scale of notation be prime, show that the rth power of every integer has the same final digit as the integer itself, and that the $(r-1)$th power of every integer has for its final digit 1.

(5.) If n be prime, and x prime to n, then either $x^{(n-1)/2} - 1$ or $x^{(n-1)/2} + 1$ is divisible by n.

(6.) If n be prime, and x prime to n, then either $x^{n(n-1)/2} - 1$ or $x^{n(n-1)/2} + 1$ is divisible by n^2.

(7.) If m and n be primes, then

$$m^{n-1} + n^{m-1} \equiv 1 \pmod{mn}.$$

(8.) If $\alpha, \beta, \gamma, \ldots$ be primes, and $N = \alpha\beta\gamma\ldots$, then

$$\Sigma(N/\alpha)^{\alpha-1} \equiv 1 \pmod{\alpha\beta\gamma\ldots}.$$

(9.) If n be an odd prime, show that

$$(a+1)^n - (a^n + 1) \equiv 0 \pmod{2n}.$$

Hence show that, if n be an odd prime and p an integer, then any integer expressed in the scale of $2n$ will end in the same digit as its $(pn - p + 1)$th power. Deduce Fermat's Theorem. (Math. Trip., 1879.)

(10.) If n be prime and $> x$, then

$$x^{n-2} + x^{n-3} + \ldots + x + 1 \equiv 0 \pmod{n}.$$

(11.) If n be an odd prime, then

$$1 + 2(n+1) + 2^2(n+1)^2 + \ldots + 2^{n-2}(n+1)^{n-2} \equiv 0 \pmod{n}.$$

(12.) If n be odd, $1^n + 2^n + \ldots + (n-1)^n \equiv 0 \pmod{n}$.

(13.) If n be prime, and $p<n$,

$$(p-1)!(n-p)!-(-1)^p\equiv 0 \text{ (mod. } n),$$

and, in particular,

$$[\{\tfrac{1}{2}(n-1)\}!]^2+(-1)^{(n-1)/2}\equiv 0 \text{ (mod. } n).$$

(Waring.)

(14.) Find in what cases one of the two $\{\frac{1}{2}(n-1)\}!\pm 1$ is divisible by n. What determines which of them is so?

(15.) If p be prime, and n not divisible by $p-1$, then

$$1^n+2^n+\ldots+(p-1)^n\equiv 0 \text{ (mod. } p).$$

(16.) If p be any odd prime, m any number >1, then

$$1^{2m}+2^{2m}+\ldots+\left(\frac{p-1}{2}\right)^{2m}\equiv 0 \text{ (mod. } p).$$

(17.) If neither a nor b be divisible by a prime of the form $4n-1$, then $a^{4n-2}-b^{4n-2}$ will not be exactly divisible by a prime of that form.

Hence show that $a^{4n-2}+b^{4n-2}$ is not divisible by any integer (prime or not) of the form $4n-1$.

Also that a^2+b^2 is not divisible by any integer of the form $4n-1$ which does not divide both a and b. Also, that any divisor of the sum of two integral squares, which does not divide each of them, is of the form $4n+1$.

(Euler.)

(18.) Show, by means of (17), that no square integer can have the form $4mn-m-n^\alpha$, where m, n, α are positive integers. (Euler.)

PARTITION OF NUMBERS.

Euler's Theory of the Enumeration of Partitions.

§ 19.] By the partition of a given integer n is meant the division of the integer into a number of others of which it is the sum. Thus $1+2+2+3+3$, $1+3+7$, are partitions of 11. There are two main classes of partitions, namely, (I.) those in which the parts may be equal or unequal; (II.) those in which the parts are all unequal. When the word "Partition" is used without qualification, class (I.) is understood.

We shall use a quadripartite symbol to denote the number of partitions of a given species. Thus P(| |) and Pu(| |) are used to denote partitions of the classes (I.) and (II.) respectively. In the first blank inside the bracket is inserted the number to be partitioned; in the second, an indication of the number of the parts; in the third, an indication of the magnitude or nature of

the parts. It is always implied, unless the contrary is stated, that the least part admissible is 1; so that $\not> m$ means any integer of the series 1, 2, . . ., m. An asterisk is used to mean any integer of the series 1, 2, . . ., ∞, or that no restriction is to be put on the number of the parts other than what arises from the nature of the partition otherwise.

Thus $P(n \mid p \mid q)$ means the number of partitions of n into p parts the greatest of which is q; $P(n \mid p \mid \not> q)$ the number of partitions of n into p parts no one of which exceeds q; $P(n \mid * \mid \not> q)$ the number of partitions of n into any number of parts no one of which is to exceed q; $Pu(n \mid \not> p \mid *)$ the number of partitions of n into p or any less number of unequal parts unrestricted in magnitude; $Pu(n \mid p \mid \text{odd})$ the number of partitions of n into p unequal parts each of which is an odd integer; $P(n \mid * \mid 1, 2, 2^2, 2^3, \ldots)$ the number of partitions of n into any number of parts, each part being a number in the series $1, 2, 2^2, 2^3, \ldots$; and so on.

The theory of partitions has risen into great importance of late in connection with the researches of Sylvester and his followers on the theory of invariants. It is also closely connected with the theory of series, as will be seen from Euler's enumeration of certain species of partitions, which we shall now briefly explain.

§ 20.] If we develop the product $(1 + zx)(1 + zx^2) \ldots (1 + zx^q)$, it is obvious that we get the term $z^p x^n$ in as many different ways as we can produce n by adding together p of the integers $1, 2, \ldots, q$, each to be taken only once. Hence we have the following equation :—

$$(1 + zx)(1 + zx^2) \ldots (1 + zx^q) = 1 + \Sigma Pu(n \mid p \mid \not> q) z^p x^n \quad (1).$$

Again, if to the product on the left of (1) we adjoin the factor $1 + z + z^2 + z^3 + \ldots$ ad ∞ (that is, $1/(1 - z)$), we shall evidently get $z^p x^n$ as often as we can produce n by adding together any number not exceeding p of the integers $1, 2, \ldots, q$. Therefore

$$(1 + zx)(1 + zx^2) \ldots (1 + zx^q)/(1 - z) = 1 + \Sigma Pu(n \mid \not> p \mid \not> q) z^p x^n \quad (2).$$

In like manner, we have

$$(1+x)(1+x^2)\dots(1+x^q) = 1 + \Sigma \mathrm{P}u(\pi \mid * \mid \not> q)x^n \quad (3);$$
$$(1+zx)(1+zx^2)\dots \text{ad } \infty = 1 + \Sigma \mathrm{P}u(n \mid p \mid *)z^p x^n \quad (4);$$
$$(1+x)(1+x^2)\dots \text{ad } \infty = 1 + \Sigma \mathrm{P}u(n \mid * \mid *)x^n \quad (5).$$

Also, as will be easily seen, we have

$$1/(1-zx)(1-zx^2)\dots(1-zx^q) = 1 + \Sigma \mathrm{P}(n \mid p \mid \not> q)z^p x^n \quad (6);$$
$$1/(1-z)(1-zx)\dots(1-zx^q) = 1 + \Sigma \mathrm{P}(n \mid \not> p \mid \not> q)z^p x^n \quad (7);$$
$$1/(1-x)(1-x^2)\dots(1-x^q) = 1 + \Sigma \mathrm{P}(n \mid * \mid \not> q)x^n \quad (8);$$
$$1/(1-zx)(1-zx^2)\dots \text{ad } \infty = 1 + \Sigma \mathrm{P}(n \mid p \mid *)z^p x^n \quad (9);$$
$$1/(1-z)(1-zx)(1-zx^2)\dots \text{ad } \infty = 1 + \Sigma \mathrm{P}(n \mid \not> p \mid *)z^p x^n \quad (10);$$
$$1/(1-x)(1-x^2)\dots \text{ad } \infty = 1 + \Sigma \mathrm{P}(n \mid * \mid *)x^n \quad (11);$$

and so on.

By means of these equations, coupled with the theorems given in chap. xxx., § 2, and Exercises XXI., a considerable number of theorems regarding the enumeration of partitions can be deduced at once.

§ 21.] *To find a recurrence-formula for enumerating the partitions of n into any number of parts none of which exceeds q; and thus to calculate a table for* $\mathrm{P}(n \mid * \mid \not> q)$.

By (8), we have

$$1/(1-x)(1-x^2)\dots(1-x^q) = 1 + \Sigma \mathrm{P}(n \mid * \mid \not> q)x^n.$$

Hence, multiplying on both sides by $1-x^q$, and replacing $1/(1-x)(1-x^2)\dots(1-x^{q-1})$ by its equivalent, we derive

$$1 + \Sigma \mathrm{P}(n \mid * \mid \not> q-1)x^n$$
$$= 1 + \Sigma\{\mathrm{P}(n \mid * \mid \not> q) - \mathrm{P}(n-q \mid * \mid \not> q\}x^n \quad (12),$$

where we understand $\mathrm{P}(0, \mid * \mid \not> q)$ to be 1.

Hence, if $n \not< q$,

$$\mathrm{P}(n \mid * \mid \not> q) = \mathrm{P}(n \mid * \mid \not> q-1) + \mathrm{P}(n-q \mid * \mid \not> q) \quad (13);$$

and, if $n < q$,

$$\mathrm{P}(n \mid * \mid \not> q) = \mathrm{P}(n \mid * \mid \not> q-1) \quad (14).$$

By means of (13) and (14) we can readily calculate a table of double entry for $\mathrm{P}(n \mid * \mid \not> q)$, as follows:—

		n																		
	A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	.	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11
	3	.	.	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44
	4	.	.	.	5	6	9	11	15	18	23	27	34	39	47	54	64	72	84	94	108
	5	.	.	.	.	7	10	13	18	23	30	37	47	57	70	84	101	119	141	164	192
	6	.	.	.	.	.	11	14	20	26	35	44	58	71	90	110	136	163	199	235	282
q	7	.	.	.	.	.	.	15	21	28	38	49	65	82	105	131	164	201	248	300	364
	8	.	.	.	.	.	.	.	22	29	40	52	70	89	116	146	186	230	288	352	434
	9	.	.	.	.	.	.	.	.	30	41	54	73	94	123	157	201	252	318	393	488
	10	.	.	.	.	.	.	.	.	.	42	55	75	97	128	164	212	267	340	423	530
	11	.	.	.	.	.	.	.	.	.	.	56	76	99	131	169	219	278	355	445	560
													F								
B		E						a											d		

Take a rectangle of squared paper BAC; and enter the values of n at the heads of the vertical columns, and the values of q at the ends of the horizontal lines. We remark, first of all, that it follows from (14) that all the values in the part of any vertical column below the diagonal AF are the same. We therefore leave all the corresponding spaces blank, the last entry in the column being understood to be repeated indefinitely.

Next, write the values of $P(1 \mid * \mid \not> 1)$, $P(2 \mid * \mid \not> 1)$, . . ., that is, 1, 1, . . ., in the row headed 1.

To fill the other rows, construct a piece of paper of the form *abcd*. Its use will be understood from the following rule, which is simply a translation of (13):—

To fill the blank immediately after the end of any step, add to the entry above that blank the number which is found at the left-hand end of the step.

Thus, to get the number 23, which stands at the end of the step lying on the fourth horizontal line, we add to 14 the number 9, which lies to the immediate right of *ab* in the same line as the blank. Again, in the ninth line 157 = 146 + 11; and so on.

By sliding *abcd* backwards and forwards, so that *bc* always lies on AD, we can fill in the table rapidly with little chance of error. We shall speak of the table thus constructed as Euler's

Table. It will be found in a considerably extended form in his *Introductio*, Lib. I., chap. xvi.

A variety of problems in the enumeration of partitions can be solved by means of Euler's Table, as we shall now show.

§ 22.] *To find by means of Euler's Table the number of partitions of n into p parts of unrestricted magnitude.*

Let us first consider $P(n \mid p \mid *)$. By (9) above, we have

$$1 + \Sigma P(n \mid p \mid *)x^n z^p = 1/(1 - zx)(1 - zx^2) \dots \text{ad } \infty,$$
$$= 1 + \Sigma x^p z^p/(1 - x)(1 - x^2) \dots (1 - x^p),$$
by Exercises XXI. (18).

Hence

$$\Sigma P(n \mid p \mid *)x^n = \Sigma x^p/(1 - x)(1 - x^2) \dots (1 - x^p),$$
$$= \Sigma P(n \mid * \mid \not> p)x^{n+p}, \qquad \text{by (8).}$$

Therefore,

$$P(n \mid p \mid *) = P(n - p \mid * \mid \not> p) \qquad (15).$$

Again,

$$1 + \Sigma Pu(n \mid p \mid *)x^n z^p = (1 + zx)(1 + zx^2) \dots \text{ad } \infty,$$
$$= 1 + \Sigma x^{\frac{1}{2}p(p+1)} z^p/(1 - x)(1 - x^2) \dots (1 - x^p),$$
by chap. XXX., § 2, Example 2.

Hence

$$\Sigma Pu(n \mid p \mid *)x^n = x^{\frac{1}{2}p(p+1)}/(1 - x)(1 - x^2) \dots (1 - x^p),$$
$$= \Sigma P(n \mid * \mid \not> p)x^{n+\frac{1}{2}p(p+1)}, \qquad \text{by (8).}$$

Therefore

$$Pu(n \mid p \mid *) = P(n - \tfrac{1}{2}p(p + 1) \mid * \mid \not> p) \qquad (16).$$

Example 1. $P(20 \mid 5 \mid *) = P(15 \mid * \mid \not> 5) = 84.$

Example 2. $Pu(20 \mid 5 \mid *) = P(5 \mid * \mid \not> 5) = 7.$

§ 23.] If we take any partition of n into p parts in which the largest part is q, and remove that part, we shall leave a partition of $n - q$ into $p - 1$ parts no one of which exceeds q. Hence we have the identity

$$P(n \mid p \mid q) = P(n - q \mid p - 1 \mid \not> q) \qquad (17);$$

and, if we make p infinite, as a particular case, we have

$$P(n \mid * \mid q) = P(n - q \mid * \mid \not> q) \qquad (18).$$

It will be observed that (18) makes the solution of a certain class of problems depend on Euler's Table.

By comparing (15) and (18), we have the theorem

$$P(n \mid * \mid q) = P(n \mid q \mid *),$$

which, however, is only a particular case of a theorem regarding conjugacy, to be proved presently.

§ 24.] *Theorems regarding conjugacy.*

$$\text{(I.)} \qquad P(n \mid \not> p \mid \not> q) = P(n \mid \not> q \mid \not> p) \qquad (19).$$

$$\text{(II.)} \qquad P(n-p \mid q-1 \mid \not> p) = P(n-q \mid p-1 \mid \not> q) \qquad (20).$$

$$\text{(III.)} \qquad P(n \mid p \mid q) = P(n \mid q \mid p) \qquad (21).$$

To prove (I.) we observe that, by (7), we have

$$1 + \Sigma P(n \mid \not> p \mid \not> q) z^p x^n = 1/(1-z)(1-zx)\dots(1-zx^q),$$

$$= 1 + \Sigma z^p \frac{(1-x^{q+1})(1-x^{q+2})\dots(1-x^{q+p})}{(1-x)(1-x^2)\dots(1-x^p)}.$$

Hence

$$\Sigma P(n \mid \not> p \mid \not> q) x^n = \frac{(1-x^{q+1})(1-x^{q+2})\dots(1-x^{q+p})}{(1-x)(1-x^2)\dots(1-x^p)},$$

$$= \frac{(1-x)(1-x^2)\dots(1-x^{q+p})}{(1-x)(1-x^2)\dots(1-x^q)(1-x)(1-x^2)\dots(1-x^p)}.$$

Since the function last written is symmetrical as regards p and q, it must also be the equivalent of $\Sigma P(n \mid \not> q \mid \not> p) x^n$. Hence Theorem (I.).

Theorem (II.) follows from (6) in the same way.

Since, by (17), we have

$$P(n \mid p \mid q) = P(n-q \mid p-1 \mid \not> q),$$
$$P(n \mid q \mid p) = P(n-p \mid q-1 \mid \not> p);$$

therefore, by (II.),

$$P(n \mid p \mid q) = P(n \mid q \mid p),$$

which establishes Theorem (III.).

The following particular cases are obtained by making p or q infinite :—

$$P(n \mid \not> p \mid *) = P(n \mid * \mid \not> p) \qquad (22);$$

$$P(n \mid p \mid *) = P(n \mid * \mid p) \qquad (23).$$

§ 25.] The following theorems enable us to solve a number of additional problems by means of Euler's Table :—

$$\begin{aligned}\mathrm{P}(n \mid p \mid \not> q) = \mathrm{P}(n-p \mid * \mid \not> p) &- \Sigma\mathrm{P}(n-\mu_1-p \mid * \mid \not> p) \\ &+ \Sigma\mathrm{P}(n-\mu_2-p \mid * \mid \not> p) \\ &- \Sigma\mathrm{P}(n-\mu_3-p \mid * \mid \not> p) \\ &\quad . \quad . \quad . \quad . \quad . \quad (24).\end{aligned}$$

Here the summations are with respect to $\mu_1, \mu_2, \ldots$; and μ_1 is any one of the numbers $q, q+1, \ldots, q+p-1$, μ_2 the sum of any two of them, μ_3 the sum of any three, and so on. The series of sums is to be continued so long as $n-\mu_r-p \not< 0$. If $\mathrm{P}(n \mid p \mid \not> q)$ come out 0 or negative, this indicates that the partition in question is impossible.

$$\begin{aligned}\mathrm{P}(n \mid \not> p \mid \not> q) = \mathrm{P}(n \mid * \mid \not> p) &- \Sigma\mathrm{P}(n-\nu_1 \mid * \mid \not> p) \\ &+ \Sigma\mathrm{P}(n-\nu_2 \mid * \mid \not> p) \\ &- \Sigma\mathrm{P}(n-\nu_3 \mid * \mid \not> p) \\ &\quad . \quad . \quad . \quad . \quad (25).\end{aligned}$$

Here $\nu_1, \nu_2, \ldots$ have the same meanings with regard to $q+1, q+2, \ldots, q+p$ as formerly $\mu_1, \mu_2, \ldots$ with regard to $q, q+1, \ldots, q+p-1$.

$$\begin{aligned}&\mathrm{P}(n \mid * \mid *) \\ &= \mathrm{P}(n-1 \mid * \mid \not> 1) + \mathrm{P}(n-2 \mid * \mid \not> 2) + \ldots + \mathrm{P}(0 \mid * \mid \not> n) \quad (26).\end{aligned}$$

The demonstrations will present no difficulty after what has already been given above.

CONSTRUCTIVE THEORY OF PARTITIONS.

§ 26.] Instead of making the theory of partitions depend on series, we might contemplate the various partitions directly, and develop their properties from their inherent character. Sylvester has recently considered the subject from this point of view, and has given what he calls a *Constructive Theory of Partitions*, which throws a new light on many parts of the subject, and greatly simplifies some of the fundamental demonstrations.* Into this

* *Amer. Jour. Math.* (1882).

theory we cannot within our present limits enter; but we desire, before leaving the subject, to call the attention of our readers to the graphic method of dealing with partitions, which is one of the chief weapons of the new theory.

By the *graph of a partition* is meant a series of rows of asterisks, each row containing as many asterisks as there are units in a corresponding part of the partition. Thus

* * *
* * * * *
* * *

is the graph of the partition 3 + 5 + 3 of the number 11.

For many purposes it is convenient to arrange the graph so that the parts come in order of magnitude, and all the initial asterisks are in one column. Thus the above may be written—

* * * * *
* * *
* * *

The graph is then said to be *regular*.

The direct contemplation of the graph at once gives us intuitive demonstrations of some of the foregoing theorems.

For example, if we turn the columns of the graph last written into rows, we have

* * *
* * *
* * *
*
*

where there are as many asterisks as before. The new graph, therefore, represents a new partition of 11, which may be said to be *conjugate* to the former partition. Thus *to every partition of n into p parts the greatest of which is q, there is a conjugate partition into q parts the greatest of which is p.* Hence

$$P(n \mid p \mid q) = P(n \mid q \mid p),$$

an old result.

Again, *to every partition of n into p parts no one of which exceeds q, there will be a conjugate partition into q or fewer parts the greatest of which is p.* Hence

$$P(n \mid p \mid \not> q) = P(n \mid \not> q \mid p) \qquad (27),$$

a new result; and so on.*

* According to Sylvester (*l.c.*), this way of proving the theorems of conjugacy originated with Ferrers.

§ 27.] The following proof, given by Franklin, of Euler's famous theorem that

$$(1-x)(1-x^2)(1-x^3)\ldots \text{ad } \infty = \sum_{p=0}^{\infty}(-)^p x^{\frac{1}{2}(3p^2\pm p)} \quad (28),*$$

is an excellent illustration of the peculiar power of the graphic method.

The coefficient of x^n in the expansion in question is obviously

$$Pu(n \mid \text{even} \mid *) - Pu(n \mid \text{odd} \mid *) \quad (29).$$

Let us arrange the graphs of the partitions (into unequal parts) regularly in descending order. Then the right-hand edge of the graph will form a series of terraces all having slopes of the same angle (this slope may, however, consist of a single asterisk), thus—

A

```
* *
* * *
* * * * *
* * * * * *
* * * * * * *
```

B

```
* * *
* * * *
* * * * * *
* * * * * * *
```

We can transform the graph A by removing the top row and placing it along the slope of the last terrace, thus—

A′

```
* * *
* * * * *
* * * * * * *
* * * * * * * *
```

We then have a regular graph A′ representing a partition into unequal parts. This process may be called *contraction*.

We cannot transform B in this way; but we may *extend* B by removing the slope of its last terrace, and placing it above the top row, thus—

B′

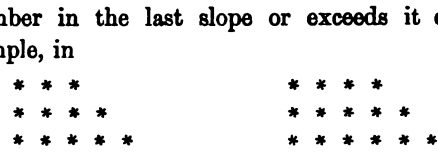
```
* *
* * *
* * * *
* * * * *
* * * * * *
```

We then have a regular graph B′ representing a partition into unequal parts.

Every graph can be transformed by contraction or by extension, except when the top row meets the slope of the last terrace; and in this case also, provided it does not happen that the number of asterisks in the top row is equal

* Euler originally discovered this theorem by induction from particular cases, and was for long unable to prove it. For other demonstrations, see Legendre, *Théorie des Nombres*, t. ii., § 15, and Sylvester (*l.c.*).

to the number in the last slope or exceeds it only by one, as, for example, in



Contraction or extension in the first of these would produce an irregular graph; contraction in the second would produce an irregular graph; and extension would produce a graph which corresponds to a partition having two parts equal. These two cases may be spoken of as *unconjugate*; they can only arise when the p parts of the partition are

$$p, \quad p+1, \quad p+2, \quad \ldots, \quad 2p-1,$$

and the number

$$n = p + (p+1) + \ldots + (2p-1) = \tfrac{1}{2}(3p^2 - p);$$

or when the p parts are

$$p+1, \quad p+2, \quad p+3, \quad \ldots, \quad 2p,$$

and

$$n = (p+1) + (p+2) + \ldots + 2p = \tfrac{1}{2}(3p^2 + p).$$

Since contraction or extension always converts a partition having an even or an odd number of parts into one having an odd or an even number of parts respectively, we see that, unless n be a number of the form $\frac{1}{2}(3p^2 \pm p)$, $Pu(n \mid \text{even} \mid *) = Pu(n \mid \text{odd} \mid *)$.

When n has one or other of the forms $\frac{1}{2}(3p^2 \pm p)$, there will be one unconjugate partition which will be even or odd according as p is even or odd; all the others will occur in pairs which are conjugate in Franklin's transformation. Hence

$$Pu(\tfrac{1}{2}(3p^2 \pm p) \mid \text{even} \mid *) - Pu(\tfrac{1}{2}(3p^2 \pm p) \mid \text{odd} \mid *) = (-1)^p \quad (30).$$

Euler's Theorem follows at once.

EXERCISES XXXVIII.

(1.) Show how to evaluate $Pu(n \mid \triangleright p \mid *)$ by means of Euler's Table.

Evaluate

(2.) $P(13 \mid 5 \mid \triangleright 3)$.
(3.) $P(13 \mid \triangleright 5 \mid \triangleright 3)$.
(4.) $P(10 \mid * \mid *)$.
(5.) $P(20 \mid 9 \mid \triangleright 3)$.

Establish the following:—

(6.) $Pu(n \mid * \mid *) = P(n - \frac{1}{2}q(q+1) \mid * \mid \nvdash q)$, where $\frac{1}{2}q(q+1)$ just $\nvdash n$.

(7.) $Pu(n \mid p \mid *) = P(n - \frac{1}{2}p(p-1) \mid p \mid *)$.

(8.) $P(n \mid p \mid *) = Pu(n + \frac{1}{2}p(p-1) \mid p \mid *)$.

(9.) $Pu(n \mid p \mid \nvdash q) = P(n - \frac{1}{2}p(p-1) \mid p \mid \nvdash q - p + 1)$.

(10.) Is the theorem $P(n-p \mid q-1 \mid *) = P(n-q \mid p-1 \mid *)$ universally true?

(11.) Show how to form a table for the values of $P(n \mid * \mid 2, 3, \ldots, q)$.
(See *Proc.* Edinb. Math. Soc., 1883-4.)

(12.) Show how to form a table for the number of partitions of n into an indefinite number of odd parts.

Establish the following:—

(13.) $P(n \mid * \mid 1, 2, 2^2, 2^3, \ldots) = 1$.

(14.) $Pu(n \mid p \mid 1, 3, \ldots, 2q-1) = P(n - p^2 + p \mid p \mid 1, 3, \ldots, 2q-1)$.

(15.) $P(n \mid p \mid 2, 4, \ldots, 2q) = P(n - p \mid p \mid 1, 3, \ldots, 2q-1)$.

(16.) $P(n \mid * \mid \text{odd}) = Pu(n \mid * \mid *)$.

(17.) $P(n \mid \nvdash p \mid 2, 4, \ldots, 2q) = P(n \mid \nvdash q \mid 2, 4, \ldots, 2p)$.

(18.) $P(n+p \mid p \mid 1, 3, \ldots, 2q+1) = P(n+q \mid q \mid 1, 3, \ldots, 2p+1)$.

(19.) $Pu(n+p^2 \mid p \mid 1, 3, \ldots, 2q+1) = Pu(n+q^2 \mid q \mid 1, 3, \ldots, 2p+1)$.

(20.) $P(n+2p \mid p \mid 2, 4, \ldots, 2q+2) = P(n+2q \mid q \mid 2, 4, \ldots, 2p+2)$.

(21.) Show that $P(n \mid p \mid *) = P(n-1 \mid p-1 \mid *) + P(n-p \mid p \mid *)$; and hence construct a table for $P(n \mid p \mid *)$. (See Whitworth, *Choice and Chance*, chap. iii.).

CHAPTER XXXVI.

Probability, or the Theory of Averages.

§ 1.] An elementary account of the Theory of Probability, or, as we should prefer to call it, the Theory of Averages, has usually found a place in English text-books on algebra. This custom is justified by several considerations. The theory in question affords an excellent illustration of the application of the theory of permutations and combinations which is the fundamental part of the algebra of discrete quantity; it forms in its elementary parts an excellent logical exercise in the accurate use of terms and in the nice discrimination of shades of meaning; and, above all, it enters, as we shall see, into the regulation of some of the most important practical concerns of modern life.

The student is probably aware that there are certain occurrences, or classes of events, of such a nature that, although we cannot with the smallest degree of certainty assert a particular proposition regarding any one of them taken singly, yet we can assert the same proposition regarding a large number N of them with a degree of certainty which increases (with or without limit, as the case may be) as the number N increases.

For example, if we take any particular man of 20 years of age, nothing could be more uncertain than the statement that he will live to be 25; but, if we consider 1000 such men, we may assert with considerable confidence that 96 per cent of them will live to be 25; and, if we take a million, we might with much greater confidence assign the proportion with even closer accuracy. In so doing, however, it would be necessary to state the limits both of habitat and epoch within which the men are to be taken; and, even with a million cases, we must not expect to be able to assign

the proportion of those who survive for 5 years with absolute accuracy, but be prepared, when we take one million with another, to find occasional small fluctuations about the indicated percentage.

We may, for illustration, indicate the limits just spoken of by saying that "man of 20" is to mean a healthy man or woman living in England in the 18th century. The "event," as it is technically called, here in question is the living for 5 years more of a man of 20; the alternative to this event is not living for 5 years more. The whole, made up of an event and its alternative or alternatives, we call its *universe*. The alternative or alternatives to an event taken collectively we often call the *Complementary Event*. The living or not living of all the men of 20 in England during the 18th century we may, following Mr. Venn,* call the *series* of the event. It will be observed that on every occasion embraced by the series the event we are considering is in question; and we express the above result of observation by saying that the probability that a man of 20 living under the assigned conditions reached the age of 25 is ·96.

We are thus led to the following abstract definition of the *Probability or Chance of an Event*:—

If on taking any very large number N *out of a series of cases in which an event* A *is in question,* A *happens on* pN *occasions, the probability of the event* A *is said to be* p.

In the framing of this definition we have, as is often done in mathematical theories, substituted an ideal for the actual state of matters usually observed in nature. In practice the number p, which for the purposes of calculation we suppose a definite quantity, would fluctuate to an extent depending on the nature of the series of cases considered and on the number N of specimen cases selected.† Moreover, the mathematical definition contains no indication of the extent or character of the series of cases.

* *Logic of Chance.*

† We might take more explicit notice of this point by wording the definition thus:—"If, *on the average,* in N out of a series of cases, &c." But, from the point of view of the ideal or mathematical theory, nothing would thus be gained.

How far the possible fluctuations of p, the extent of the series, and the magnitude of N will affect the bearing of any conclusion on practice must be judged by the light of circumstances. It is obvious, for instance, that it would be unwise to apply to the 14th century the probability of the duration of human life deduced from statistics taken in the 18th. This leads us also to remark that the application of the theory of probability is not merely historical, as the definition might suggest. Into most of the important practical applications there enters an element of *induction.** Thus we do in fact apply in the 19th century a table of mortality statistics deduced from observations in the 18th century. The warranty for this extension of the series of cases by induction must be sought in experience, and cannot in most cases be obtained *a priori*.

There are, however, some cases where the circumstances are so simple that the probability of the event can be deduced, without elaborate collecting and sifting of observations, merely from our definition of the circumstances under which the event is to take place. The best examples of such cases are games of hazard played with cards, dice, &c. If, for example, we assert regarding the tossing of a halfpenny that out of a large number of trials heads will come up nearly as often as tails—in other words, that the probability of heads is $\frac{1}{2}$, what we mean thereby is that all the causes which tend to bring up heads are to neutralise the causes that tend to bring up tails. In every series of cases in question, the assumption, well or ill justified, is made that this counterbalancing of causes takes place. That this is really the right point of view will be best brought home to us if we reflect that undoubtedly a machine could be constructed which would infallibly toss a halfpenny so as always to land it head-up on a thickly sanded floor, provided the coin were always placed the same way into the machine; also, that the coin might have two heads or two tails; and so on.

In cases where the statement of probability rests on grounds so simple as this, the difficulty regarding the extension of the

* In the proper, logical sense of the word.

series by induction is less prominent. The ideal theory in such cases approximates more closely than usual to the actual circumstances. It is for this reason that the illustrations of the elementary rules of probability are usually drawn from games of hazard. The reader must not on that account suppose that the main importance of the theory lies in its application to such cases; nor must he forget that its other applications, however important, are subject to restrictions and limitations which are not apparent in such physically simple cases as the theory of cards and dice.

Before closing this discussion of the definition of probability as a mathematical quantity, it will be well to warn the learner that probability is not an attribute of any particular event happening on any particular occasion. It can only be predicated of an event happening or conceived to happen on a very large number of "occasions," or, in popular language, of an event "on the average" or in the "long run." Unless an event can happen, or be conceived to happen, a great many times, there is no sense in speaking of its probability, or at least no sense that appears to us to be admissible in the following theory. The idea conveyed by the definition here adopted would be better expressed by substituting the word *frequency* for the word *probability*; but, after the above caution, we shall adhere to the accepted term.

§ 2.] The following corollaries and extensions may be added to the definition.

Cor. 1. *If the probability of an event be p, then out of* N *cases in which it is in question it will happen* pN *times,* N *being any very large number.*

This is merely a transposition of the words of the definition.

As an example, let it be required to find the number out of 5000 men of 20 years of age who will on the average live to be 25. The probability of a man of 20 living to be 25 may be taken to be ·96; hence the number required is $\cdot 96 \times 5000 = 4800$.

Cor. 2. *If the probability of an event be p, the probability of its failing is* $1 - p$.

For out of a large number N of cases the event will happen on pN occasions; hence it will fail to happen on N $- p$N

$= (1-p)$N occasions. Hence, by the definition, the probability of the failing of the event is $1-p$.

Cor. 3. *If the universe of an event be made up of n alternatives, or, in other words, if an event must happen and that in one out of n ways, and if the respective probabilities of its happening in these ways be* $p_1, p_2, \ldots, p_n$, *then* $p_1 + p_2 + \ldots + p_n = 1$.

For on every one of N occasions the event will happen; and it will happen in the first way on p_1N occasions, in the second on p_2N occasions, and so on. Hence $N = p_1N + p_2N + \ldots + p_nN$; that is, $1 = p_1 + p_2 + \ldots + p_n$.

Cor. 4. *If an event is certain to happen, its probability is* 1; *if it is certain not to happen, its probability is* 0.

For in the former case the event happens on 1.N cases out of N cases; in the latter on 0.N cases out of N.

The probability of every event is thus a positive number lying between 0 and 1.

Cor. 5. *If an event must happen in one out of n ways all equally probable, or if one out of n events must happen and all are equally probable, then the probability of each way of happening in the first case, or of each event happening in the second, is* $1/n$.

This follows at once from Cor. 3 by making $p_1 = p_2 = \ldots = p_n$.

As a particular case, it follows that, if an event be equally likely to happen or to fail, its probability is $\frac{1}{2}$.

Definition.—*The ratio of the probability of the happening of an event to the probability of its failing to happen is called the odds in favour of the event, and the reciprocal of this ratio is called the odds against it.*

Thus, if the probability of an event be p, the odds in favour are $p : 1-p$; the odds against $1-p : p$. Also, if the odds in favour be $m : n$, the probability of the event is $m/(m+n)$. If the probability of the event be $\frac{1}{2}$, that is, if it be equally likely to happen or to fail, the odds in favour are 1 : 1, and are said to be *even*.

Cor. 6. *If the universe of an event can be analysed into* $m+n$ *cases each of which in the long run will occur equally often,* and if*

* This is usually expressed by saying that all the cases are equally likely.

in m of these cases the event will happen and in the remaining n fail to happen, the probability of the event is $m/(m+n)$.

After what has been said this will be obvious.

DIRECT CALCULATION OF PROBABILITIES.

§ 3.] The following examples of the calculation of probabilities require no special knowledge beyond the definition of probability and the principles of chap. xxiii.

Example 1. There are 5 men in a company of 20 soldiers who have made up their minds to desert to the enemy whenever they are put on outpost duty. If 3 men be taken from the company and sent on outpost duty, what is the probability that all of them desert?

The 3 men may be chosen from among the 20 in ${}_{20}C_3$ ways, all of which are equally likely. Three deserters may be chosen from among the 5 in ${}_5C_3$ ways, all equally likely. The probability of the event in question is therefore ${}_5C_3/{}_{20}C_3 = \frac{5.4.3}{1.2.3} \Big/ \frac{20.19.18}{1.2.3} = 1/114$.

Example 2. If n people seat themselves at a round table, what is the chance that two named individuals be neighbours?

There are (see chap. xxiii., § 4) $(n-1)!$ different ways, all equally likely, in which the people may seat themselves. Among these we may have A and B or B and A together along with the $(n-2)!$ different arrangements of the rest; that is, we have $2(n-2)!$ cases favourable to the event and all equally likely. The required chance is therefore $2(n-2)!/(n-1)! = 2/(n-1)$.

When $n=3$, this gives chance $=1$, as it ought to do. The odds against the event are in general $n-3$ to 2; the odds will therefore be even when the number of people is 5.

Example 3. If a be a prime integer, and $n=a^r$, and if any integer $I \not> n$ be taken at random, find the chance that I contains a as a factor s times and no more.

The integer I must be of the form λa^s, where λ is any integer less than a^{r-s} and prime to a^{r-s}. Now, by chap. xxxv., § 8, the number of integers less than a^{r-s} and prime to it is $a^{r-s}(1-1/a)$. Also the number of integers $\not> n$ is a^r. Hence the required chance is $a^{r-s}(1-1/a)/a^r = a^{-s}(1-1/a) = 1/a^s - 1/a^{s+1}$.

Example 4. Find the probability that two men A and B of m and n years of age respectively both survive for p years.

The mortality tables (see § 15 below) give us the numbers out of 100,000 individuals of 10 years of age who complete their mth, nth, $m+p$th, $n+p$th years. Let these numbers be l_m, l_n, l_{m+p}, l_{n+p}. The probabilities that A and B live to be $m+p$ and $n+p$ years of age respectively are l_{m+p}/l_m, l_{n+p}/l_n respectively. Consider now two large groups of men numbering M and N respectively. We suppose A to be always selected from the first and B always

from the second. In this way we could select altogether MN pairs of men who may be alive or dead after p years have elapsed. The number out of the M men *living* after p years is Ml_{m+p}/l_m, by § 2, Cor. 1. Similarly the number living out of the N men is Nl_{n+p}/l_n. Out of these we could form $MNl_{m+p}l_{n+p}/l_m l_n$ pairs. This last number will be the number of pairs of survivors out of the MN pairs with which we started. Hence the probability required is $l_{m+p}l_{n+p}/l_m l_n = (l_{m+p}/l_m)(l_{n+p}/l_n)$; in other words, it is the product of the probabilities that the two men singly each survive for p years. The student should study this example carefully, as it furnishes a direct proof of a result which would usually be deduced from the law for the multiplication of probabilities. See below, § 6.

Example 5. A number of balls is to be drawn from an urn, 1, 2, . . ., n being all equally likely. What is the probability that the number drawn be even?

We can draw 1, 2, . . ., n respectively in ${}_nC_1$, ${}_nC_2$, . . ., ${}_nC_n$ ways respectively. Hence we may consider the universe of the event as consisting of ${}_nC_1 + {}_nC_2 + \ldots + {}_nC_n = (1+1)^n - 1 = 2^n - 1$ equally likely cases. The number of these in which the drawing is even is ${}_nC_2 + {}_nC_4 + \ldots = \frac{1}{2}\{(1+1)^n + (1-1)^n - 2\} = \frac{1}{2}(2^n - 2) = 2^{n-1} - 1$. The number of ways in which an odd drawing can be made is ${}_nC_1 + {}_nC_3 + \ldots = \frac{1}{2}\{(1+1)^n - (1-1)^n\} = \frac{1}{2}2^n = 2^{n-1}$. Hence the chance that the drawing be even is $(2^{n-1}-1)/(2^n-1)$, that it be odd $2^{n-1}/(2^n-1)$. The sum of these is unity, as it ought to be; since, if the drawing is not odd, it must be even. In general, an odd drawing is more likely than an even drawing, the odds in its favour being $2^{n-1} : 2^{n-1} - 1$; but the odds become more nearly even as n increases.

Example 6. A white rook and two black pawns are placed at random on a chess-board in any of the positions which they might occupy in an actual game. Find the ratio of the chance that the rook can take one or both of the pawns to the chance that either or both of the pawns can take the rook.

Let us look at the board from the side of white; and calculate in the first place the whole number of possible arrangements of the pieces. No black pawn can lie on any of the front squares; hence we may have the rook on any of these 8 and the two pawns on any two of the remaining 56; in all, $8 \times {}_{56}C_2 = 8 \times 56 \times 55$ arrangements. Again, we may have the rook on any one of the 56 squares and the two pawns on any two of the remaining 55 squares; in all, $56 \times 55 \times 54$ arrangements. The universe may therefore be supposed to contain $62 \times 56 \times 55$ equally likely cases.

Instead of calculating the chance that the rook can take either or both of the pawns, it is simpler, as often happens, to calculate the chance of the complementary event, namely, that the rook can take neither of the pawns. If the rook lie on one of the front row of squares, neither of the pawns can lie on the corresponding column, that is, the pawns may occupy any two out of 49 squares; this gives $8 \times 49 \times 48$ arrangements. If the rook lies in any one of the remaining 56 squares, neither of the pawns must lie in the row or column belonging to that square; hence there are for the two pawns 42×41 positions. We thus have $56 \times 42 \times 41$ arrangements. Altogether we have

$8 \times 49 \times 48 + 56 \times 42 \times 41 = 56 \times 49 \times 42$ arrangements in which the rook can take neither pawn. Hence the chance that the rook can take neither pawn is $56 \times 49 \times 42/62 \times 56 \times 55 = 1029/1705$. The chance that the rook can take one or both of the pawns is therefore $1 - 1029/1705 = 676/1705$.

Consider now the attack on the rook. If he is on a side square, he can only be attacked by either of the two pawns from one square. For the side squares we have therefore only 24×54 arrangements in which the rook can be taken. There remain 36 squares on each of which the rook can be taken from two squares, that is, in 6 ways. For the 36 squares we therefore have $36 \times 2 + 36 \times 4 \times 53$ arrangements in which the rook can be taken by one or by both the pawns. Altogether there are 9000 arrangements in which the rook may be taken. Hence the chance that he be in danger is $9000/62 \times 56 \times 55 = 225/4774$. The ratio of the two chances is 9464 : 1125.

§ 4.] A considerable number of interesting examples can be solved by the method of chap. xxiii., § 15. Let there be r bags, the first of which contains $a_1, b_1, c_1, \ldots, k_1$ counters, marked with the numbers $\alpha_1, \beta_1, \gamma_1, \ldots, \kappa_1$; the second, $a_2, b_2, c_2, \ldots k_2$, marked $\alpha_2, \beta_2, \gamma_2, \ldots, \kappa_2$; and so on. If a counter be drawn from each bag, what is the chance that the sum of the numbers drawn is n?

By chap. xxiii., § 15, the number of ways in which the sum of the drawings can amount to n is the coefficient, A_n say, of x^n in the distribution of the product

$$\begin{aligned}&(a_1 x^{\alpha_1} + b_1 x^{\beta_1} + \ldots + k_1 x^{\kappa_1})\\ &\times (a_2 x^{\alpha_2} + b_2 x^{\beta_2} + \ldots + k_2 x^{\kappa_2})\\ &\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ &\times (a_r x^{\alpha_r} + b_r x^{\beta_r} + \ldots + k_r x^{\kappa_r}).\end{aligned}$$

Again, the whole number of drawings possible is the sum of all the coefficients; that is to say,

$$\begin{aligned}&(a_1 + b_1 + \ldots + k_1)\\ &\times (a_2 + b_2 + \ldots + k_2)\\ &\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ &\times (a_r + b_r + \ldots + k_r) = \mathrm{D}, \text{ say.}\end{aligned}$$

Hence the required chance is A_n/D.

Example 1. A throw has been made with three dice. The sum is known to be 12; required the probability that the throw was 4, 4, 4.

The number of ways in which 12 can be thrown with three dice is the coefficient of x^{12} in

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^3,$$

that is to say, of x^9 in

$$(1+x+x^2+x^3+x^4+x^5)^3.$$

Now the coefficients in $(1+x+\ldots+x^5)^2$ up to the term in x^9 are (see chap. iv., § 15) $1+2+3+4+5+6+5+4+3+2$. Hence the coefficient of x^9 in the cube of the multinomial is $5+6+5+4+3+2=25$.* The required probability is therefore 1/25.

Example 2. One die has 3 faces marked 1, 2 marked 2, and 1 marked 3; another has 1 face marked 1, 2 marked 2, and 3 marked 3. What is the most probable throw with the two dice, and what the chance of that throw?

The numbers of ways in which the sums 2, 3, 4, 5, 6 can be made are the coefficients of x^2, x^3, x^4, x^5, x^6 in the expansion of $(3x+2x^2+x^3)(x+2x^2+3x^3)$. Now this product is equal to

$$3x^2+8x^3+14x^4+8x^5+3x^6.$$

The sum that will occur oftenest in the long run is therefore 4. The whole number of different ways in which the different throws may turn out is $(3+2+1)(1+2+3)=36$. Hence the probability of the sum 4 is $14/36 = 7/18$.

Example 3. An urn contains m counters marked with the numbers 1, 2, . . ., m. A counter is drawn and replaced r times; what is the chance that the sum of the numbers drawn is n?†

The whole number of possible different drawings is m^r.

The number of those which give the sum n is the coefficient of x^n in $(x+x^2+\ldots+x^m)^r$, that is to say, of x^{n-r} in $(1+x+\ldots+x^{m-1})^r$. Now $1+x+\ldots+x^{m-1}=(1-x^m)/(1-x)$. We have therefore to find the coefficient of x^{n-r} in

$$(1-x^m)^r(1-x)^{-r} = \{1-{}_rC_1x^m+{}_rC_2x^{2m}-{}_rC_3x^{3m}+\ldots\} \times \left\{1+\frac{r}{1}x+\frac{r(r+1)}{1.2}x^2+\frac{r(r+1)(r+2)}{1.2.3}x^3+\ldots\right\}.$$

The coefficient in question is

$$A_{n-r}=\frac{r(r+1)\ldots(n-1)}{(n-r)!}-\frac{r(r+1)\ldots(n-m-1)r}{(n-r-m)!1!}+\frac{r(r+1)\ldots(n-2m-1)r(r-1)}{(n-r-2m)!2!}-\ldots.$$

The required probability is A_{n-r}/m^r.

Example 4. If m odd and n even integers ($n<m-1$) be written down at random, show that the chance that no two odd integers are adjacent is $n!(n+1)!/(m+n)!(n-m+1)!$.

In order to find in how many different ways we can write down the integers so that no two odd ones come together, we may suppose the m odd integers written down in any one of the $m!$ possible ways, and consider the $m-1$ spaces between them together with the two spaces to the right and left of the row. The problem now is to find in how many ways we can fill the n even

* We might also have found the coefficient of x^9 by expanding $(1-x^6)^3(1-x)^{-3}$, as in Example 4 below.

† This is generally called Demoivre's Problem. For an interesting account of its history see Todhunter, *Hist. Prob.*, pp. 59, 85.

integers into the spaces so that there shall always be one at least in every one of the $m-1$ spaces. A little consideration will show that the number of ways, irrespective of order, is the coefficient of x^n in

$$(1+x+x^2+\ldots \text{ ad } \infty)^2(x+x^2+\ldots \text{ ad } \infty)^{m-1};$$

that is, of x^{n-m+1} in $(1+x+x^2+\ldots)^2(1+x+x^2+\ldots)^{m-1}$;

that is, of x^{n-m+1} in $(1-x)^{-(m+1)}$.

This coefficient is

$$\frac{(m+1)(m+2)\ldots(n+1)}{(n-m+1)!}=\frac{(n+1)!}{m!(n-m+1)!}.$$

If we remember that every distribution of the n integers among the $m+1$ spaces can be permutated in $n!$ ways, we now see that the number of ways in which the $m+n$ integers can be arranged as required is

$$n!m!(n+1)!/m!(n-m+1)!=n!(n+1)!/(n-m+1)!.$$

The whole number of ways in which the $m+n$ integers can be arranged is $(m+n)!$, hence the probability required is $n!(n+1)!/(n-m+1)!(m+n)!$.

ADDITION AND MULTIPLICATION OF PROBABILITIES.

§ 5.] In many cases we have to consider the probabilities of a set of events which are of such a nature that the happening of any one of them upon any occasion excludes the happening of any other upon that particular occasion. A set of events so related are said to be *mutually exclusive.* The set of events considered may be merely different ways of happening of the same event, provided these ways of happening are mutually exclusive.

In such cases the following rule, which we may call the *Addition Rule,* applies :—

If the probabilities of n mutually exclusive events be $p_1, p_2, \ldots, p_n$, *the chance that one out of these n events happens on any particular occasion on which all of them are in question is* $p_1+p_2+\ldots+p_n$.

To prove this rule, consider any large number N of occasions where all the events are in question. Out of these N occasions the n events will happen on $p_1\mathrm{N}, p_2\mathrm{N}, \ldots, p_n\mathrm{N}$ occasions respectively. There is no cross classification here, since no more than one of the events can happen on any one occasion. Out of N occasions, therefore, one or other of the n events will happen on $p_1\mathrm{N}+p_2\mathrm{N}+\ldots+p_n\mathrm{N}=(p_1+p_2+\ldots+p_n)\mathrm{N}$ occasions. Hence the probability that one out of the n events happens on any one occasion is $p_1+p_2+\ldots+p_n$.

It should be observed that the reasoning would lose all force if the events were not mutually exclusive, for then it might be that on the p_1N occasions on which the first event happens one or more of the others happen. We shall give the proper formula in this case presently.

As an illustration of the application of this rule, let us suppose that a throw is made with two ordinary dice, and calculate the probability that the throw does not exceed 8. There are 7 ways in which the event in question may happen, namely, the throw may be 2, 3, 4, 5, 6, 7, or 8; and these ways are of course mutually exclusive. Now (see § 4, Example 1) the probabilities of these 7 throws are 1/36, 2/36, 3/36, 4/36, 5/36, 6/36, 5/36 respectively. Hence the probability that a throw with two dice does not exceed 8 is $(1+2+3+4+5+6+5)/36 = 26/36 = 13/18$.

§ 6.] When a set of events is such that the happening of any one of them in no way affects the happening of any other, we say that the events are *mutually independent.* For such a set of events we have the following *Multiplication Rule*:—

If the respective probabilities of n independent events be $p_1, p_2, \ldots, p_n$, *the probability that they all happen on any occasion in which all of them are in question is* $p_1 p_2 \ldots p_n$.

In proof of this rule we may reason as follows:—Out of any large number N of cases where all the events are in question, the first event will happen on p_1N occasions. Out of these p_1N occasions the second event will also happen on $p_2(p_1\mathrm{N}) = p_1 p_2 \mathrm{N}$ occasions; so that out of N there are $p_1 p_2$N occasions on which both the first and second events happen. Continuing in this way, we show that out of N occasions there are $p_1 p_2 \ldots p_n$N occasions on which all the n events happen. The probability that all the n events happen on any occasion is therefore $p_1 p_2 \ldots p_n$.

It should be noticed that the above reasoning would stand if the events were not independent, provided p_2 denote the probability that event 2 happen after event 1 has happened, p_3 the probability that 3 happen after 1 and 2 have happened, and so on.

It must be observed, however, that the probability calculated is then that the events happen in the order 1, 2, 3, ... n. Hence the following conclusion:—

Cor. *If the events* 1, 2, . . ., *n be interdependent and* p_1 *denote the probability of* 1, p_2 *the probability that* 2 *happen after* 1 *has happened,* p_3 *the probability that* 3 *happen after* 1 *and* 2 *have happened, and so on, then the probability that the events* 1, 2, . . ., *n happen in the order indicated is* $p_1 p_2 \ldots p_n$.

As an illustration of the multiplication rule, let us suppose that a die is thrown twice, and calculate the probability that the result is such that the first throw does not exceed 3 and the second does not exceed 5.

The probability that the first throw does not exceed 3 is, by the addition rule, 3/6 ; the probability that the second does not exceed 5 is 5/6. The result of the first throw in no way affects the result of the second ; hence the probability that the result of the two throws is as indicated is, by the multiplication rule, $(3/6) \times (5/6) = 5/12$.

As an example of the effect of a slight alteration in the wording of the question, consider the following :—A die has been thrown twice : what is the probability that one of the throws does not exceed 3 and the other does not exceed 5 ?

Since the particular throws are now not specified, the event in question happens—1st, if the first throw does not exceed 3 and the second does not exceed 5 ; 2nd, if the first throw is 4 or 5 and the second does not exceed 3. These cases are mutually exclusive, and the respective probabilities are 5/12 and 1/6. Hence, by the addition rule, the probability of the event in question is 7/12.

§ 7.] The following examples will illustrate the application of the addition and multiplication of probabilities.

Example 1. One urn, A, contains m balls, pm being white, $(1-p)m$ black; another, B, contains n balls, qn white, $(1-q)n$ black. A person selects one of the two urns at random, and draws a ball ; calculate the chance that it be white ; and compare with the chance of drawing a white ball when all the $m+n$ balls are in one urn.

There are two ways, mutually exclusive, in which a white ball may be drawn, namely, from A or from B.

The chance that the drawer selects the urn A is 1/2, and if he selects that urn the chance of a white ball is p. Hence the chance that a white ball is drawn from A is (§ 6, Cor.) $\frac{1}{2}p$. Similarly the chance that a white ball is drawn from B is $\frac{1}{2}q$. The whole chance of drawing a white ball is therefore $(p+q)/2$.

If all the balls be in one urn, the chance is $(pm+qn)/(m+n)$.

Now $(pm+qn)/(m+n) > = < (p+q)/2$,

according as $2(pm+qn) > = < (p+q)(m+n)$,

according as $(m-n)(p-q) > = < 0$.

Hence the chance of drawing a white ball will be unaltered by mixing if either the numbers of balls in A and B be equal, or the proportion of white balls in each be the same.

If the number of balls be unequal, and the proportions of white be unequal, then the mixing of the balls will increase the chance of drawing a white if the urn which contains most balls have also the larger proportion of white; and will diminish the chance of drawing a white if the urn which contains most balls have the smaller proportion of white.

De Morgan* has used a particular case of this example to point out the danger of a fallacious use of the addition rule. Let us suppose the two urns to be as follows: A (3 wh., 4 bl.); B (4 wh., 3 bl.). We might then with some plausibility reason thus:—The drawer must select either A or B. If he select A, the chance of white is 3/7; if he select B, the chance of white is 4/7. Hence, by the addition rule, the whole chance of white is $3/7 + 4/7 = 1$. In other words, white is certain to be drawn, which is absurd. The mistake consists in not taking account of the fact that the drawer has a choice of urns and that the chance of his selecting A must therefore be multiplied into his chance of drawing white after he has selected A. The chance should therefore be $3/14 + 4/14 = 1/2$.

The necessity for introducing the factor 1/2 will be best seen by reasoning directly from the fundamental definition. Let us suppose the drawer to make the experiment any large number N of times. In the long run the one urn will be selected as often as the other. Hence out of N times A will be selected N/2 times. Out of these N/2 times white will be drawn from A $(3/7)(N/2) = N(3/14)$ times. Similarly, we see that white will be drawn from B $N(4/14)$ times. Hence, on the whole, out of N trials white will be drawn $(3/14 + 4/14)N$ times. The chance is therefore $3/14 + 4/14$.

Example 2. Four cards are drawn from an ordinary pack of 52; what is the chance that they be all of different suits?

We may treat this as an example of § 6, Cor. The chance that the first card drawn be of one of the 4 suits is, of course, 1. The chance, after one suit is thus represented, that the next card drawn be of a different suit is, since there are now only 3 suits allowable and only 51 cards to choose from, 3.13/51. After two cards of different suits are drawn, the chance that the next is of a different suit is 2.13/50. Finally, the chance that the last card is of a different suit from the first three is 13/49. By the principle just mentioned the whole chance is therefore $3.13.2.13.13/51.50.49 = 13^3/17.25.49 = 1/10$ roughly.

Example 3. How many times must a man be allowed to toss a penny in order that the odds may be 100 to 1 that he gets at least one head?

Let x be the number of tosses. The complementary event to "one head at least" is "all tails." Since the chance of a tail each time is 1/2, and the result of each toss is independent of the result of every other, the chance of "all tails" in x tosses is $(1/2)^x$. The chance of one head at least is therefore $1 - (1/2)^x$. By the conditions of the question, we must therefore have

$$1 - (1/2)^x = 100/101;$$

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* Art. "Theory of Probability," *Ency. Metrop.* Republished *Ency. Pure Math.* (1847), p. 399.

hence

$$2^x = 101,$$
$$x = \log 101 / \log 2,$$
$$= 2{\cdot}0043/{\cdot}3010,$$
$$= 6{\cdot}6 \ldots .$$

It appears, therefore, that in 6 tosses the odds are less than 100 to 1, and in 7 tosses more.

Example 4. A man tosses 10 pennies, removes all that fall head up; tosses the remainder, and again removes all that fall head up; and so on. How many times ought he to be allowed to repeat this operation in order that there may be an even chance that before he is done all the pennies have been removed?

Let x be the number of times, then it is clearly necessary and sufficient for his success that each of the 10 pennies shall have turned up head at least once. The chance that each penny come up head at least once in x trials is $1-(1/2)^x$. Hence the chance that each of the 10 has turned up heads at least once is $\{1-(1/2)^x\}^{10}$. By the conditions of the problem we must therefore have

$$\{1-(1/2)^x\}^{10} = 1/2;$$
$$(1/2)^x = 1-(1/2)^{1/10} = {\cdot}06697;$$
$$x = -\log {\cdot}06697/\log 2,$$
$$= 3{\cdot}9 \text{ very nearly.}$$

Hence he must have 4 trials to secure an even chance.

Example 5. A man is to gain a shilling on the following conditions. He draws twice (replacing each time) out of an urn containing one white and one black ball. If he draws white twice he wins. If he fails a black ball is added, he tries twice again, and wins if he draws white twice. If he fails another black ball is added; and so on, *ad infinitum*. What is his chance of gaining the shilling? (Laurent, *Calcul des Probabilités* (1873), p. 69.)

The chances of drawing white in the various trials are $1/2^2, 1/3^2, \ldots 1/n^2, \ldots$ The chances of failing in the various trials are $1-1/2^2, 1-1/3^2, \ldots, 1-1/n^2, \ldots$ Hence the chance of failing in all the trials is $(1-1/2^2)(1-1/3^2)\ldots(1-1/n^2)\ldots$ ad ∞.

Now

$$\operatorname*{L}_{n=\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$
$$= \operatorname*{L}_{n=\infty}\frac{\{1.3\}\{2.4\}\ldots\{(n-3)(n-1)\}\{(n-2)n\}\{(n-1)(n+1)\}}{1^2.2^2\ldots n^2},$$
$$= \operatorname*{L}_{n=\infty}\frac{n(n+1)}{2n^2},$$
$$= \operatorname*{L}_{n=\infty}\frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}.$$

The chance of failing to gain the shilling is therefore 1/2. Hence the chance of gaining the shilling is 1/2.

We might have calculated the chance of gaining the shilling directly, by

observing that it is the sum of the chances of the following events: 1°, gaining in the first trial; 2°, failing in 1st and gaining in 2nd; 3°, failing in 1st and 2nd and gaining in the 3rd; and so on. In this way the chance presents itself as the following infinite series:—

$$\frac{1}{2^2}+\left(1-\frac{1}{2^2}\right)\frac{1}{3^2}+\ldots+\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)\frac{1}{(n+1)^2}+\ldots.$$

The sum of this series to infinity must therefore be 1/2. That this is so may be easily verified. The present is one example among many in which the theory of probability suggests interesting algebraical identities.

Example 6. A and B cast alternately with a pair of ordinary dice. A wins if he throws 6 before B throws 7, and B if he throws 7 before A throws 6. If A begin, show that his chance of winning: B's = 30 : 31. (Huyghens, *De Ratiociniis in Ludo Aleæ*, 1657.)

Let p and q be the chances of throwing and of failing to throw 6 at a single cast with two dice; r and s the corresponding chances for 7.

A may win in the following ways: 1°, A succeed at 1st throw; 2°, A fail at 1st, B fail at 2nd, A succeed at 3rd; and so on. His chance is therefore represented by the following infinite series:—

$$p+qsp+qsqsp+\ldots=p\{1+(qs)+(qs)^2+\ldots\},$$
$$=p/(1-qs).$$

B may win in the following ways:—1°, A fail at 1st, B succeed at 2nd; 2°, A fail at 1st, B fail at 2nd, A fail at 3rd, B succeed at 4th; and so on. His chance is therefore

$$qr+qsqr+qsqsqr+\ldots=qr\{1+(qs)+(qs)^2+\ldots\},$$
$$=qr/(1-qs).$$

Hence A's chance : B's = $p : qr$.

Now (see § 4, Example 1) $p=5/36$, $q=31/36$, $r=6/36$; hence

$$\text{A's chance : B's} = 5/36 : 6.31/36^2,$$
$$= 30 : 31.$$

For Huyghens' own solution see Todhunter, *Hist. Prob.*, p. 24.

Example 7. A coin is tossed $m+n$ times ($m>n$). Prove that the chance of at least m consecutive heads appearing is $(n+2)/2^{m+1}$.

The event in question happens if there appear—1st, exactly m; 2nd, exactly $m+1$; . . .; $(n+1)$th, exactly $m+n$ consecutive heads.

Now a run of exactly m consecutive heads may commence with the 1st, 2nd, 3rd, $n-1$th, nth, $n+1$th throw. Since $m>n$, there cannot be more than one run of m or more consecutive heads, so that the complication due to repetition of runs does not occur in the present problem. The chances of the first and last of these cases are each $1/2^{m+1}$, the chances of the others $1/2^{m+2}$. Hence the chance of a run of exactly m consecutive heads is $2/2^{m+1}+(n-1)/2^{m+2}=(n+3)/2^{m+2}$.

In like manner, we see that the chance of a run of $m+1$ consecutive heads is $(n+2)/2^{m+3}$; and so on, up to $m+n-2$. Also the chances of a run of exactly $m+n-1$ and of exactly $m+n$ consecutive heads are $1/2^{m+n-1}$ and $1/2^{m+n}$ respectively.

Hence the chance p of a run of at least m heads is given by

$$p=\frac{n+3}{2^{m+2}}+\frac{n+2}{2^{m+3}}+\ldots+\frac{5}{2^{m+n}}+\frac{4}{2^{m+n+1}}+\frac{1}{2^{m+n}}.$$

The summation of the series on the left-hand side is effected (see chap. XX., § 13) by multiplying by $(1-1/2)^2=1/4$. We thus find

$$\tfrac{1}{4}p=\frac{n+3}{2^{m+2}}+\frac{n+2}{2^{m+3}}+\frac{n+1}{2^{m+4}}+\ldots+\frac{4}{2^{m+n+1}}$$
$$-\frac{2(n+3)}{2^{m+3}}-\frac{2(n+2)}{2^{m+4}}-\ldots-\frac{2.5}{2^{m+n+1}}-\frac{2.4}{2^{m+n+2}}$$
$$+\frac{n+3}{2^{m+4}}+\ldots+\frac{6}{2^{m+n+1}}+\frac{5}{2^{m+n+2}}+\frac{4}{2^{m+n+3}}+\frac{1}{2^{m+n+2}};$$

$$\tfrac{1}{4}p=\frac{n+3}{2^{m+2}}-\frac{n+4}{2^{m+3}}-\frac{3}{2^{m+n+2}}+\frac{2}{2^{m+n+2}}+\frac{1}{2^{m+n+2}},$$

$$=\frac{n+2}{2^{m+3}}.$$

Hence $p=(n+2)/2^{m+1}$.

GENERAL THEOREMS REGARDING THE PROBABILITY OF COMPOUND EVENTS.

§ 8.] *The probability that an event, whose probability is p, happen on exactly r out of n occasions in which it is in question is ${}_nC_r p^r q^{n-r}$, where $q=1-p$ is the probability that the event fail.*

The probability that the event happen on r *specified* occasions and fail on the remaining $n-r$ is by the multiplication rule $ppq\,pqq\ldots$ where there are r p's and $n-r$ q's, that is, $p^r q^{n-r}$. Now the occasions are not specified; in other words, the happening, and failing, may occur in any order. There are as many ways of arranging the r happenings and $n-r$ failings as there are permutations of n things r of which are alike and $n-r$ alike, that is to say, $n!/r!(n-r)! = {}_nC_r$. There are therefore ${}_nC_r$ mutually exclusive ways in which the event with which we are concerned may happen; and the probability of each of these is $p^r q^{n-r}$. Hence, by the addition rule, the probability in question is ${}_nC_r p^r q^{n-r}$.

It will be observed that the probabilities that the event happen exactly n, $n-1$, . . ., 2, 1, 0 times respectively, are the 1st, 2nd, 3rd, . . ., $(n+1)$th terms of the expansion of $(p+q)^n$.

Since, if we make n trials, the event must happen either 0,

or 1, or 2, . . ., or n times, the sum of all these probabilities ought to be unity. This is so; for, since $p+q=1$, $(p+q)^n=1$.

It will be seen without further demonstration that the proposition just established is merely a particular case of the following general theorem :—

If there be m events A, B, C, . . . *one but not more of which must happen on every occasion, and if their probabilities be p, q, r, . . . respectively, the probability that on n occasions* A *happen exactly α times,* B *exactly β times,* C *exactly γ times, . . . is*

$$n!\,p^{\alpha}q^{\beta}r^{\gamma}\ldots/\alpha!\,\beta!\,\gamma!\ldots,$$

where $\alpha+\beta+\gamma+\ldots=n$.

It should be observed that the expression just written is the general term in the expansion of the multinomial $(p+q+r+\ldots)^n$.

Example 1. The faces of a cubical die are marked 1, 2, 2, 4, 4, 6; required the probability that in 8 throws 1, 2, 4 turn up exactly 3, 2, 3 times respectively.

By the general theorem just stated the probability is

$$\frac{8!}{3!\,2!\,3!}\left(\frac{1}{6}\right)^3\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right)^3=\frac{7.5.2}{3^8},$$

$$=\frac{1}{94} \text{ approximately.}$$

Example 2. Out of n occasions in which an event of probability p is in question, on what number of occasions is it most likely to happen?

We have here to determine r so that ${}_nC_r\,p^rq^{n-r}$ may be a maximum.

Now $${}_nC_r\,p^rq^{n-r}/{}_nC_{r-1}p^{r-1}q^{n-r+1}=(n-r+1)p/rq.$$

Hence the probability will increase as r increases, so long as

$$(n-r+1)p>rq,$$

that is, $$(n+1)p>r(p+q),$$

that is $$r<(n+1)p.$$

If $(n+1)p$ be an integer, $=s$ say, then the event will be equally likely to happen on $s-1$ or on s occasions, and more likely to happen $s-1$ or s times than any other number of times.

If $(n+1)p$ be not an integer, and s be the greatest integer in $(n+1)p$, then the event is most likely to happen on s occasions.*

* When n is very large, $(n+1)p$ differs inappreciably from np. Hence out of a very large number n of occasions an event is most likely to happen on pn occasions. This, of course, is simply the fundamental principle of § 2, Cor. 1, arrived at by a circuitous route starting from itself in the first instance.

As a numerical instance, suppose an ordinary die is thrown 20 times, what is the number of aces most likely to appear?

Here $n=20\,;\quad p=1/6\,;\quad (n+1)p=3\frac{1}{2}$.

The most likely number of aces is therefore 3.

§ 9.] *The probability that an event happen on at least r out of n occasions where it is in question is*

$$ {}_nC_r p^r q^{n-r} + {}_nC_{r+1} p^{r+1} q^{n-r-1} + \ldots + {}_nC_{n-1} p^{n-1} q + p^n \ldots (1). $$

For an event happens *at least r times* if it happen either exactly r; or exactly $r+1$; . . .; or exactly n times. Hence the probability that it happens at least r times is the sum of the probabilities that it happens exactly r, exactly $r+1$, . . ., exactly n times; and this, by § 8, gives the expression (1).

Another expression for the probability just found may be deduced as follows:—Suppose we watch the sequence of the happenings and failings in a series of different cases. After we have observed the event to have happened just r times, we may withdraw our attention and proceed to consider another case; and so on. Looking at the matter in this way, we see that the r happenings may be just made up on the rth, or on the $r+1$th, . . ., or on the nth occasion.

If the r happenings have been made up in just s occasions, then the event must have happened on the sth occasion and on any $r-1$ of the preceding $s-1$ occasions. The probability of this contingency is

$$ p \times {}_{s-1}C_{r-1} p^{r-1} q^{s-r} = {}_{s-1}C_{s-r} p^r q^{s-r}. $$

Hence the probability that the event happen at least r times in n trials is

$$ p^r + {}_rC_1 p^r q + {}_{r+1}C_2 p^r q^2 + \ldots + {}_{n-1}C_{n-r} p^r q^{n-r} $$
$$ = p^r \{ 1 + {}_rC_1 q + {}_{r+1}C_2 q^2 + \ldots + {}_{n-1}C_{n-r} q^{n-r} \} \quad (2). $$

As the two expressions (1) and (2) are outwardly very different, it may be well to show that they are really identical. To do this, we have to prove that

$$ 1 + {}_rC_1 q + {}_{r+1}C_2 q^2 + \ldots + {}_{n-1}C_{n-r} q^{n-r} $$
$$ = p^{n-r} \left\{ 1 + {}_nC_1 \left(\frac{q}{p}\right) + {}_nC_2 \left(\frac{q}{p}\right)^2 + \ldots + {}_nC_{n-r} \left(\frac{q}{p}\right)^{n-r} \right\}, $$
$$ = (1-q)^{n-r} \left\{ 1 + {}_nC_1 \left(\frac{q}{1-q}\right) + {}_nC_2 \left(\frac{q}{1-q}\right)^2 + \ldots + {}_nC_{n-r} \left(\frac{q}{1-q}\right)^{n-r} \right\}. $$

The expression last written is, up to the $(n-r)$th power of q, identical with

$$(1-q)^{n-r}\{1+q/(1-q)\}^n=(1-q)^{n-r}/(1-q)^n=(1-q)^{-r}.$$

Now, as may be readily verified,

$$(1-q)^{-r}=1+{}_rC_1q+{}_{r+1}C_2q^2+\ldots+{}_{n-1}C_{n-r}q^{n-r}+\ldots.$$

The required identity is therefore established.

Example. A and B play a game which must be either lost or won; the probability that A gains any game is p, that B gains it $1-p=q$; what is the chance that A gains m games before B gains n? (Pascal's Problem.)*

The issue in question must be decided in $m+n-1$ games at the utmost. The chance required is in fact the chance that A gains m games at least out of $m+n-1$, that is, by (1) above,

$$p^{m+n-1}+{}_{m+n-1}C_1p^{m+n-2}q+\ldots+{}_{m+n-1}C_mp^mq^{n-1} \qquad (1').$$

We might adopt the second way of looking at the question given above, and thus arrive at the expression

$$p^m\{1+{}_mC_1q+{}_{m+1}C_2q^2+\ldots+{}_{m+n-2}C_{n-1}q^{n-1}\} \qquad (2').$$

for the required chance.

§ 10.] The results just arrived at may be considerably generalised. Let us consider n independent events $A_1, A_2, \ldots, A_n$, whose respective probabilities are $p_1, p_2, \ldots, p_n$.

In the first place, in contrast to §§ 8, 9, let us calculate *the chance that one at least of the n events happen.*

The complementary event is that none of the n events happen. The probability of this is $(1-p_1)(1-p_2)\ldots(1-p_n)$. Hence the probability that one at least happen is

$$1-(1-p_1)(1-p_2)\ldots(1-p_n)$$
$$=\Sigma p_1-\Sigma p_1p_2+\Sigma p_1p_2p_3-\ldots+(-)^{n-1}p_1p_2\ldots p_n \qquad (1).$$

Next let us find *the probability that one and no more of the n events happen.*

The probability that any particular event, say A_1, and none of the others happen is $p_1(1-p_2)(1-p_3)\ldots(1-p_n)$. Hence the required probability is

$$\Sigma p_1(1-p_2)(1-p_3)\ldots(1-p_n)$$
$$=\Sigma p_1-{}_2C_1\Sigma p_1p_2+{}_3C_2\Sigma p_1p_2p_3-\ldots+(-)^{n-1}{}_nC_{n-1}p_1p_2\ldots p_n \qquad (2).$$

* Famous in the history of mathematics. It was first solved for the particular case $p=q$ by Pascal (1654). The more general result (1') above was given by John Bernoulli (1710). The other formula (2') seems to be due to Montmort (1714). See Todhunter, *Hist. Prob.*, p. 98.

For the products two and two arise from $-\Sigma p_1(p_2 + p_3 + \ldots + p_n)$, and each pair will come in once for every letter in it. Again, the products three and three arise from $\Sigma p_1(p_2p_3 + p_2p_4 + \ldots)$; hence each triad will come in once for every pair of letters that can be selected from it; and so on.

By precisely similar reasoning, we can show that *the probability that r and no more of the n events happen is*

$$\begin{aligned}\Sigma p_1p_2\ldots p_r(1-p_{r+1})(1-p_{r+2})\ldots(1-p_n) &= \Sigma p_1p_2\ldots p_r - {}_{r+1}C_1\Sigma p_1p_2\ldots p_{r+1} \\ &\quad + {}_{r+2}C_2\Sigma p_1p_2\ldots p_{r+2} \\ &\quad \cdots\cdots\cdots \\ &\quad (-)^s{}_{r+s}C_s\Sigma p_1p_2\ldots p_{r+s} \\ &\quad \cdots\cdots\cdots \\ &\quad (-)^{n-r}{}_nC_{n-r}p_1p_2\ldots p_n \qquad (3).\end{aligned}$$

We can now calculate *the probability that r at least out of the n events happen.*

To do so we have merely to sum all the values of (3) obtained by giving r the values $r, r+1, r+2, \ldots, n$ successively.

In this summation the coefficient of $\Sigma p_1p_2\ldots p_{r+s}$ is

$$(-)^s\{{}_{r+s}C_s - {}_{r+s}C_{s-1} + {}_{r+s}C_{s-2} - \ldots (-)^{s-1}{}_{r+s}C_1 + (-1)^s\}.$$

Now the expression within the brackets is the coefficient of x^s in $(1+x)^{r+s} \times (1+x)^{-1}$, that is to say, in $(1+x)^{r+s-1}$. This coefficient is ${}_{r+s-1}C_s$. Hence the coefficient of $\Sigma p_1p_2\ldots p_{r+s}$ is $(-)^s{}_{r+s-1}C_s$.

The probability that r at least out of the n events happen is therefore

$$\begin{aligned}&\Sigma p_1p_2\ldots p_r - {}_rC_1\Sigma p_1p_2\ldots p_{r+1} \\ &\quad + {}_{r+1}C_2\Sigma p_1p_2\ldots p_{r+2} \\ &\quad \cdots\cdots\cdots \\ &\quad (-)^s{}_{r+s-1}C_s\Sigma p_1p_2\ldots p_{r+s} \\ &\quad \cdots\cdots\cdots \\ &\quad (-)^{n-r}{}_{n-1}C_{n-r}p_1p_2\ldots p_n \qquad (4).\end{aligned}$$

Since the happening of the same event on n different occasions may be regarded as the happening of n different events whose

probabilities are all equal, the formulæ (3) and (4) above ought, when $p_1 = p_2 = \ldots = p_n$ each $= p$, to reduce to ${}_nC_r p^r q^{n-r}$ and the expression (1) or (2) of § 9 respectively.

If the reader observe that, when $p_1 = p_2 = \ldots = p_n = p$, $\Sigma p_1 p_2 \ldots p_r = {}_nC_r p^r$, &c., he will have no difficulty in showing that (3) is actually identical with ${}_nC_r p^r q^{n-r}$ in the particular case in question.

The particular result derived from (4) is more interesting. We find for the probability, that an event of probability p will happen r times at least out of n occasions, the expression

$$ {}_nC_r p^r - {}_rC_1 \, {}_nC_{r+1} p^{r+1} + \ldots (-)^s {}_{r+s-1}C_s \, {}_nC_{r+s} p^{r+s} \ldots (-)^{n-r} {}_{n-1}C_{n-r} p^n \quad (5). $$

Here we have yet another expression equivalent to (1) and (2) of § 9. It is not very difficult to transform either of the two expressions of § 9 into the one now found; the details may be left to the reader.

Example. The probabilities of three independent events are p, q, r; required the probability of happening—

1st. Of one of the events but not more;
2nd. Of two but not more;
3rd. Of one at least;
4th. Of two at least;
5th. Of one at most;
6th. Of two at most.

The results are as follows:—

1st. $p+q+r-2(pq+pr+qr)+3pqr$;
2nd. $pq+pr+qr-3pqr$;
3rd. $p+q+r-(pq+pr+qr)+pqr$;
4th. $pq+pr+qr-2pqr$;
5th. $1-(pq+pr+qr)+2pqr$;
6th. $1-pqr$.

The first four are particular cases of preceding formulæ; 5 is complementary to 4; and 6 is complementary to "of all three."

§ 11.] The *Recurrence* or *Finite Difference Method* for solving problems in the theory of probability possesses great historical and practical interest, on account of the use that has been made of it in the solution of some of the most difficult questions in the subject. The spirit of the method may be explained thus.

Suppose, for simplicity, that the required probability is a function of one variable x; and let us denote it by u_x. Reasoning from the data of the problem, we deduce a relation connecting the values of u_x for a number of successive values of x; say the relation

$$f(u_{x+2},\ u_{x+1},\ u_x) = 0 \qquad \text{(A)}.$$

We then discuss the analytical problem of finding a function u_x which will satisfy the equation (A).

It is not by any means necessary to solve the equation (A) completely. Since we know that our problem is definite, all that we require is a form for u_x which will satisfy (A) and at the same time agree with the conditions of the problem in certain particular cases. The following examples will sufficiently illustrate the method from an elementary point of view.

Example 1. A and B play a game in which the probabilities that A and B win are α and β respectively, and the probability that the game be drawn is γ. To start with, A has m and B has n counters. Each time the game is won the winner takes a counter from the loser. If A and B agree to play until one of them loses all his counters, find their respective chances of winning in the end.*

Let u_x and v_x denote the chances that A and B win in the end when each has x counters. If we put $m+n=p$, the respective chances at any stage of the game are u_x and v_{p-x}.

Consider A's chance when he has $x+1$ counters. The next round he may, 1st, win; 2nd, lose; 3rd, draw the game. The chances of his ultimately winning on these hypotheses are αu_{x+2}; βu_x; γu_{x+1} respectively. Hence, by the addition rule,

$$u_{x+1} = \alpha u_{x+2} + \beta u_x + \gamma u_{x+1}.$$

If we notice that $\alpha+\beta+\gamma=1$ (for the game must be either won, lost, or drawn), we deduce from the equation just written

$$\alpha u_{x+2} - (\alpha+\beta) u_{x+1} + \beta u_x = 0 \qquad (1).$$

It is obvious that $u_x = A\lambda^x$, where A and λ are constants, will be a solution of (1), provided

$$\alpha\lambda^2 - (\alpha+\beta)\lambda + \beta = 0 \qquad (2),$$

that is, provided $\lambda=1$ or $\lambda=\beta/\alpha$. Hence $u_x = A$ and $u_x = B(\beta/\alpha)^x$ are both solutions of (1); and it is further obvious that $u_x = A + B(\beta/\alpha)^x$ is a solution of (1).

We have now the means of solving our problem, for it is clear from (1) that, if we knew two particular values of u_x, say u_0 and u_1, then all other

* First proposed by Huyghens in a particular case; and solved by James Bernoulli. See Todhunter, *Hist. Prob.*, p. 61.

values could be calculated by the recurrence formula (1) itself. The solution $u_x = A + B(\beta/\alpha)^x$, containing two undetermined constants A and B, is therefore sufficiently general for our purpose.* We may in fact determine A and B most simply by remarking that when A has none of the counters his chance is 0, and when he has all the counters his chance is 1. We thus have

$$A + B = 0, \quad A + B(\beta/\alpha)^p = 1,$$

whence $$A = \alpha^p/(\alpha^p - \beta^p), \quad B = -\alpha^p/(\alpha^p - \beta^p).$$

We therefore have

$$u_x = \alpha^{p-x}(\alpha^x - \beta^x)/(\alpha^p - \beta^p);$$

and, in like manner,

$$v_x = \beta^{p-x}(\alpha^x - \beta^x)/(\alpha^p - \beta^p).$$

The chances at the beginning of the game are given by

$$u_m = \alpha^n(\alpha^m - \beta^m)/(\alpha^p - \beta^p),$$
$$u_n = \beta^m(\alpha^n - \beta^n)/(\alpha^p - \beta^p).$$

Cor. 1. *If* $\alpha = \beta$, *then* (see chap. xxv., § 12)

$$u_m = m/p, \quad u_n = n/p.$$

The odds on A *in this particular case are* m *to* n.

It might be supposed that when the skill of the players is unequal this could be compensated by a disparity of counters. There is, however, a limit, as the following proposition will show :—

Cor. 2. *The utmost disparity of counters cannot reduce the odds in* A*'s favour to less than* $\alpha - \beta$ *to* β.

For, if we give A 1 counter, and B n counters, the odds in A's favour are $\alpha^n(\alpha - \beta)/\beta(\alpha^n - \beta^n) : 1$; that is, $(\alpha - \beta)/\beta\{1 - (\beta/\alpha)^n\} : 1$. Now, if $\alpha > \beta$, this can be diminished by increasing n ; but, since $\lim_{n=\infty} (\beta/\alpha)^n = 0$, it cannot become less than $(\alpha - \beta)/\beta : 1$, that is, $\alpha - \beta : \beta$.

Hence we see that, if A be twice as skilful as B($\alpha = 2\beta$), we cannot by any disparity of counters (so long as we give him any at all) make the odds in his favour less than even.

Example 2. A pack of n different cards is laid face downwards. A person names a card ; and that card and all above it are removed and shown to him. He then names another ; and so on, until none are left. Required the chance that during the operation he names the top card once at least.†

Let u_n be the chance of succeeding when there are n cards ; so that u_{n-1} is the chance of succeeding when there are $n - 1$; and so on. At the first trial the player may name the 1st, 2nd, 3rd, . . ., or the nth card, the chance of each of these events being $1/n$. Now his chances of ultimately succeeding in the n cases just mentioned are 1, u_{n-2}, u_{n-3}, . . ., u_1, 0 respectively. Hence

$$u_n = 1/n + u_{n-2}/n + u_{n-3}/n + \ldots + u_2/n + u_1/n.$$

We have therefore

$$nu_n = 1 + u_1 + u_2 + \ldots + u_{n-2} \qquad (1).$$

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* This piece of reasoning may be replaced by the considerations of chap. xxxi., § 8.

† *Reprint of Problems from the Ed. Times*, vol. xlii., p. 69.

From (1) we deduce

$$(n-1)u_{n-1}=1+u_1+u_2+\ldots+u_{n-3} \qquad (2).$$

From (1) and (2)

$$nu_n-(n-1)u_{n-1}=u_{n-2},$$

that is,

$$n(u_n-u_{n-1})=-(u_{n-1}-u_{n-2}) \qquad (3).$$

Hence

$$(n-1)(u_{n-1}-u_{n-2})=-(u_{n-2}-u_{n-3}),$$
$$(n-2)(u_{n-2}-u_{n-3})=-(u_{n-3}-u_{n-4}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$3(u_3-u_2)=-(u_2-u_1).$$

Hence, multiplying together the last $n-2$ equations, we deduce

$$\tfrac{1}{2}n\,!(u_n-u_{n-1})=(-1)^{n-2}(u_2-u_1).$$

Since $u_1=1$, $u_2=\frac{1}{2}$, this gives

$$u_n-u_{n-1}=(-1)^{n-1}/n\,! \qquad (4).$$

Hence, again,

$$u_{n-1}-u_{n-2}=(-1)^{n-2}/(n-1)\,!,$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$u_2-u_1=(-1)^1/2\,!,$$
$$u_1-0=1.$$

From the last n equations we derive, by addition,

$$u_n=1-1/2\,!+1/3\,!-\ldots+(-1)^{n-1}/n\,! \qquad (5).$$

Introducing the sub-factorial notation of chap. xxiii., § 18, we may write the result obtained in (5) in the form $u_n=1-n\,|\,/n\,!$.

From Whitworth's Table * we see that the chance when $n=8$ is ·632119. When $n=\infty$ the chance is $1-1/e=$ ·632121; so that the chance does not diminish greatly after the number of cards reaches 8.

EVALUATION OF PROBABILITIES WHERE FACTORIALS OF LARGE NUMBERS ARE INVOLVED.

§ 12.] In many cases, as has been seen, the calculation of probabilities depends on the evaluation of factorial functions. When the numbers involved are large, this evaluation, if pursued directly, would lead to calculations of enormous length,† and the greater part of this labour would be utterly wasted, since all that is required is usually the first few significant figures of the probability. The difficulty which thus arises is evaded by the use of Stirling's Theorem regarding the approximate value of $x\,!$

* *Choice and Chance*, chap. iv.

† In some cases the process of chap. xxxv., § 11, Examples 2 and 3 is useful.

when x is large. In its modern form this theorem may be stated thus—

$$x! = \sqrt{(2\pi x)}x^x e^{-x}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \ldots\right)$$

(see chap. xxx., § 17).

From this it appears that, if x be a large number, $x!$ may be replaced by $\sqrt{(2\pi x)}x^x e^{-x}$, the error thereby committed being of the order $1/12x$th of the value of $x!$.

As an example of the use of Stirling's Theorem, let us consider the following problem :—A pack of $4n$ cards consists of 4 suits, each consisting of n cards. The pack is shuffled and dealt out to four players; required the chance that the whole of a particular suit falls to one particular player. The chance in question is easily found to be given by

$$p = (3n)!\,n!/(4n)!.$$

Hence, by Stirling's Theorem, we have

$$p = \frac{\sqrt{(2\pi 3n)}\,(3n)^{3n}e^{-3n}\sqrt{(2\pi n)}\,n^n e^{-n}}{\sqrt{(2\pi 4n)}\,(4n)^{4n}e^{-4n}},$$

the error being comparable with $1/11n$th of p. Hence, approximately,

$$p = \sqrt{(3\pi n/2)}\,(27/256)^n.$$

Example. Let $4n = 52$, $n = 13$, then

$$p = \sqrt{(3 \times 3{\cdot}1416 \times 13/2)}\,(27/256)^{13}.$$

This can be readily evaluated by means of a table of logarithms. We find

$$p = 156/10^{14}.$$

The event in question is therefore not one that would occur often in the experience of one individual.

Exercises XXXIX.

(1.) A starts at half-past one to walk up Princes Street; what is the probability that he meet B, who may have started to walk down any time between one and two o'clock? Given that it takes A 12 minutes to walk up, and B 10 minutes to walk down.

(2.) A bag contains 3 white, 4 red, and 5 black balls. Three balls are drawn; required the probability—1st, that all three colours; 2nd, that only two colours; 3rd, that only one colour, may be represented.

(3.) A bag contains m white and n black balls. One is drawn and then a second; what is the chance of drawing at least one white—1st, when the first ball is replaced; 2nd, when it is not replaced?

(4.) If n persons meet by chance, what is the probability that they all have the same birthday, supposing every fourth year to be a leap year?

(5.) If a queen and a knight be placed at random on a chess-board, what is the chance that one of the two may be able to take the other?

(6.) Three dice are thrown; show that the cast is most likely to be 10 or 11, the probability of each being $\frac{1}{8}$.

(7.) There are three bags, the first of which contains 1, 2, 1 counters, marked 1, 2, 3 respectively; the second 1, 4, 6, 4, 1, marked 1, 2, 3, 4, 5 respectively; the third 1, 6, 15, 20, marked 1, 2, 3, 4 respectively. A counter is drawn from each bag; what is the probability of drawing 6 exactly, and of drawing some number not exceeding 6?

(8.) Six men are bracketed in an examination, the extreme difference of their marks being 6. Find the chance that their marks are all different.

(9.) From $2n$ tickets marked 0, 1, 2, . . ., $(2n-1)$, 2 are drawn; find the probability that the sum of the numbers is $2n$.

(10.) A pack of 4 suits of 13 cards each is dealt to 4 players. Find the chance—1st, that a particular player has no card of a named suit; 2nd, that there is one suit of which he has no card. Show that the odds against the dealer having all the 13 trumps is 158,753,389,899 to 1.

(11.) If I set down any r-permutation of n letters, what is the chance that two assigned letters be adjacent?

(12.) There are 3 tickets in a bag, marked 1, 2, 3. A ticket is drawn and replaced four times in succession; show that it is 41 to 40 that the sum of the numbers drawn is even.

(13.) What is the most likely throw with n dice, when $n > 6$?

(14.) Out of a pack of n cards a card is drawn and replaced. The operation is repeated until a card has been drawn twice. On an average how many drawings will there be?

(15.) Ten different numbers, each $\not> 100$, are selected at random and multiplied together; find the chance that the product is divisible by 2, 3, 4, 5, 6, 7, 8, 9, 10 respectively.

(16.) A undertakes to throw at least one six in a single throw with six dice; B in the same way to throw at least two sixes with twelve dice; and C to throw at least three sixes with eighteen dice. Which has the best chance of succeeding? (Solved by Newton; see *Pepys' Diary and Correspondence*, ed. by Mynors Bright, vol. vi., p. 179.)

(17.) A pitcher is to be taken to the well every day for 4 years. If the odds be 1000 : 1 against its being broken on any particular day, show that the chance of its ultimately surviving is rather less than $\frac{1}{4}$.

(18.) Five men toss a coin in order till one wins by tossing head; calculate their respective chances of winning.

(19.) A and B, of equal skill, agree to play till one is 5 games ahead. Calculate their respective chances of winning at any stage, supposing that the game cannot be drawn. (Pascal and Fermat.)

(20.) What are the odds against throwing 7 twice at least in 3 throws with 2 dice?

(21.) Show that the chance of throwing doublets with 2 dice, 1 of which is loaded and the other true, is the same as if both were true.

(22.) A and B throw for a stake; A's die is marked 10, 13, 16, 20, 21, 25, and B's 5, 10, 15, 20, 25, 30. The highest throw is to win and equal throws to go for nothing; show that A's chance of winning is 17/33.

(23.) A pack of $2n$ cards, n red, n black, is divided at random into 2 equal parts and a card is drawn from each; find the chance that the 2 drawn are of the same colour, and compare with the chance of drawing 2 of the same colour from the undivided pack.

(24.) $4m$ cards, numbered in 4 sets of m, are distributed into m stacks of 4 each, face up; find the chance that in no stack is a higher one of any set above one with a lower number in the same set.

(25.) Out of m men in a ring 3 are selected at random; show that the chance that no 2 of them are neighbours is

$$(m-4)(m-5)/(m-1)(m-2).$$

(26.) If m things be given to a men and b women, prove that the chance that the number received by the group of men is odd is

$$\{\tfrac{1}{2}(b+a)^m - \tfrac{1}{2}(b-a)^m\}/(b+a)^m.$$

(Math. Trip., 1881.)

(27.) A and B each take 12 counters and play with 3 dice on this condition, that if 11 is thrown A gives a counter to B, and if 14 is thrown B gives a counter to A; and he wins the game who first obtains all the counters. Show that A's chance is to B's as

$$244,140,625 : 282,429,536,481.$$

(Huyghens. See Todh., *Hist. Prob.*, p. 25.)

(28.) A and B play with 2 dice; if 7 is thrown A wins, if 10 B wins, if any other number the game is drawn. Show that A's chance of winning is to B's as 13 : 11. (Huyghens. See Todh., *Hist. Prob.*, p. 23.)

(29.) In a game of mingled chance and skill, which cannot be drawn, the odds are 3 to 1 that any game is decided by skill and not by luck. If A beats B 3 games out of 2, show that the odds are 3 to 1 that he is the better player. If B beats C 2 games out of 3, show that the chance of A's winning 3 games running from C is 103/352.

(30.) There are m posts in a straight line at equal distances of a yard apart. A man starts from any one and walks to any other; prove that the average distance which he will travel after doing this at random a great many times is $\frac{1}{3}(m+1)$ yards.

(31.) The chance of throwing f named faces in n casts with a $\overline{p+1}$-faced die is

$$\left\{(p+1)^n - \frac{f}{1}p^n + \frac{f(f-1)}{2!}(p-1)^n - \ldots\right\}/(p+1)^n.$$

(Demoivre, *Doctrine of Chances.*)

(32.) If n cards be thrown into a bag and drawn out successively, the chance that one card at least is drawn in the order that its number indicates is

$$1 - 1/2! + 1/3! - \ldots (-1)^{n-1}/n!.$$

(This is known as the *Treize Problem.* It was originally solved by Montmort and Bernoulli.)

(33.) A and B play a game in which their respective chances of winning are α and β. They start with a given number of counters p divided between them; each gives up one to the other when he loses; and they play till one is ruined. Show that inequality of counters can be made to compensate for inequality of skill, provided α/β is less than the positive root of the equation $x^p - 2x^{p-1} + 1 = 0$. If p be large, show that, to a second approximation, this root is $2 - \frac{1}{2^{p-1}} - \frac{p-1}{2^{2p-1}}$.

MATHEMATICAL MEASURE OF THE VALUE OF AN EXPECTATION.

§ 13.] If a man were asked what he would pay for the privilege of tossing a halfpenny once and no more, with the understanding that he is to receive £50 if the coin turn up head, and nothing if it turn up tail, he might give various estimates, according as his nature were more or less sanguine, of what is sometimes called the value of his expectation of the sum of £50.

It is obvious, however, that in the case where only one trial is to be allowed the expectation has in reality no definite value whatever—the player may get £50 or he may get nothing; and no more can be said.

If, however, the player be allowed to repeat the game a large number of times on condition of *paying the same sum each time* for his privilege, then it will be seen that £25 is an equitable payment to request from the player; for it is assumed that the game is to be so conducted that, in the long run, the coin will turn up heads and tails equally often; that is to say, that in a very large number of games the player will win about as often as he loses. With the above understanding, we may speak of £25 as the value of the player's expectation of £50; and it will be observed that the value of the expectation is the sum expected multiplied by the probability of getting it.

This idea of the value of an expectation may be more fully illustrated by the case of a lottery. Let us suppose that there are prizes of the value of £a, £b, £c, . . ., the respective probabilities of obtaining which by means of a single ticket are p, q, r, . . . If the lottery were held a large number N of times, the holder of a single ticket would get £a on pN occa-

sions, £b on qN occasions, £c on rN occasions, ... Hence the holder of a single ticket in each of the N lotteries would get £$(pNa + qNb + rNc + \ldots)$. If, therefore, he is to pay the same price £t for his ticket each time, we ought to have, for equity,

$$Nt = pNa + qNb + rNc + \ldots,$$

that is,

$$t = pa + qb + rc + \ldots .$$

Hence the price of his ticket is made up of parts corresponding to the various prizes, namely, pa, qb, rc, . . . These parts are called the *values of the expectations of the respective prizes*; and we have the rule that *the value of the expectation of a sum of money is that sum multiplied by the chance of getting it.*

The student must, however, remember the understanding upon which this definition has been based. It would have no meaning if the lottery were to be held once for all.

Example. A player throws a six-faced die, and is to receive 20s. if he throws ace the first throw; half that sum if he throws ace the second throw; quarter that sum if he throws ace the third throw; and so on. Required the value of his expectation.

The player may get 20, $20/2$, $20/2^2$, $20/2^3$, . . . shillings. His chances of getting these sums are $1/6$, $5/6^2$, $5^2/6^3$, $5^3/6^4$, . . . Hence the respective values of the corresponding parts of his expectation are $20/6$, $20.5/6^2.2$, $20.5^2/6^3.2^2$, $20.5^3/6^4.2^3$, . . . shillings. The whole value of his expectation is therefore

$$\frac{20}{6}\left\{1+\frac{5}{12}+\left(\frac{5}{12}\right)^2+\left(\frac{5}{12}\right)^3+\ldots \text{ ad } \infty\right\} = \frac{20}{6}\Big/\left(1-\frac{5}{12}\right) = \frac{40}{7} \text{ shillings,}$$

that is, 5s. 8½d.

§ 14.] It is important to notice that the rule which directs us to add the component parts of an expectation applies whether the separate contingencies be mutually exclusive or not. *Thus, if p_1, p_2, p_3, . . . be the whole probabilities of obtaining the separate sums a_1, a_2, a_3, . . ., then the value of the expectation is $p_1a_1 + p_2a_2 + p_3a_3 + \ldots$, even if the expectant may get more than one of the sums in question.* Observe, however, that p_1 must be the *whole* probability of getting a_1, that is, the probability of getting the sum a_1 irrespective of getting or failing to get the other sums.

If the expectant may get any number of the sums a_1, a_2,

. . ., a_n, we might calculate his expectation by dividing it into the following mutually exclusive contingencies :—a_1, a_2, . . ., a_n; $a_1 + a_2$, $a_1 + a_3$, &c.; $a_1 + a_2 + a_3$, &c.; . . .; $a_1 + a_2 + \ldots + a_n$. Hence the value of his expectation is

$$\begin{aligned}
&\Sigma a_1 p_1 (1 - p_2)(1 - p_3) \ldots (1 - p_n) \\
&+ \Sigma (a_1 + a_2) p_1 p_2 (1 - p_3) \ldots (1 - p_n) \\
&+ \Sigma (a_1 + a_2 + a_3) p_1 p_2 p_3 (1 - p_4) \ldots (1 - p_n) \\
&\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&+ (a_1 + a_2 + \ldots + a_n) p_1 p_2 p_3 \ldots p_n.
\end{aligned}$$

By the general principle above enunciated the value in question is also $\Sigma a_1 p_1$. The comparison of the values gives a curious algebraic identity, which the student may verify either in general or in particular cases.

Example. A man may get one or other or both of the sums a and b. The chance of getting a is p, and of getting b is q. Required the value of his expectation.

He may get a alone, or b alone, or $a+b$; and the respective chances are $p(1-q)$, $q(1-p)$, pq. Hence the value of his expectation is $ap(1-q)+bq(1-p)+(a+b)pq$, which reduces to $ap+bq$, as it ought to do by the general principle.

N.B.—If the man were to get one or other, but not both of the sums a and b, and his respective chances were p and q, the value of his expectation would still be $ap+bq$; but p and q would no longer have the same meanings as in last case.

LIFE CONTINGENCIES.

§ 15.] The best example of the mathematical theory of the value of expectations is to be found in the valuation of benefits which are contingent upon the duration or termination of one or more human lives. The data required for such calculations are mainly of two kinds—1st, knowledge, or forecast as accurate as may be, of the interest likely to be yielded by investment of capital on good and easily convertible security; 2nd, statistics regarding the average duration of human life, usually embodied in what are called Mortality Tables.

The table printed below illustrates the arrangement of mortality statistics most commonly used in the calculation of life contingencies :—

The H^M Table of the Institute of Actuaries.

Age. x	Number Living. l_x	Decrement. d_x	Age. x	Number Living. l_x	Decrement. d_x	Age. x	Number Living. l_x	Decrement. d_x
10	100,000	490	40	82,284	848	70	38,124	2371
11	99,510	397	41	81,436	854	71	35,753	2433
12	99,113	329	42	80,582	865	72	33,320	2497
13	98,784	288	43	79,717	887	73	30,823	2554
14	98,496	272	44	78,830	911	74	28,269	2578
15	98,224	282	45	77,919	950	75	25,691	2527
16	97,942	318	46	76,969	996	76	23,164	2464
17	97,624	379	47	75,973	1041	77	20,700	2374
18	97,245	466	48	74,932	1082	78	18,326	2258
19	96,779	556	49	73,850	1124	79	16,068	2138
20	96,223	609	50	72,726	1160	80	13,930	2015
21	95,614	643	51	71,566	1193	81	11,915	1883
22	94,971	650	52	70,373	1235	82	10,032	1719
23	94,321	638	53	69,138	1286	83	8,313	1545
24	93,683	622	54	67,852	1339	84	6,768	1346
25	93,061	617	55	66,513	1399	85	5,422	1138
26	92,444	618	56	65,114	1462	86	4,284	941
27	91,826	634	57	63,652	1527	87	3,343	773
28	91,192	654	58	62,125	1592	88	2,570	615
29	90,538	673	59	60,533	1667	89	1,955	495
30	89,865	694	60	58,866	1747	90	1,460	408
31	89,171	706	61	57,119	1830	91	1,052	329
32	88,465	717	62	55,289	1915	92	723	254
33	87,748	727	63	53,374	2001	93	469	195
34	87,021	740	64	51,373	2076	94	274	139
35	86,281	757	65	49,297	2141	95	135	86
36	85,524	779	66	47,156	2196	96	49	40
37	84,745	802	67	44,960	2243	97	9	9
38	83,943	821	68	42,717	2274	98	0	
39	83,122	838	69	40,443	2319			

In the first column are entered the ages 10, 11, 12, . . . Opposite 10 is entered an arbitrary number 100,000 of children that reach their tenth birthday; opposite 11 the number of these that reach their eleventh birthday; opposite 12 the number that reach their twelfth birthday; and so on. We shall denote these numbers by l_{10}, l_{11}, l_{12}, . . . In a third column are entered the differences, or "decrements," of the numbers in the second column; these we shall denote by d_{10}, d_{11}, d_{12}, . . . It is obvious that d_x gives the number out of the 100,000 that die between their xth and $x+1$th birthdays. It is impossible here to discuss the methods employed in constructing a table of mortality, or

to indicate the limits of its use; we merely remark that in applying it in any calculation the assumption made is that the lives dealt with will fall according to the law indicated by the numbers in the table. This law, which we may call the Law of Mortality, is of course only imperfectly indicated by the table itself; for although we are told that d_x die between the ages of x and $x+1$, we are not told how these deaths are distributed throughout the intervening year. For rough purposes it is sufficient to assume that the distribution of deaths throughout each year is uniform; although the variation of the decrements from one part of the table to another shows that uniform decrease * is by no means the general law of mortality.

§ 16.] By means of a Mortality Table a great many interesting problems regarding the duration of life may be solved which do not involve the consideration of money. The following are examples.

Example 1. By the probable duration n of the life of a man of m years of age is meant the number of years which he has an even chance of adding to his life. To find this number.

By hypothesis we have $l_{m+n}/l_m = 1/2$. Hence $l_{m+n} = l_m/2$. $l_m/2$ will in general lie between two numbers in the table, say l_p and l_{p+1}. Hence $m+n$ must lie between p and $p+1$. We can get a closer approximation by the rule of proportional parts (see chap. xxi., § 13).

Example 2. To find the "mean duration" or "expectancy of life" for a man of m years of age.

By this is meant the average N (arithmetical mean) of the number of additional years of life enjoyed by all men of m years of age.

Let us take as specimen lives the l_m men of the table who pass their mth birthday; suppose them all living at a particular epoch; and trace their lives till they all die.

In the first year $l_m - l_{m+1}$ die. If we suppose these deaths to be equally distributed through the year, as many of the $l_m - l_{m+1}$ will live any assigned amount over half a year as will live by the same amount under half a year. Hence the $l_m - l_{m+1}$ lives that have failed will contribute $\frac{1}{2}(l_m - l_{m+1})$ years to the united life of the l_m specimen lives. Again, each of the l_{m+1} who live through the year will contribute one year to the united life. Hence the whole contribution to the united life during the first year is $\frac{1}{2}(l_m - l_{m+1}) + l_{m+1} = \frac{1}{2}(l_m + l_{m+1})$. Similarly, the contribution during the second year is $\frac{1}{2}(l_{m+1} + l_{m+2})$; and so on. Hence the united life is

$$\frac{1}{2}(l_m + l_{m+1}) + \frac{1}{2}(l_{m+1} + l_{m+2}) + \ldots = \frac{1}{2}l_m + l_{m+1} + l_{m+2} + \ldots \quad (1),$$

* Demoivre's hypothesis.

the series continuing so long as the numbers in the table have any significant value.

If we now divide the united life by the number of original lives, we find for the mean duration

$$N = \tfrac{1}{2} + (l_{m+1} + l_{m+2} + \ldots)/l_m \qquad (2).$$

Owing to our assumption regarding the uniform distribution of deaths over the intervals between the tabular epochs, this expression is of course merely an approximation.

Example 3. A and B, whose ages are a and b respectively, are both living at a particular epoch; find the chance that A survive B.

The compound event whose chance is required may be divided into mutually exclusive contingencies as follows:—

1st. B may die in the first year, and A survive;
2nd. ,, second ,, ;

and so on.

The 1st contingency may be again divided into two:—

(α) A and B may both die within the year, B dying first;
(β) B may die within the year, and A live beyond the year.

The chance that A and B both die within the first year is $(l_a - l_{a+1})(l_b - l_{b+1})/l_a l_b$. Since the deaths are equally distributed through the year, if A and B both die during the year, one is as likely to survive as the other; hence the chance of A surviving B on the present hypothesis is $\frac{1}{2}$. The chance of the contingency (α) is therefore $(l_a - l_{a+1})(l_b - l_{b+1})/2l_a l_b$. The chance of ($\beta$) is obviously $l_{a+1}(l_b - l_{b+1})/l_a l_b$.

Hence the whole chance of the 1st contingency, being the sum of the chances of (α) and (β), is $(l_a + l_{a+1})(l_b - l_{b+1})/2l_a l_b$.

In like manner, we can show that the chance of the 2nd contingency is $(l_{a+1} + l_{a+2})(l_{b+1} - l_{b+2})/2l_a l_b$.

Hence the whole chance that A survive B is given by

$$S_{a,b} = \{(l_a + l_{a+1})(l_b - l_{b+1}) + (l_{a+1} + l_{a+2})(l_{b+1} - l_{b+2}) + \ldots\}/2l_a l_b \qquad (1).$$

The reader will have no difficulty in seeing that (1) may be written in the following form, which is more convenient for arithmetical computation:—

$$S_{a,b} = \tfrac{1}{2} + \{\sum_{r=1}^{r=\infty} l_{a+r}(l_{b+r-1} - l_{b+r+1}) - l_a l_{b+1}\}/2l_a l_b \qquad (2),$$

where ∞ stands for the greatest age in the table for which a significant value of l_x is given.

If we denote by $S_{b,a}$ the chance that B survive A, we have, of course, $S_{b,a} = 1 - S_{a,b}$.

If $a = b$, it will be found that (2) gives $S_{a,b} = 1/2$; as it ought to do.

§ 17.] Let us now consider the following money problem in life contingencies:—*What should an Insurance Office ask for undertaking to pay an annuity of* £1 *to a man of m years of age, the first*

payment to be made $n+1$ years hence, the second $n+2$ years hence; and so on, for t years, if the annuitant live so long.*

We suppose that the office makes no charges for the use of the shareholders' capital, for management, and for "margin" to cover the uncertainty of the data of even the best tables of mortality. Allowances on this head are not matters of pure calculation, and differ in different offices, as is well known. We suppose also that the rate of interest on the invested funds of the office is £i per £1, so that the present value, v, of £1 due one year hence is £$1/(1+i)$. The solution of the problem is then a mere matter of average accounting.

Let ${}_{n|t}a_m$ denote the present value of the annuity; and let us suppose that the office sells an annuity of the kind in question † to every one of l_m men of m years of age supposed to be all living at the present date.

The office receives at once ${}_{n|t}a_m l_m$ pounds. On the other hand, it will be called upon to pay

$$\begin{array}{cccc} £l_{m+n+1}, & £l_{m+n+2}, & \ldots, & £l_{m+n+t}, \\ n+1, & n+2, & \ldots, & n+t \end{array}$$

years hence respectively. Reducing all these sums to present value, and balancing outgoings and incomings on account of the l_m lives, we have, by chap. xxii., § 3,

$${}_{n|t}a_m l_m = v^{n+1} l_{m+n+1} + v^{n+2} l_{m+n+2} + \ldots + v^{n+t} l_{m+n+t}.$$

Hence

$$\begin{aligned} {}_{n|t}a_m &= (v^{n+1} l_{m+n+1} + v^{n+2} l_{m+n+2} + \ldots + v^{n+t} l_{m+n+t})/l_m, \\ &= v^n \sum_{r=1}^{r=t} l_{m+n+r} v^r / l_m \qquad (1). \end{aligned}$$

The same result might be arrived at by using the theory of expectation.

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* This is what is meant by saying that the annuity begins to run n years hence.

† The annuity need not necessarily be sold to the person ("nominee") on whose life it is to depend. The life of the nominee merely concerns the definition of the "status" of the annuity, that is, the conditions under which it is to last.

The annuity whose value we have just calculated would be technically described as a *deferred temporary* annuity.

If the annuity be an *immediate* temporary annuity, that is, if it commence to run at once, and continue for t years provided the nominee live so long, we must put $n = 0$. Then, using the actuarial notation, we have

$$ {}_{t|}a_m = \sum_{r=1}^{r=t} l_{m+r} v^r / l_m \qquad (2). $$

If the annuity be *complete*, that is, if it is to run during the whole life of the nominee, the summation must be continued as long as the terms of the series have any significant value; this we may indicate by putting $t = \infty$. Then, according as the annuity is or is not deferred, we have

$$ {}_{n|}a_m = v^n \sum_{r=1}^{r=\infty} l_{m+n+r} v^r / l_m \qquad (3). $$

$$ a_m = \sum_{r=1}^{r=\infty} l_{m+r} v^r / l_m \qquad (4). $$

§ 18.] The function a_m, which gives the value of an immediate complete annuity on a life of m years, is of fundamental importance in the calculation of contingencies which depend on a single life. Its values have been deduced from various tables of mortality, and tabulated. By means of such tables we can readily solve a variety of problems. Thus, for example, ${}_{n|}a_m$, ${}_{t|}a_m$, ${}_{n|t}a_m$ can all be found from the annuity tables; for we have

$$ {}_{n|}a_m = v^n l_{m+n} a_{m+n} / l_m \qquad (5); $$

$$ {}_{t|}a_m = a_m - v^t l_{m+t} a_{m+t} / l_m \qquad (6); $$

$$ {}_{n|t}a_m = (v^n l_{m+n} a_{m+n} - v^{n+t} l_{m+n+t} a_{m+n+t}) / l_m \qquad (7); $$

as the reader may easily verify by means of formulæ (1) to (4).

These results may also be readily established *a priori* by means of the theory of expectation.

§ 19.] Let us next find $a_{k,m}$ *the present value of an immediate complete annuity of* £1 *on the joint lives of two nominees of* k *and* m *years of age respectively.*

The understanding here is that the annuity is to be paid so

long as both nominees are living and to cease when either of them dies.

The present values of the expectations of the 1st, 2nd, 3rd, . . . instalments are

$$vl_{k+1}\,l_{m+1}/l_k\,l_m, \quad v^2 l_{k+2}\,l_{m+2}/l_k\,l_m, \quad v^3 l_{k+3}\,l_{m+3}/l_k\,l_m, \text{ \&c., ...}$$

Hence we have

$$\begin{aligned} a_{k,m} &= (vl_{k+1}\,l_{m+1} + v^2 l_{k+2}\,l_{m+2} + \ldots)/l_k\,l_m, \\ &= \sum_{r=1}^{r=\infty} v^r l_{k+r}\,l_{m+r}/l_k\,l_m \qquad (1). \end{aligned}$$

Just as in § 18, we obviously have

$$\begin{aligned} {}_n|a_{k,m} &= v^n a_{k+n,m+n}\,l_{k+n}\,l_{m+n}/l_k\,l_m; \\ |_t a_{k,m} &= a_{k,m} - v^t a_{k+t,m+t}\,l_{k+t}\,l_{m+t}/l_k\,l_m; \\ {}_n|_t a_{k,m} &= (v^n a_{k+n,m+n}\,l_{k+n}\,l_{m+n} \\ &\qquad - v^{n+t} a_{k+n+t,m+n+t}\,l_{k+n+t}\,l_{m+n+t})/l_k\,l_m; \end{aligned}$$

and it will now be obvious that all these formulæ can be easily extended to the case of an annuity on the joint lives of any number of nominees.

Tables for $a_{k,m}$ have been calculated; and, by combining them with tables for a_m, a large number of problems can be solved.

Example 1. To find the present value of an immediate annuity on the last survivor of two lives m and n, usually denoted by $a_{\overline{m,n}}$.

Let p_r, q_r be the probabilities that the nominees are living r years after the present date; then the probability that one at least is living r years hereafter is $p_r + q_r - p_r q_r$.

Hence

$$\begin{aligned} a_{\overline{m,n}} &= \sum_1^\infty v^r(p_r + q_r - p_r q_r), \\ &= \Sigma v^r p_r + \Sigma v^r q_r - \Sigma v^r p_r q_r, \\ &= a_m + a_n - a_{m,n}. \end{aligned}$$

This is also obvious from the consideration that, if we paid an annuity on each of the lives, we should pay £1 too much for every year that both lives were in existence.

Example 2. Find the present value $a_{\overline{k,m,n}}$ of an annuity to be paid so long as any one of three nominees shall be alive, the respective ages being k, m, n.

If p_s, q_s, r_s be the chances that the respective nominees be alive after s years, then

$$\begin{aligned} a_{\overline{k,m,n}} &= \Sigma v^s\{1 - (1-p_s)(1-q_s)(1-r_s)\}, \\ &= \Sigma v^s(p_s + q_s + r_s - q_s r_s - r_s p_s - p_s q_s + p_s q_s r_s), \\ &= a_k + a_m + a_n - a_{m,n} - a_{n,k} - a_{k,m} + a_{k,m,n}. \end{aligned}$$

The numerical solution of this problem would require a table of annuities on three joint lives, or some other means of calculating $a_{k,m,n}$.

§ 20.] A contract of life insurance is of the following nature:—A man A agrees to make certain payments to an insurance office, on condition that the office pay at some stated time after his death a certain sum to his heirs. As regards A, he enters into the contract knowing that he may pay less or more than the value of what his heirs ultimately receive according as he lives less or more than the average of human life; *his* advantage is that he makes the provision for his heirs a *certainty*, so far as his life is concerned, instead of a contingency. As regards the office, it is their business to see that the charge made for A's insurance is such that they shall not ultimately lose if they enter into a large number of contracts of the kind made with A; but, on the contrary, earn a certain percentage to cover expenses of management, interest on shareholders' capital, &c.

The usual form of problem is as follows:—

What annual premium P_m *must a man of m years of age pay (in advance) during all the years of his life, on condition that the office shall pay the sum of* £1 *to his heirs at the end of the year in which he dies?*

P_m is to be the "net premium," that is, we suppose no allowance made for profit, &c., to the office. Suppose that the office insures l_m lives of m years, and let us trace the incomings and outgoings on account of these lives alone. The office receives in premiums $£P_m l_m$, $£P_m l_{m+1}$, . . . at the beginning of the 1st, 2nd, . . . years respectively. It pays out on lives failed $£(l_m - l_{m+1})$, $£(l_{m+1} - l_{m+2})$, . . . at the end of the 1st, 2nd, . . . years respectively. Hence, to balance the account, we must have, when all these sums are reduced to present value,

$$P_m(l_m + l_{m+1}v + l_{m+2}v^2 + \ldots)$$
$$= (l_m - l_{m+1})v + (l_{m+1} - l_{m+2})v^2 + (l_{m+2} - l_{m+3})v^3 + \ldots \quad (1),$$

the summation to be continued as long as the table gives significant values of l_x.

Since $d_m = l_m - l_{m+1}$, we deduce from (1)

$$P_m = \frac{d_m v + d_{m+1}v^2 + d_{m+2}v^3 + \ldots}{l_m + l_{m+1}v + l_{m+2}v^2 + \ldots} \quad (2).$$

Dividing by l_m, we deduce from (1)

$$P_m\{1 + (l_{m+1}v + l_{m+2}v^2 + l_{m+3}v^3 + \ldots)/l_m\} = v + v(l_{m+1}v + l_{m+2}v^2 + \ldots)/l_m - (l_{m+1}v + l_{m+2}v^2 + \ldots)/l_m.$$

Hence

$$P_m(1 + a_m) = v + va_m - a_m,$$
$$P_m = v - a_m/(1 + a_m) \qquad (3).$$

The last equation shows that the premium for a given life can be deduced from the present value of an immediate complete annuity on the same life. In other words, life insurance premiums can be calculated by means of a table of life annuities.

§ 21.] It is not necessary to enter further here into the details of actuarial calculations; but the mathematical student will find it useful to take a glance at two methods which are in use for calulating annuities and life insurances. They are good specimens of methods for dealing with a mass of statistical information.

Recurrence Method for Calculating Life Annuities.

The reader will have no difficulty in showing, by means of the formulæ of § 17, that

$$a_m = v(1 + a_{m+1})l_{m+1}/l_m \qquad (1).$$

From this it follows that we can calculate the present value of an annuity on a life of m years from the present value on a life of $m + 1$ years. We might therefore begin at the bottom of the table of mortality, calculate backwards step by step, and thus gradually construct a life annuity table, without using the complicated formula (4) of § 17 for each step.

A similar process could be employed to calculate a table for two joint lives differing by a given amount.

Columnar or Commutation Method.

Let us construct a table as follows:—

In the 1st column tabulate l_x;
„ 2nd „ d_x;
„ 3rd „ $v^x l_x = D_x$, say;
„ 4th „ $v^{x+1} d_x = C_x$, say.

Next form the 5th column by adding the numbers in the 3rd column from the bottom upwards. In other words, tabulate in the 5th column the values of

$$N_x = D_{x+1} + D_{x+2} + D_{x+3} + \ldots .$$

In like manner, in the 6th column tabulate

$$M_x = C_x + C_{x+1} + C_{x+2} + \ldots .$$

All this can be done systematically, the main part of the labour being the multiplications in calculating D_x and C_x.

From a table of this kind we can calculate annuities and life premiums with great ease. Referring to the formulæ above, the reader will see that we have

$$a_m = N_m/D_m \quad (2);$$
$${}_n|a_m = N_{m+n}/D_m \quad (3);$$
$${}_{|t}a_m = (N_m - N_{m+t})/D_m \quad (4);$$
$${}_n|_t a_m = (N_{m+n} - N_{m+n+t})/D_m \quad (5);$$
$$P_m = M_m/N_{m-1} \quad (6).$$

§ 22.] In the foregoing chapter the object has been to illustrate as many as possible of the elementary mathematical methods that have been used in the Calculus of Probabilities; and at the same time to indicate practical applications of the theory.

All matter of debatable character or of doubtful utility has been excluded. Under this head fall, in our opinion, the theory of *a priori* or inverse probability, and the applications to the theory of evidence. The very meaning of some of the propositions usually stated in parts of these theories seems to us to be doubtful. Notwithstanding the weighty support of La Place, Poisson, De Morgan, and others, we think that many of the criticisms of Mr. Venn on this part of the doctrine of chances are unanswerable. The mildest judgment we could pronounce would be the following words of De Morgan himself, who seems, after all, to have "doubted":—"My own impression, derived from this [a point in the theory of errors] and many other circumstances connected with the analysis of probabilities, is, that mathematical results have outrun their interpretation."*

* "An Essay on Probabilities and on their Application to Life Contingencies and Insurance Offices" (De Morgan), *Cabinet Cyclopædia*, App., p. xxvi.

The reader who wishes for further information should consult the elementary works of De Morgan (just quoted) and of Whitworth (*Choice and Chance*); also the following, of a more advanced character:—Laurent, *Traité du Calcul des Probabilités*, (Paris, 1873); Meyer, *Vorlesungen über Wahrscheinlichkeitsrechnung* (Leipzig, 1879); Articles, "Annuities," "Insurance," "Probabilities," *Encyclopædia Britannica*, 9th edition.

The classical works on the subject are Montmort's *Essai d'Analyse sur les Jeux de Hazards*, 1708, 1714; James Bernoulli's *Ars Conjectandi*, 1713; Demoivre's *Doctrine of Chances*, 1718, 1738, 1756; Laplace's *Théorie Analytique des Probabilités*, 1812, 1820; and Todhunter's *History of the Theory of Probability*, 1865. The work last mentioned is a mine of information on all parts of the subject; a perusal of the preface alone will give the reader a better idea of the historical development of the subject than any note that could be inserted here. Suffice it to say that few branches of mathematics have engaged the attention of so many distinguished cultivators, and few have been so fruitful of novel analytical processes, as the theory of probability.

Exercises XL.

(1.) A bag contains 4 shillings and 4 sovereigns. Three coins are drawn; find the value of the expectation.

(2.) A bag contains 3 sovereigns and 9 shillings. A man has the option, 1st, of drawing 2 coins at once, or, 2nd, of drawing first one coin and afterwards another, provided the first be a shilling. Which had he better do?

(3.) One bag contains 10 sovereigns, another 10 shillings. One is taken out of each and placed in the other. This is done twice; find the probable value of the contents of each bag thereafter.

(4.) A player throws n coins and takes all that turn up head; all that do not turn up head he throws up again, and takes all the heads as before; and so on r times. Find the value of his expectation; and the chance that all will have turned up head in r throws at most. (St. John's Coll., Camb., 1870.)

(5.) Two men throw for a guinea, equal throws to divide the stake. A uses an ordinary die, but B, when his turn comes, uses a die marked 2, 3, 4, 5, 6, 6; show that B thereby increases the value of his expectation by 5/18ths.

(6.) The *Jeu des Noyaux* was played with 8 discs, black on one side and

white on the other. A stake S was named. The discs were tossed up by the player; if the number of blacks turned up was odd the player won S, if all were blacks or all whites he won 2S, otherwise he lost S to his opponent. Show that the expectations of the player and opponent are 131S/256 and 125S/256 respectively. (Montmort. See Todh., *Hist. Prob.*, p. 95.)

(7.) A promises to give B a shilling if he throws 6 at the first throw with 2 dice, 2 shillings if he throws 6 at the second throw, and so on, until a 6 is thrown. Calculate the value of B's expectation.

(8.) A man is allowed one throw with 2 ordinary dice and is to gain a number of shillings equal to the greater of the two numbers thrown; what ought he to pay for each throw? Generalise the result by supposing that each die has n faces.

(9.) A bag contains a certain number of balls, some of which are white. I am to get a shilling for every ball so long as I continue to draw white only (the balls drawn not being replaced). But an additional ball not white having been introduced, I claim as a compensation to be allowed to replace every white ball I draw. Show that this is fair.

(10.) A person throws up a coin n times; for every sequence of m ($m \neq n$) heads or m tails he is to receive $2^m - 1$ shillings; prove that the value of his expectation is $n(n+3)/4$ shillings.

(11.) A manufacturer has n sewing machines, each requiring one worker, and each yielding every day it works q times the worker's wages as net profit. The machines are never all in working order at once; and it is equally likely that 1, 2, 3, . . ., or any number of them, are out of repair. The worker's wages must be paid whether there is a machine for him or not. Prove that the most profitable number of workers to engage permanently is the integer next to $nq/(q+1) - \frac{1}{2}$. (Math. Trip., 1875.)

(12.) A blackleg bets £5 to £4, £7 to £6, £9 to £5 against horses whose chances of winning are $\frac{5}{9}$, $\frac{1}{3}$, $\frac{1}{7}$ respectively. Calculate the most and the least that he can win, and the value of his expectation.

(13.) The odds against n horses which start for a race are $a:1$; $a+1:1$; . . ., $a+n-1:1$. Show that it is possible for a bookmaker, by properly laying bets of different amounts, to make certain to win if $n > (a+1)(e+1)$, and impossible if $n < a(e-1)$, where e is the Napierian base.

(14.) If A_p denote the value of an annuity to last during the joint lives of p persons of the same age, prove that the value of an equal annuity, to continue so long as there is a survivor out of n persons of that age, may be found by means of the formula

$$nA_1 - \frac{n(n-1)}{2!}A_2 + \frac{n(n-1)(n-2)}{3!}A_3 - \ldots \pm A_n.$$

(15.) M is a number of married couples, the husbands being m years of age, the wives n years of age. What is the number of living pairs, widows, widowers, and dead pairs after t years?

Work out the case where $M = 500$, $m = 40$, $n = 30$.

(16.) If $S_{a,b}$ have the meaning of § 16, show that

$$2l_a l_b S_{a,b} - 2l_{a+1} l_{b+1} S_{a+1,b+1} = (l_a + l_{a+1})(l_b - l_{b+1}).$$

(17.) Find the probability that a man of 80 survive one or other of two men of 90 and 95 respectively.

(18.) If $a_{l,m,n,\ldots}$ denote the present value of an immediate complete annuity of £1 on the joint lives of a set of men of l, m, n, . . . years of age respectively, show that the present value of an immediate annuity of £1 which is to continue so long as there is a survivor out of k men whose ages are l, m, n, . . . respectively is

$$\Sigma a_l - \Sigma a_{l,m} + \Sigma a_{l,m,n} - \ldots .$$

(19.) What annual premium must a married couple of ages m and n respectively pay in order that the survivor of them may enjoy an annuity of £1 when the other dies?

(20.) Calculate the annual premium to insure a sum to be paid n years hence, or on the death of the nominee, if he dies within that time.

(21.) Show how to calculate the annual premium for insuring a sum which diminishes in arithmetical progression as the life of the nominee lengthens.

(22.) An annuity, payable so long as either A (m years of age) or B (n years of age) survive C (p years of age), is to be divided equally between A and B so long as both are alive, and is to go to the survivor when one of them dies. Show that the present values of the interests of A and B are

$$a_m - \tfrac{1}{2}a_{m,n} - a_{m,p} + \tfrac{1}{2}a_{m,n,p}$$

and

$$a_n - \tfrac{1}{2}a_{m,n} - a_{n,p} + \tfrac{1}{2}a_{m,n,p}$$

respectively.

(23.) If the population increase in a geometrical progression whose ratio is r, show that the proportion of men of n years of age in any large number of the community taken at random is $(l_n/r^n)/\sum_0^\infty (l_n/r^n)$.

RESULTS OF EXERCISES.

I.

(1.) 504000. (2.) 1210809600. (3.) 720. (4.) 12. (5.) 6. (8.) 5040; 64864800. (9.) 1235520. (10.) 6188; 3003; 3185. (11.) 408408; 18 ways of setting together on the front, 10 ways of setting at equal distances all round. (12.) $({}_{19}C_4\,{}_{12}C_4 + {}_{18}C_4\,{}_{12}C_3\,{}_9C_1 + {}_{17}C_4\,{}_{12}C_2\,{}_9C_2 + {}_{16}C_4\,{}_{12}C_1\,{}_9C_3 + {}_{15}C_4\,{}_9C_4)_4P_4^2$. (13.) ${}_{10}C_2\,{}_{20}C_5\,{}_{30}C_{10}\,{}_{60}C_{20}$. (14.) 172800. (15.) 267148. (16.) 1814400, if clock and counter-clock order be not distinguished. (17.) $2(2n^2-3n+2)(2n-2)!$. (18.) 960. (19.) ${}_9C_4\,{}_7C_3\,{}_7P_7$; ${}_9C_4\,{}_7C_3\,{}_4^1P_4\,{}_3P_3$. (20.) $52!/(13!)^4$; $39!/(13!)^3$. (21.) $32!/(12!)^2 8!$. (22.) $64!/(2!)^6(8!)^2 32!$. (23.) 26; 136. (24.) 286; 84. (25.) $(p+q)!/p!q!$; $(p+qr)/p!(qr)!$; a little over 6 years.

II.

(1.) $448266240x^2$. (2.) -2093. (3.) $2^n.1.3\ldots(2n-1)/n!$. (4.) $(-)^{n+r}(2n)!/(n+r)!(n-r)!$. (5.) $2^{2n}.1.3\ldots(4n-1)/(2n)!$. (6.) If n be even, the middle term is $\{n!/(\frac{1}{2}n)!\}x^{n/2}$; if n be odd, the two middle terms are $\{n!/\frac{1}{2}(n-1)!\frac{1}{2}(n+1)!\}\{2x^{(n-1)/2} + \frac{1}{2}x^{(n+1)/2}\}$. (11.) $(2\sqrt{3}+3)^{2m} + (2\sqrt{3}-3)^{2m} - 1$; $(2\sqrt{3}+3)^{2m+1} - (2\sqrt{3}-3)^{2m+1}$. (15.) $\frac{1}{2}n(n+1)$. (16.) $2^{n-1}(2+n)$. (27.) $r+1$. (28.) 10. (29.) $\frac{1}{3}(n^3+11n)$. (32.) 190274064. (33.) $\Sigma a^7 + 7\Sigma a^6 b + 21\Sigma a^5 b^2 + 42\Sigma a^5 bc + 35\Sigma a^4 b^3 + 105\Sigma a^4 b^2 c + 210\Sigma a^4 bcd + 140\Sigma a^3 b^3 c + 210\Sigma a^3 b^3 c^2 + 420\Sigma a^3 b^2 cd + 630\Sigma a^2 b^2 c^2 d$. (37.) $23!/(4!)^5 5^3$.

III.

(1.) 944. (2.) 20. (3.) $(n+1)(n+2)(n+3)(n+4)(n+5)/5!$ if the separate numbers thrown be attended to; $5n+1$ if the sum of the numbers thrown be alone attended to. (4.) 231. (6.) ${}_{p+1}C_n$. (7.) 62. (8.) ${}_{15}C_6$. (11.) $(2n)!/2^n n!$. (13.) $(N+a+b+c-3)!/a!b!c!$. (15.) 1 or 0 according as n is even or odd; $\{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\}/2^{n+1}\sqrt{5}$. (17.) ${}_{2m-1}C_{r-1}\,{}_{n-1}C_{r-1}$. (18.) 116280.

V.

(1.) x/y must not lie between 1 and b^2/a^2. (2.) x must lie between $\frac{1}{2}(7-\sqrt{53})$ and $\frac{1}{2}(7+\sqrt{53})$. (3.) x between $(dc-b^2)/(ad-bc)$ and $(d^2-ab)/(ad-bc)$, and y between $(ab-c^2)/(ad-bc)$ and $(a^2-cd)/(ad-bc)$. (15.) Greater. (17.) Less. (39.) 3^{13}.

VI.

(1.) $3abc$. (2.) $abc/3\sqrt{3}$. (4.) $d^m/3^{m-1}$ is a minimum value if m do not lie between 0 and 1, otherwise a maximum. (5.) Minimum when $apx^p = bqy^q = crz^r$. (7.) There is a maximum or minimum when $(x+l)\log a = (y+m)\log b = (z+n)\log c$, according as $\log a \log b \log c$ is positive or negative. (8.) $x = (nb/ma)^{1/(m+n)}$. (9.) $x=1$, $x=38/15$ give maxima; $x=2$, $x=3$ minima. (10.) $\frac{1}{4}abc$. (11.) Minimum when $x = mc/(m-n)$, $y = nc/(m-n)$. (15.) Minimum $2\sqrt{(ab)}/(a+b)$.

VII.

(1.) 3, ∞. (2.) 9/4. (3.) $\log 13/7$. (4.) $\frac{1}{2}n(n+1)$. (5.) 0. (6.) $a^{m+n-p-q}m/p$. (7.) $a^{m-n}m/n$. (8.) nq, ∞, mp according as $p \gtrless q$. (9.) $(m^2 - mn + n^2)/(m^2 + mn + n^2)$. (10.) $1/2a$. (11.) $a^{(q-p)/pq}q/p$. (12.) $a^{\frac{1}{2}}$. (13.) $16a/9$. (14.) 1. (15.) p. (16.) $\frac{1}{2}n(n-1)z^{n-2}$. (17.) $a^{m+n-p-q}q^2m(m-n)/n^2p(p-q)$. (18.) $(n-1)/2a$. (19.) $\log a$. (20.) 1. (21.) 1. (22.) 1. (23.) 1. (24.) ∞. (25.) ∞ if $x = 1+0$, 0 if $x = 1-0$. (26.) $e^{\frac{1}{2}}$. (27.) 0 if n be negative, if n be positive 0 or ∞ according as $a \gtrless 1$. (28.) 1. (29.) 1. (30.) 0 or ∞ according as $m \gtrless n$. (31.) ∞ or 0 according as $a \gtrless 1$. (32.) 1. (33.) $e^{\frac{1}{2}}$. (34.) ∞. (35.) $\sqrt{(ab)}$. (36.) $\mathrm{Exp}(2\sqrt[3]{3})$. (37.) ∞ or 0 according as $\lambda_1(a_{r-1} - b_{r-1})$ is positive or negative. (38.) 1. (39.) 0. (40.) 1. (41.) 2. (42.) 1. (43.) 1. (44.) 1. (45.) $\frac{1}{4}\pi$. (46.) 0. (47.) $\frac{1}{4}\cos a$. (48.) 0. (49.) -2. (50.) 1. (51.) $\frac{1}{2}$. (52.) 1. (53.) 1. (54.) 0. (55.) 0. (56.) 1. (57.) $\log m/\log n$. (58.) 1. (59.) 1. (60.) 1. (61.) 1. (62.) e^{-2m^2n}. (63.) e^{-2m^2/n^2}. (64.) $e^{2/\pi}$. (65.) $2/\pi$. (74.) See chap. xxx., § 23.

VIII.

(1.) Div. (2.) Div. (3.) Conv. if x be positive. (4.) Conv. (5.) Div. (6.) Div. if $\mathrm{mod}\, x \geqslant a$; conv. if $\mathrm{mod}\, x > a$. (7.) Conv. if $x < 4$; div. if $x < 4$. (8.) Conv. (9.) Div., $(x<1)$. (10.) Conv. (11.) Div. (12.) Conv. if $a > 1$; div. if $a \geqslant 1$. (13.) Div. (14.) Div. (15.) Abs. conv. (16.) Div.

IX.

(1.) $(-)^{r-2}3.1.1.3 \ldots (2r-5)/2.4.6.8 \ldots 2r$. (2.) $1.3 \ldots (2r-1)/2.4 \ldots 2r$. (3.) $3.7.11 \ldots (4r-1)/4.8.12 \ldots 4r$. (4.) $2.1.4.7 \ldots (3r-5)2^{2\,3}/12.24.36.48 \ldots 12r$. (5.) $(-)^{r-1}1.2 \ldots (3r-4)a^{1\,3-r}/r!$. (6.) $-1.2.5 \ldots (3r-4)a^{1-3r}/3.6.9 \ldots 3r$. (7.) $-(n-1)(2n-1) \ldots (nr-n-1)/r!$. (8.) $1.4.7 \ldots (3r/2-2)/(r/2)!$ if r be even; 0 if r be odd. (9.) $(-)^n n(n+1) \ldots (n+\frac{1}{3}(r-n)-1)/\{\frac{1}{3}(r-n)\}!$. (10.) $1 + \frac{3}{3}(x/a) + \frac{3}{6}(x/a)^2 + \frac{3\,2}{6\,1}(x/a)^3$. (11.) The first. (12.) The third. (13.) The fourth and fifth. (14.) The eighth. (15.) If $n=1$, the 2nd and 3rd; if $n=2$, the 2nd; if $n \prec 3$, the 1st. (19.) If $m=0$, $S=a$; if $m=1$, $S=b$; if $n>1$, $S=0$: if $n<1$ $(\neq 0)$ the series is divergent. (22.) $1 - \sqrt[3]{2}$. (23.) If $m \prec 1$, $S = m(m-1)2^{m-2}$; if $m=0$, $S=0$.

X.

(1.) $\Sigma 1/a^r(c-a)(a-b)$. (2.) 0. (3.) $\Sigma 1/a^{r-m-2}/(c-a)(a-b)$. (4.) $2r+1+1/2^{r+1}$. (5.) r, if r be even ; $r-1$, if $r=4t+1$; $r+1$, if $r=4t-1$. (6.) ${}_nH_r q^r - {}_mC_1 \cdot {}_mH_{r-1}\, pq^{r-1} + {}_mC_2 \cdot {}_nH_{r-2}\, p^2q^{r-2} + \ldots$ (15.) $\frac{1}{6}(n+1)(n+2)(n+3)$. (19.) $1-1.3\ldots(2n-1)/2^n n!$. (20.) $7.10\ldots(3n+1)/3.6\ldots(3n-3)$.

XI.

(2.) 275/128. (3.) 869699/256. (4.) 48 ; 0. (5.) 11989305/2048. (6.) $(-)^r\{(r-1)+(r+5)/2^{r+2}\}$. (10.) 1·0001005084 ; 1·0004000805. (11.) $2mx$. (12.) $1+2x(1-r^n)/(1-r)$. (13.) $1+(-)^{n-1}x/2^n$.

XII.

(1.) ·367879. (2.) ·04165. (5.) $(1-x)^2e^x$. (6.) $3(e-1)$. (7.) $e+1$. (8.) $1/e$. (9.) $15e$.

XIII.

(4.) 917. (5.) $2\log\{(x-1)/(x+1)\}+\log\{(x+2)/(x-2)\}$. (6.) $\log(12e)$. (7.) $(1+1/x)\log(1+x)-1$. (8.) $\frac{1}{4}(x-x^{-1})\log\{(1+x)/(1-x)\}+\frac{1}{2}$. (9.) When $x=1$ the sum is $18-24\log 2$. (10.) $\frac{3}{4}$. (12.) $\Sigma\{x^{3n-2}/(3n-2)+x^{3n-1}/(3n-1)-2x^{3n}/3n\}$.

XXV.

(1.) $\frac{1}{2}n(n+1)+\frac{1}{6}(r-2)n(n+1)(n-1)$. (2.) $\frac{1}{4}n(n+1)(n+4)(n+5)$. (3.) $3/4-1/2n-1/2(n+1)$. (4.) $1/15-1/5(5n+3)$. (5.) $1/12-1/4(2n+1)(2n+3)$. (6.) $1/18-1/3(n+1)(n+2)(n+3)$. (7.) $a/2+b/4-a/(n+2)-b/2(n+1)(n+2)$. (8.) $1/8-(4n+3)/8(2n+1)(2n+3)$. (9.) $7/36-(3n+7)/(n+1)(n+2)(n+3)$. (10.) $11/180-(6n+11)/12(2n+1)(2n+3)(2n+5)$. (11.) $3/4+n-(2n+3)/2(n+1)(n+2)$. (12.) $u_n=(n+1)^3(n+3)(n+5)/n(n+1)\ldots(n+6)$; apply § 3, Example 4. (13.) $\sin\theta\sec(n+1)\theta\sec\theta$. (14.) $\cot(\theta/2^n)/2^n-\cot\theta$. (15.) $\tan^{-1}na^n$. (16.) $\tan^{-1}1+\tan^{-1}1/2-\tan^{-1}1/n-\tan^{-1}1/(n+1)$. (17.) $(m+n)!/(m+1)(n-1)!$. (18.) $\{1/(m-1)!-(n+1)!/(m+n-1)!\}/(m-2)$. (19.) $(-)^n{}_{m-1}C_n$. (21.) $\{m-1-(n+1)!/m^{\lfloor n\rfloor}\}/(m-2)$. (22.) $(a^{\lfloor n+r\rfloor}/c^{\lfloor n\rfloor}-a^{\lfloor r+1\rfloor})/(a-c+r+1)$. (23.) $(a^{\lfloor n+2\rfloor}/c^{\lfloor n+r+1\rfloor}-a/c^{\lfloor r\rfloor})/(a-c-r+1)$. (24.) $\{(a-1)^{\lfloor m-1\rfloor}/c^{\lfloor m-1\rfloor}-(a+n)^{\lfloor m-1\rfloor}/(c+n+1)^{\lfloor m-1\rfloor}\}/(m-1)(a-c-1)$. (25.) Deduce from (24). (26.) Deduce from (24). (27.) $2m\{1-(-)^n2^n(m-1)(m-2)\ldots(m-n)/1.3\ldots(2n-1)\}/(2m-1)$.

XXVI.

(1.) $2^{n+1}+\frac{1}{2}(3^{n+1}-3)$. (2.) $\frac{7}{2}\{1+(-1)^n\}+6-3\{i^{n+1}+(-i)^{n+1}\}-\frac{1}{2}i\{i^n-(-i)^n\}$. (3.) $11\{1-(4x)^{n+1}\}/\{1-4x\}-9\{1-(3x)^{n+1}\}/\{1-3x\}$; $(2+3x)/(1-7x+12x^2)$, $x<\frac{1}{4}$. (4.) $3\{1-(2x)^{n+1}\}/\{1-2x\}+2\{1-(3x)^{n+1}\}/\{1-3x\}$; $(5-13x)/(1-5x+6x^2)$, $x<\frac{1}{3}$. (5.) $\frac{1}{2}\{1-(3x)^{n+1}\}/\{(1-3x)+\frac{1}{2}\{1-(5x)^{n+1}\}/\{1-5x\}$; $(1-4x)/(1-8x+15x^2)$, $x<\frac{1}{4}$. (6.) $3\{1-(2x)^{n+1}\}/\{1-2x\}-2\{1-x^{n+1}\}/\{1-x\}$; $(1+x)/(1-3x+2x^2)$, $x<\frac{1}{2}$.

XXVII.

(1.) $(1+2x^3)/(1-x^3)^2$. (2.) $-[\log\{(1-x)/(1+x+x^2)\} - \sqrt{3}\tan^{-1}\{\sqrt{3}x/(2+x)\}]/3x$; $\frac{1}{3}\{e^x + 2e^{-x/2}\cos(\sqrt{3}x/2)\}$. (4.) $\frac{1}{3}[e^{-x} + e^{x/2}\{\cos(\sqrt{3}x/2) + \sqrt{3}\sin(\sqrt{3}x/2)\}]$. (5.) $\frac{1}{3}(2^m + 2\cos.m\pi/3)$; $\frac{2}{3}3^{m/2}\cos.m\pi/6$. (6.) $1/2 - 1/(n+2)!$. (7.) $\{2^{m+3} - 1 - (m+3)(m+4)/2\}/(m+1)(m+2)(m+3)$. (8.) $1/(1+x) - \log(1+x)$. (9.) $\frac{1}{2}\cos\theta - \frac{1}{4}\cos 2\theta$. (10.) $1 - (2n+3)/(n+2)^2$. (11.) $2 - 4\log 2$. (14.) $\sin m\pi/m\pi$; $\cosh m\pi$.

XXVIII.

The partial quotients are as follows:—

(1.) 0, 4, 1, 6, 2. (2.) 0, 2, 4, 8, 16. (3.) 1, 15, 1, 1, 1, 3, 1, 14, 1, 1, 5. (4.) 31, 1, 1, 1, 1, 1, 1, 1, 1, 3. (5.) 2, 1, 2, 1, 1, 4, 1, 1, 6, 3, 12, 3, 5, 1, 2. (6.) 0, 126, 1, 1, 2, 1, 1, 6. (7.) 1, $\dot{2}$. (8.) 2, $\dot{4}$. (9.) 3, $\dot{3}$, $\dot{6}$. (10.) 3, $\dot{6}$. (11.) 3, $\dot{2}$, $\dot{6}$. (12.) 1, $\dot{4}$, $\dot{2}$. (13.) 2, $\dot{1}$, $\dot{2}$. (14.) 3, $\dot{1}$, $\dot{5}$. (16.) 0, 2, $\dot{1}$; 0, $\dot{1}$. (17.) a, $\dot{2}$, $2\dot{a}$; $a-1$, $\dot{2}$, $2(\dot{a}-1)$.

XXIX.

(1.) The 1st, 2nd, 3rd, . . . convergents are 1, 2/3, 9/13, 20/29, 29/42, 78/113, . . .: the errors corresponding less than 1/3, 1/39, 1/377, 1/1218, 1/4746, 1/17515, . . . (2.) 972/1393. (3.) 2177/528. (4.) Transits at the same node will occur 8, 243, . . . years after: after 8 years Venus will be less than 1°·5 from the node. (5.) Transits at the same node will occur 13, 33, . . . years after.

XXXI.

(1.) 10, $\dot{2}\dot{0}$;
0, $\dot{1}\dot{0}$,
$\dot{1}$.

(2.) 0, 1, $1\dot{2}\dot{6}$, $\dot{2}$;
0, 0, $\dot{6}\dot{3}$, $\dot{6}\dot{3}$;
64, $\dot{6}\dot{3}$, $\dot{1}$.

(3.) 1, $\dot{5}$, 3, 1, 8, 1, 3, 5, $\dot{2}$;
0, $\dot{1}\dot{2}$, 13, 8, 12, 12, 8, 13, $\dot{1}\dot{2}$;
$\dot{1}\dot{2}$, 5, 7, 20, 3, 20, 7, $\dot{5}$.

(4.) 0, 7, $\dot{1}$, 4, 3, 1, 2, 2, 1, 3, 4, 1, $\dot{1}\dot{4}$;
0, 0, $\dot{7}$, 5, 7, 5, 4, 6, 4, 5, 7, 5, $\dot{7}$;
61, $\dot{1}$, 12, 3, 4, 9, 5, 5, 9, 4, 3, $\dot{1}\dot{2}$.

(5.) $\dot{1}$, 2, 10, 2, $\dot{1}$;
$\dot{1}\dot{0}$, 15, 25, 25, $\dot{1}\dot{5}$;
$\dot{2}\dot{5}$, 20, 5, 20, $\dot{2}\dot{5}$.

(6.) $\dot{2}$, $\dot{4}$;
$\dot{2}$, $\dot{2}$;
$\dot{2}$, $\dot{1}$.

(7.) $2 + \underset{\dot{1}+}{\cfrac{1}{}}\ \underset{1+}{\cfrac{1}{}}\ \underset{\dot{3}+}{\cfrac{1}{}}$.

(8.) $\dot{3} + \underset{\dot{6}+}{\cfrac{1}{}}$; $-\underset{\dot{6}+}{\cfrac{1}{}}\ \underset{\dot{3}+}{\cfrac{1}{}}$.

(10.) $a+\dfrac{1}{2a+}$; $a+(\alpha^{n-1}-\beta^{n-1})/(\alpha^n-\beta^n)$, α and β being the roots of $x^2-2ax-1=0$. (11.) $\frac{1}{2}\{a+\sqrt{(a^2+4)}\}$; $(\alpha^{n+1}-\beta^{n+1})/(\alpha^n-\beta^n)$, where α and β are the roots of $x^2-ax-1=0$. (12.) $\frac{1}{2}\{a-\sqrt{(a^2-4)}\}$; $(\alpha^n-\beta^n)/(\alpha^{n+1}-\beta^{n+1})$, where α and β are the roots of $x^2-ax+1=0$. (13.) $\{-ab+\sqrt{(a^2b^2+4ab)}\}/2a$; if α, β be the roots of $x^2-(ab+2)x+1=0$, then $p_{2n}=b(\alpha^n-\beta^n)/(\alpha-\beta)$, $q_{2n}=(\alpha^{n+1}-\beta^{n+1}-\alpha^n+\beta^n)/(\alpha-\beta)$, and $p_{2n-1}=(p_{2n}-p_{2n-2})/b$, $q_{2n-1}=(q_{2n}-q_{2n-2})/b$. (14.) $-1+\sqrt{[\{3(\alpha^n-\beta^n)+2(\alpha^{n-1}-\beta^{n-1})\}/(\alpha^{n+1}-\beta^{n+1})]}$, where α and β are the roots of $x^2-x-1=0$. (20.) $-\frac{1}{2}n+\sqrt{[\{(\frac{1}{4}n^2+n)(\alpha^{n-1}-\beta^{n-1})+(\frac{1}{4}n^2+1)(\alpha^{n-2}-\beta^{n-2})\}/(\alpha^n-\beta^n)]}$, where α and β are the roots of $x^2-x-1=0$.

XXXII.

(1.) $3+7t$, $2-5t$. (2.) $17t+7$, $16t+5$. (3.) $2206-7t$, $11t-3309$. (4.) $1013t-3021756$, $1367t-4077746$. (5.) 13. (6.) 280. (7.) 6. (8.) If 25 fr. = 20s., 41. (9.) Buy 300 of each and spend 1021d. (10.) 69. (12.) 19. (13.) 715. (14.) 697.

XXXIV.

(1.) Converges. (2.) Converges. (3.) Oscillates. (4.) Converges. (5.) Converges. (6.) Converges. (7.) Converges if $k>2$, oscillates if $k>2$. (8.) Converges. (9.) Oscillates. (10.) Oscillates. (15.) Each of the fractions converges to 1. (23.) e. (24.) $1/(1-e)$. (25.) $\log_e 2$. (26.) $(3-e)/(e-2)$.

XXXIX.

(1.) 11/30. (2.) 3/11, 29/44, 3/44. (3.) $m(m+2n)/(m+n)^2$, $m(m+2n-1)/(m+n)(m+n-1)$. (4.) $(365.4^n+1)/(1461)^n$. (5.) 4/9. (7.) 55/672, 299/2688. (8.) 1/42. (9.) $(n-1)/n(2n-1)$. (10.) $(39!)^2/26!52!$, $4(39!)^2/26!52!$. (11.) $2(r-1)/n(n-1)$. (13.) $7n/2$, or, if this be not integral, the two integers on either side of it. (14.) $\sum_{r=2}^{n+1} r(r-1)n(n-1)\ldots(n-r+2)/n^r$. (18.) 16/31, 8/31, 4/31, 2/31, 1/31. (19.) The chances in A's favour are 6/10, 7/10, 8/10, 9/10, when he is 1, 2, 3, 4 up respectively. (20.) 25 to 2. (23.) $(1-1/n)/2$, $(1-1/n)/(2-1/n)$.

XL.

(1.) £1 : 11 : 6. (2.) His expectations are 11s. 6d. and 10s. 4½d. respectively. (3.) £8 : 5 : 9½, £2 : 4 : 2½. (4.) $n(1-1/2^r)$, $(1-1/2^r)^n$. (7.) 7s. 2½d.; $(n+1)(4n-1)/6n$. (12.) £6, £1, £4 : 2 : 2½.

INDEX OF PROPER NAMES,
PARTS I. AND II.

The Roman numeral refers to the parts, the Arabic to the page.

ABEL, ii. 119, 122, 127, 130, 263
Adams, ii. 219, 227
Alkhayami, ii. 423
Allardice, i. 441
Archimedes, ii. 415
Argand, i. 222, 254
Arndt, ii. 478
BABBAGE, ii. 156
Bernoulli, James, ii. 204, 208, 252, 377, 379, 559, 577
Bernoulli, John, ii. 251, 274, 342, 377, 556
Bertrand, ii. 112, 118, 159
Bezout, i. 358
Blissard, i. 84
Bombelli, i. 201
Bonnet, ii. 118, 159
Boole, ii. 209, 372
Bourguet, ii. 159, 229
Briggs, i. 529
Briot and Bouquet, ii. 370
Brouncker, ii. 327, 421, 451, 488
Burckhardt, ii. 508
Bürgi, i. 530
CARDANO, i. 253
Catalan, ii. 159, 196, 227, 229, 390
Cauchy, i. 77, 254; ii. 42, 47, 83, 97, 98, 110, 118, 127, 147, 164, 202, 204, 215, 263, 316, 320, 370
Cayley, ii. 33, 287, 289, 301, 347
Clausen, ii. 322, 475
Clerk-Maxwell, ii. 301
Cossali, i. 191
Cotes, i. 247
Cramer, ii. 370
DASE, ii. 508
De Gua, ii. 370
De Morgan, i. 254, 346; ii. 112, 118, 360, 370, 394, 550, 576
Demoivre, i. 239, 247; ii. 274, 282, 375, 377, 379, 381, 385, 546, 564, 569, 577
Desboves, ii. 63
Descartes, i. 201
Diophantos, ii. 445
Dirichlet, ii. 95, 126, 133, 424, 445
Du Bois Reymond, ii. 119, 130, 131, 160
Durège, ii. 370
ELY, ii. 209, 320
Euclid, i. 47, 272
Euler, i. 254; ii. 81, 97, 164, 228, 256, 274, 309, 315, 317, 318, 319, 320, 321, 324, 334, 339, 341, 342, 382, 392, 421, 466, 468, 484, 487, 498, 511, 522, 523, 525, 527, 528, 535
FAVARO, ii. 421
Fermat, ii. 450, 471, 518, 522, 563
Ferrers, ii. 534
Fibonacci, i. 202
Fort, ii. 77
Fourier, ii. 122
Franklin, ii. 88, 535
Frost, ii. 99, 370, 371
GALOIS, ii. 477
Gauss, i. 46, 254; ii. 81, 160, 309, 321, 445, 495, 514, 522, 525
Glaisher, i. 172, 530; ii. 216, 333, 347, 371, 384, 395, 508
Goldbach, ii. 395
Grassmann, i. 254
Gray, ii. 219
Greenhill, ii. 289
Gregory, ii. 97
Gregory, James, ii. 309, 327
Grillet, ii. 59
Gronau, ii. 289
Gross, ii. 513
Gudermann, ii. 287, 289
Günther, ii. 288, 421
HAMILTON, i. 254
Hankel, i. 5, 254
Harriot, i. 201
Heilermann, ii. 490
Heine, ii. 95, 499
Heis, ii. 289
Herigone, i. 201
Hermite, ii. 445
Hero, i. 83
Hindenburg, ii. 467
Horner, i. 346
Houel, ii. 288

Hutton, i. 201
Huyghens, ii. 421, 552, 559, 564
Jacobi, ii. 445
Jensen, ii. 160
Jordan, i. 76 ; ii. 32
Kohn, ii. 119
Kramp, ii. 4, 374
Kronecker, ii. 213
Kummer, ii. 119, 160, 445
Lagrange, i. 57, 451 ; ii. 370, 421, 423, 425, 450, 522, 525
Laisant, ii. 289, 312, 334
Lambert, i. 176 ; ii. 288, 321, 421, 489, 495
Laplace, ii. 50, 577
Laurent, ii. 160, 551, 577
Legendre, ii. 445, 484, 495, 535
Leibnitz, ii. 309, 377
Lionnet, ii. 225, 228
Lock, ii. 247
Longchamps, ii. 97
Macdonald, i. 530
Machin, ii. 309
Malmsten, ii. 80, 118
Mascheroni, ii. 81
Mayer, F. C., ii. 288
Mercator, ii. 288
Mertens, ii. 127
Metius, ii. 415
Meyer, ii. 577
Möbius, ii. 371, 466, 476
Montmort, ii. 379, 381, 556, 564, 577, 578
Muir, i. 358 ; ii. 311, 443, 466, 469, 474, 475, 476, 490
Napier, i. 171, 201, 254, 529
Netto, ii. 32
Newton, i. 201, 436, 472, 474, 479 ; ii. 256, 306, 327, 349, 362, 370, 375, 563
Nicolai, ii. 81
Oughtred, i. 201, 256
Pacioli, i. 202
Pascal, i. 67 ; ii. 556, 563
Paucker, ii. 119
Peacock, i. 254
Pfaff, ii. 311
Pringsheim, ii. 161
Puiseaux, ii. 370
Purkiss, ii. 61
Pythagoras, ii. 503
Raabe, ii. 118, 348
Recorde, i. 216
Reynaud and Duhamel, ii. 49
Riemann, i. 254 ; ii. 241, 301
Rudolf, i. 200
Salmon, i. 440
Sang, i. 530
Scheubel, i. 201
Schlömilch, ii. 45, 51, 80, 98, 160, 186, 335, 349, 478, 495
Seidel, ii. 131, 478
Serret, i. 76 ; ii. 32, 416, 425, 443, 453
Shanks, ii. 310
Sharp, ii. 309
Smith, Henry, ii. 445, 471
Sprague, i. 531 ; ii. 88
Stainville, ii. 311
Stern, ii. 421, 469, 477, 478, 489, 497
Stevin, i. 171, 201
Stifel, i. 81, 200
Stirling, ii. 344, 375, 378, 395, 561
Stolz, ii. 157, 161, 370
Sutton, i. 531
Sylvester, i. 48, 176 ; ii. 466, 475, 528, 533
Tait, ii. 229
Tartaglia, i. 191
Tchebichef, ii. 159
Thomæ, ii. 160, 370
Todhunter, ii. 247, 252, 546, 552, 556, 577
Van Ceulen, ii. 309
Vandermonde, ii. 9
Venn, ii. 539
Viète, i. 201 ; ii. 252
Vlacq, i. 530
Wallace, ii. 288, 290, 291
Wallis, ii. 327, 421, 450, 499, 509
Waring, ii. 391, 525, 527
Weierstrass, i. 230 ; ii. 138, 144, 161
Whitworth, ii. 25, 33, 537, 561, 577
Wilson, ii. 523
Wolstenholme, i. 443 ; ii. 17, 33, 348, 519
Wronski, ii. 188

THE END.

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